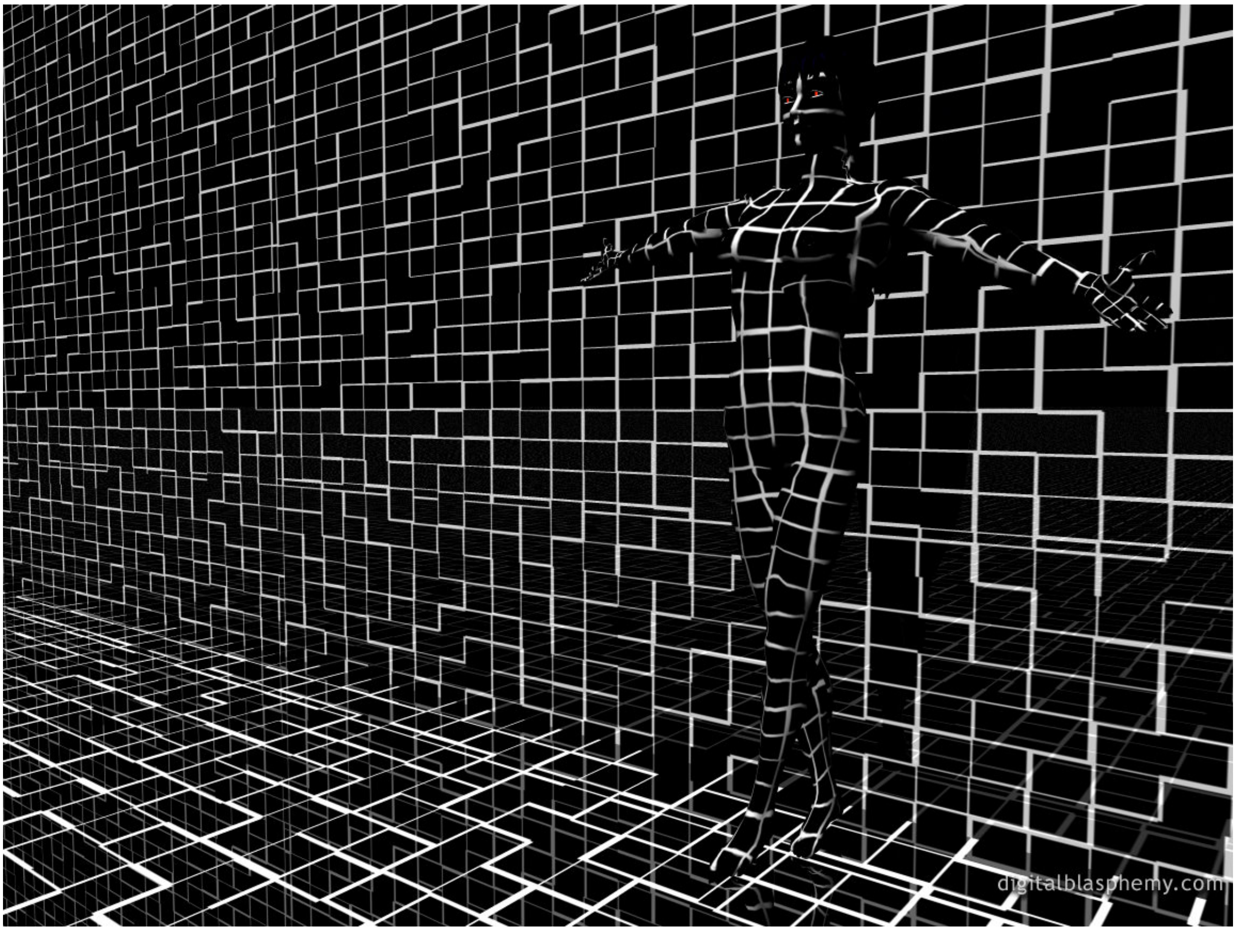
Year 1 Pure Mathematics Examination Revision

TWENTY-ONE TODAY

Four 50 minute Revision Sessions
of 21 Short, Sharp Questions
(Full detailed answers included)



PURE MATHEMATICS YEAR 1 REVISION

Twenty-One Today

NUMBER 1

Twenty-One short, sharp questions on all aspects of the Year 1 course
You may use a calculator

Question 1

Rationalise the denominator,

$$\frac{1}{\sqrt{7} - 1}$$

Question 2

$$f(x) = 4x^3 - 7x^2 + ax + 13$$

$f(x)$ is divisible by $(x - 1)$

Find the value of the constant, a

Question 3

A circle has centre (8, 15) and its circumference passes through the origin.

Determine the equation of the circle.

Question 4

Use the rules of logs to simplify

$$log_m 45 - 2\ log_m 3$$

Question 5

Using your answer to Question 4, or otherwise, state the value of m if

$$log_m 45 - 2\ log_m 3 = 1$$

Question 6

$$f(x) = \frac{12}{x^2}$$

What is the integral of $f(x)$?

Don't forget the constant of the integration !

Question 7

$$f(x) = \frac{x^3}{3} + 5x^2 + 27x - 100$$

Prove that $f(x)$ is an increasing function.

HINT : Differentiate, complete the square, explain why derivative is always positive.

Question 8

Find the equation of the curve which passes through $(3, -1)$ and for which

$$\frac{dy}{dx} = 2x + 5$$

Question 9

Which two of the following lines are mutually perpendicular ?

(a) $y = \frac{1}{\sqrt{2}}x + 3$

(b) $y = \frac{1}{2}x + 3$

(c) $y = -2x + 3$

(d) $y = \sqrt{2}\,x + 3$

Question 10

Consider these three facts about a function $f(x)$

$$f(x) = x^3 + x^2 + px + q$$

$f(x)$ is divisible by $(x - 1)$

$f(x)$ Has remainder -7 when divided by $(x - 2)$

Determine the value of p and the value of q.

Question 11
Solve

$$7^{x} = 102$$

Give your answer to three significant figures.

Question 12
By first expanding the brackets, solve

$$2^{x}(2^{x} - 5) + 6 = 0$$

Question 13
What is the reverse of the process known as integration called ?

Question 14
The endpoints of the diameter of a circle are $(-8,\ 1)$ and $(4,\ 5)$
Find the equation of the circle.

Question 15
Provide a counter-example to show that the following is not true;

$$1 + x^2 < (1 + x)^2$$

Question 16
Solve the following equation over the interval $0 \leqslant \theta \leqslant 360°$

$$2\cos^2\theta - 5\cos\theta + 2 = 0$$

Question 17
Without using the binomial theorem or logarithms solve,

$$(2x + 1)^5 = 32$$

Question 18
Expand the brackets

$$\left(\frac{x}{3} - 1\right)^3$$

Question 19

Sketch the curve of the equation $y = 25 - x^2$, giving the coordinates of any points where it crosses axes.

Question 20

Using Question 19, solve the inequality

$$25 - x^2 \leqslant 0$$

Question 21

Determine the area enclosed by the curve $y = 25 - x^2$ and the x-axis.

Twenty-One Today
N U M B E R 1
Solutions

Answer 1

$$\frac{1}{\sqrt{7}-1} \times \frac{(\sqrt{7}+1)}{(\sqrt{7}+1)} = \frac{\sqrt{7}+1}{7-\sqrt{7}+\sqrt{7}-1}$$

$$= \frac{\sqrt{7}+1}{6}$$

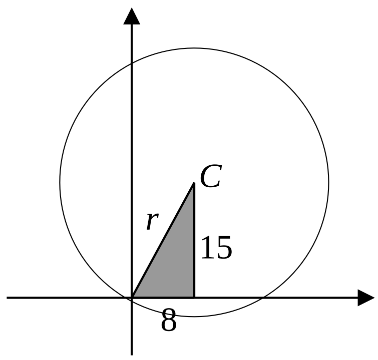
Answer 2
By the factor theorem, $f(1) = 0$
So,

$$4 \times 1^3 - 7 \times 1^2 + a \times 1 + 13 = 0$$

$$4 - 7 + a + 13 = 0$$

$$a = -10$$

Answer 3



$$r^2 = 15^2 + 8^2$$

$$= 225 - 64$$

$$= 289$$

$$r = 17$$

Equation of the circle is,

$$(x - 8)^2 + (y - 15)^2 = 17^2$$

Answer 4

$$log_m 45 - 2\, log_m 3 = log_m 45 - log_m 3^2$$

$$= log_m \left(\frac{45}{9} \right)$$

$$= log_m 5$$

Answer 5

$$\log_m 45 - 2\log_m 3 = 1$$
$$\log_m 5 = 1$$
$$\therefore\ m^1 = 5 \qquad \text{i.e. } m = 5$$

Answer 6

$$f(x) = 12x^{-2}$$
$$\int f(x)\,dx = \frac{12x^{-1}}{-1} + c$$
$$= -\frac{12}{x} + c$$

Answer 7

$$f'(x) = \left[x^2 + 10x\right] + 27$$
$$= \left[(x+5)^2 - 25\right] + 27$$
$$= (x+5)^2 + 2$$

The smallest that $(x+5)^2$ can be is zero.
Even then, 2 is added on.
$\therefore f'(x)$ is always positive.
i.e. $f(x)$ is an increasing function.

Answer 8
By integration,

$$y = x^2 + 5x + c$$
$$-1 = 3^2 + 5 \times 3 + c$$
$$c = -25$$
$$\therefore y = x^2 + 5x - 25$$

Answer 9
Lines (b) and (c) are mutually perpendicular because the product of their gradients is – 1
That is,

$$\frac{1}{2} \times (-2) = -1$$

Answer 10

By the factor theorem,

$$f(1) = 0$$

$$p + q = -2$$

By the remainder theorem,

$$f(2) = -7$$

$$2p + q = -19$$

Solve these two equations simultaneously to obtain,

$$p = -17 \quad \text{and} \quad q = 15$$

Answer 11

Method 1

$$7^x = 102$$

take the log_e of both sides

$$ln\, 7^x = ln\, 102$$

$$x\, ln\, 7 = ln\, 102$$

$$x = \frac{ln\, 102}{ln\, 7}$$

$$x = 2.38$$

Method 2

$$log_a b = c \quad \Leftrightarrow \quad a^c = b$$

The "jump out of logs" manoeuvre used backwards gives

$$7^x = 102 \;\Leftrightarrow\; log_7 102 = x$$

$$x = 2.38$$

Answer 12

$$(2^x)^2 - 5(2^x) + 6 = 0$$

Let $z = 2^x$

$$z^2 - 5z + 6 = 0$$

$$(z - 3)(z - 2) = 0$$

Either	or
$2^x = 3$	$2^x = 2$
$x = 1.58$	$x = 1$

Answer 13

Differentiation

Answer 14

$$\begin{aligned} Diameter &= \sqrt{(4 - (-8))^2 + (5 - 1)^2} \\ &= \sqrt{12^2 + 4^2} \\ &= \sqrt{160} \\ &= 4\sqrt{10} \\ \therefore\ Radius &= 2\sqrt{10} \\ Midpoint &= \left(\frac{-8 + 4}{2},\ \frac{1 + 5}{2} \right) \\ &= (-2,\ 3) \end{aligned}$$

Equation of circle : $(x + 2)^2 + (y - 3)^2 = 40$

Answer 15

Let $x = -1$, for example.

The statement given then becomes, $2 < 0$, which is not true.

Answer 16

$$2\cos^2\theta - 5\cos\theta + 2 = 0$$

$$(2\cos\theta - 1)(\cos\theta - 2) = 0$$

Either or

$\cos\theta = \frac{1}{2}$ (*two solutions*) $\cos\theta = 2$ (*no solutions*)

$$\theta = 60°,\ 300°$$

Answer 17

$$\begin{aligned} (2x + 1)^5 &= 32 \\ 2x + 1 &= 2 \\ 2x &= 1 \\ x &= \frac{1}{2} \end{aligned}$$

Answer 18

$$\left(\frac{x}{3} - 1\right)^3 = +1 \times \left(\frac{x}{3}\right)^3 \times (-1)^0$$
$$+3 \times \left(\frac{x}{3}\right)^2 \times (-1)^1$$
$$+3 \times \left(\frac{x}{3}\right)^1 \times (-1)^2$$
$$+1 \times \left(\frac{x}{3}\right)^0 \times (-1)^3$$
$$= \frac{x^3}{27} - \frac{x^2}{3} + x - 1$$

Answer 19

Crosses x-axis at $(-5,\ 0)$ or $(5,\ 0)$
Crosses y-axis at $(0, 25)$

Answer 20

$x \leqslant -5$ or $x \geqslant 5$

Answer 21

By symmetry,

$$2\int_0^5 25 - x^2\, dx = 2\left[25x - \frac{x^3}{3}\right]_0^5$$
$$= 2\left[25 \times 5 - \frac{5^3}{3}\right]$$
$$= 166\frac{2}{3}$$

Twenty-One Today

N U M B E R 2

Twenty-One short, sharp questions on all aspects of the Year 1 course
You May Use A Calculator

Question 1

Find

$$\int_4^{25} \sqrt{x}\ dx$$

Question 2

Differentiate $y = 4x^3 + 5x$

Question 3

$$f(x) = \frac{12}{x^2}$$

Determine $f'(-2)$

Question 4

$$f(x) = x^2 - 9 \qquad x \in \mathbb{R}$$

Explain why the function $f(x)$ does NOT have an inverse.

Question 5

$$g(x) = \frac{2x^2 + 9x - 18}{(x+6)(x+1)} \qquad x \in \mathbb{R},\ x \neq 1,\ 6$$

(i) Simplify $g(x)$ as far as possible.

(ii) Hence, give an expression for $g^{-1}(x)$ also simplified as far as possible.

Question 6

Sketch the curve of the equation $y = x^2 - 16$, giving the coordinates of any points where it crosses axes.

Question 7

Using Question 6, solve the inequality,

$$x^2 - 16 > 0$$

Question 8

Determine the gradient of the line with equation,

$$2y - 5x - 1 = 0$$

Question 9

The vectors $5\boldsymbol{a} + k\boldsymbol{b}$ and $8\boldsymbol{a} + 56\boldsymbol{b}$ are parallel.
Find the value of k

Question 10

What is the value of the discriminant of $x^2 + 12x + 37 = 0$?

Question 11

The equation $x^2 + 2kx + 4k = 0$, where k is a non-zero integer, has equal roots. Find the value of k.

Question 12

Definition :

A unit vector in the direction of $\boldsymbol{a}$ *is* $\dfrac{\boldsymbol{a}}{|\boldsymbol{a}|}$

Given that $\boldsymbol{a} = 6\boldsymbol{i} + 8\boldsymbol{j}$ find

(i) $|\boldsymbol{a}|$

(ii) a unit vector in the direction of $\boldsymbol{a}$

Question 13
Sketch the curve with equation

$$y = x^2 - 4x + 3$$

Annotate your sketch with the coordinates of the three points where the curve cuts the axes.

Question 14
Beginning "LHS = ", show that

$$\frac{3x^2 + 4\sqrt{x}}{2x} = 1.5x + 2x^{-\frac{1}{2}}$$

Question 15
By considering Question 14, determine $g'(x)$

$$g(x) = \frac{3x^2 + 4\sqrt{x}}{2x}$$

Question 16

Consider the Arithmetic Sequence 8, 1, x, y, ...

Determine the value of x and the value of y.

Question 17

The formula for the n^{th} term of an Arithmetic progression is given in the Exam Formulae book:

$$U_n = a + (n - 1)d$$

along with two formula to sum such a series:

$$Sum_{AP} = \frac{1}{2}n(a + l) = \frac{n}{2}\{2a + (n - 1)d\}$$

where a is the first term

n is the number of terms

d is the common difference

l is the last term

Use two of these formulae to calculate;

$$4 + 15 + 26 + ... + 213$$

Question 18

Determine the value of this series
(which is not in Arithmetic or Geometric Progression)

$$\sum_{1}^{4} \left(m^2 - 1 \right)$$

Question 19

The point $(3, \; 5)$ is on the curve $y = x^2 - 2x + c$
State the value of c.

Question 20

Given that $10000\sqrt{10} = 10^k$, find the value of k

Question 21

Solve the simultaneous equations

$$y = 2x - 5$$
$$x^2 + y^2 = 25$$

Twenty-One Today
N U M B E R   2
Solutions

Answer 1

78

Answer 2

$$\frac{dy}{dx} = 12x^2 + 5$$

Answer 3

$$f'(-2) = 3$$

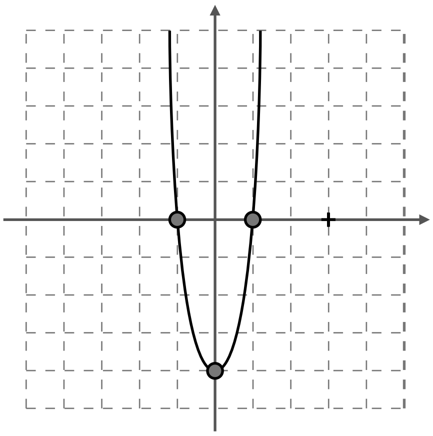
Answer 4

Only one-to-one functions have an inverse.
$f(x)$ is NOT one-to-one

Answer 5

(i) $g(x) = \dfrac{2x - 3}{x + 1}$ **(ii)** $g^{-1}(x) = \dfrac{x + 3}{2 - x}$

Answer 6



On y-axis : (0, – 16)   On x-axis : (– 4, 0) and (4, 0)

Answer 7

$x < -4$ or $x > 4$

Answer 8

2.5

Answer 9

$k = 35$

Answer 10

-4

Answer 11

Consider the discriminant, D

$$D = 0 \quad \text{gives equal roots}$$

$$b^2 - 4ac = 0$$

$$(2k)^2 - 4 \times 1 \times (4k) = 0$$

$$4k^2 - 16k = 0$$

$$4k(k - 4) = 0$$

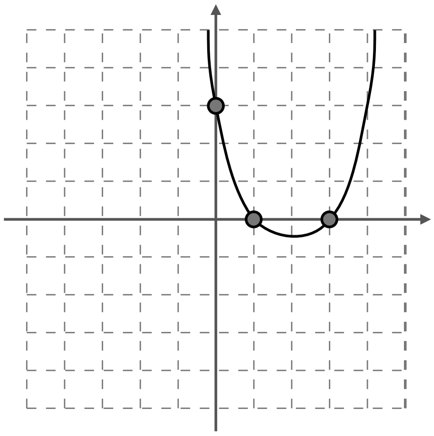
Either $k = 0$ or $k = 4$

Reject $k = 0$ as told k is Non zero $\therefore\ k = 4$

Answer 12

(i) 10 **(ii)** $0.6\,\boldsymbol{i} + 0.8\,\boldsymbol{j}$

Answer 13



On y-axis : (0, 3) On x-axis : (1, 0) and (3, 0)

Answer 14

$$\text{LHS} = \frac{3x^2 + 4\sqrt{x}}{2x}$$

$$= \frac{3x^2}{2x} + \frac{4x^{\frac{1}{2}}}{2x}$$

$$= 1.5x + 2x^{-\frac{1}{2}}$$

$$= \text{RHS} \qquad \square$$

Answer 15

$$g' 9(x) = 1.5 - x^{-\frac{3}{2}}$$

Answer 16

$x = -6, \; y = -13$

Answer 17

$$U_n = a + (n - 1)d$$
$$213 = 4 + (n - 1) \times 11$$
$$n - 1 = \frac{209}{11}$$
$$n = 20$$

$$Sum = \frac{1}{2} n (a + l)$$
$$= \frac{20}{2} (4 + 213)$$
$$= 10 \times 217$$
$$= 2170$$

Answer 18

$0 + 3 + 8 + 15$

$= 26$

Answer 19

$c = 2$

Answer 20

$k = 4.5$

Answer 21

$$x^2 + (2x - 5)^2 = 25$$
$$x^2 + 4x^2 - 20x + 25 = 25$$
$$5x^2 - 20x = 0$$
$$5x(x - 4) = 0$$
$$x = 0 \quad \text{or} \quad x = 4$$

Thus, the solutions are, $(0, -5)$ or $(4, 3)$

Twenty-One Today

NUMBER 3

Twenty-One short, sharp questions on all aspects of the Year 1 course
You may use a calculator

Question 1

$$y = 4\sqrt{x}$$

Find the coordinates of the only point on this curve with a gradient of 8

Question 2

$$g(x) = x^3 - 7x^2 - x + 23$$

Find the remainder when $g(x)$ is divided by $(x + 2)$

Question 3

Give the exact solution to the equation

$$ln(11x - 5) = 3$$

Question 4

The distance between the points $(-2,\ 10)$ and $(x,\ 6)$ is $\sqrt{65}$

Find the two possible values of x

Question 5

(i) Work out the value of the discriminant of

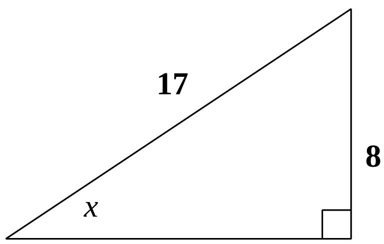
$$f(x) = x^2 + x + 2$$

(ii) What does your part **(i)** answer tell you about the number of roots of $f(x)$?

Question 6

If $\sin\theta = \frac{8}{17}$ state the exact value of $\cos\theta$ and the exact value of $\tan\theta$

HINT :



Question 7

Simplify;

$$\frac{101\,!}{99\,!}$$

Question 8

Differentiate, $f(x) = \dfrac{x^3 + 9x}{3x^2}$

HINT : Use the 'wedge' technique

Question 9

Integrate, $f(x) = \dfrac{x^4 + 5x^2}{x}$

Question 10

Solve over the interval $0 \leqslant \theta \leqslant 360°$

$$2\,cos\,(\theta) = \sqrt{3}$$

Question 11

Given that the point A has position vector $6\boldsymbol{i} - 3\boldsymbol{j}$ and the point B has position vector $8\boldsymbol{i} + 5\boldsymbol{j}$ find the vector, $\overrightarrow{AB}$

Question 12

Find the equation of the normal to the curve with equation $y = 11 - 4\sqrt{x}$ at the point where $x = 4$

Question 13

Expand the brackets, $y = (3 - 2x)^3$

Question 14

Solve the equation, $log_3 x = -\frac{1}{2}$ leaving your answer as an exact value.

Question 15

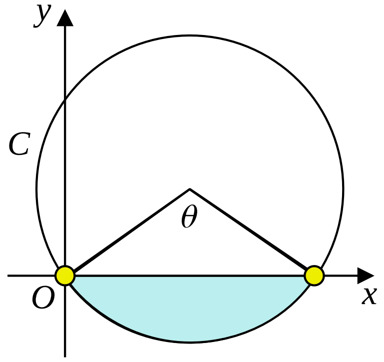
Solve the equation, $log_7 (y + 3) + log_7 (2y + 1) = 1, \quad y \in \mathbb{R}, \quad y > -\frac{1}{2}$

Question 16

Solve the equation, $9^x - 3^{x+1} - 18 = 0$

Question 17

A sketch of a circle, *C*, along with its equation is presented below;



$$(x - 3)^2 + (y - 4)^2 = 5^2$$

Find the coordinates of the two points where *C* crosses the *x*-axis.

Question 18

Considering further the circle, *C*, presented in Question 17, find, in degrees, the size of the angle marked θ.

Question 19

Find the set of values of *x* for which

$$12 + 4x > x^2$$

Question 20

Evaluate, $f(x) = \int_1^2 \frac{24}{x^3}\,dx$

Question 21

Solve, for $0 \leqslant \theta \leqslant 360°$, the equation, $2\cos^2\theta + \sin\theta - 1 = 0$

Twenty-One Today
N U M B E R 3
Solutions

Answer 1

$$y = 4x^{\frac{1}{2}}$$
$$\frac{dy}{dx} = 2x^{-\frac{1}{2}}$$
$$\therefore \frac{2}{\sqrt{x}} = 8$$
$$x = \frac{1}{16} \quad \text{or} \quad 0.0625$$
$$\therefore y = 4 \times \sqrt{\frac{1}{16}}$$
$$= 1$$

The coordinates of the only point on the curve with gradient 8 are (0.0625, 1)

Answer 2

By the remainder theorem, the remainder is given by $g(-2)$

$$g(-2) = (-2)^3 - 7 \times (-2)^2 - (-2) + 23$$
$$= -8 - 28 + 2 = 23$$
$$= -11$$

Alternatively, the long division could be carried out.

Answer 3

Method 1

$$log_a b = c \quad \Leftrightarrow \quad a^c = b$$

This “jump out of logs” manoeuvre gives

$$log_e(11x - 5) = 3 \Leftrightarrow e^3 = 11x - 5$$
$$x = \frac{e^3 + 5}{11}$$

Method 2

$$ln(11x - 5) = 3$$

Exponentiate both sides

$$e^{ln(11x-5)} = e^3$$
$$e^3 = 11x - 5$$
$$x = \frac{e^3 + 5}{11}$$

Answer 4

$$Distance = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$65 = (x - (-2))^2 + (6 - 10)^2$$

$$65 = (x + 2)^2 + (-4)^2$$

$$65 = x^2 + 4x + 20$$

$$x^2 + 4x - 45 = 0$$

$$(x + 9)(x - 5) = 0$$

$$x = -9 \text{ or } x = 5$$

Answer 5

(i) Discriminant of $a x^2 + bx = c = 0$ is $D = b^2 - 4ac$

$$\therefore \text{ for } f(x),\ D = 1 - 4 \times 1 \times 2$$

$$= -7$$

(ii) As $D < 0$, $f(x)$ has no roots

Answer 6

$$cos^2\theta + sin^2\theta = 1$$

$$cos^2\theta + \left(\frac{8}{17}\right)^2 = 1$$

$$cos^2\theta = \frac{289 - 64}{289}$$

$$= \frac{225}{289}$$

$$\therefore cos\,\theta = \frac{15}{17}$$

$$tan\,\theta = \frac{sin\,\theta}{cos\,\theta}$$

$$= \frac{8}{15}$$

Answer 7

$$\frac{101!}{99!} = \frac{101 \times 100 \times 99!}{99!}$$

$$= 10100$$

Answer 8

$$f(x) = \frac{x^3 + 9x}{3x^2}$$
$$= \frac{x^3}{3x^2} + \frac{9x}{3x^2}$$
$$= \frac{x}{3} + 3x^{-1}$$
$$\text{So, } f'(x) = \frac{1}{3} - 3x^{-2}$$
$$= \frac{1}{3} - \frac{3}{x^2}$$

Answer 9

$$\int f(x)\ dx = \int \frac{x^4 + 5x^2}{x}\ dx$$
$$= \int \frac{x^4}{x} + \frac{5x^2}{x}\ dx$$
$$= \int x^3 + 5x\ dx$$
$$= \frac{x^4}{4} + \frac{5x^2}{2} + c$$

Answer 10

$$2\cos\theta = \sqrt{3}$$
$$\cos\theta = \frac{\sqrt{3}}{2}$$
$$\theta = 30°,\ 330°$$

Answer 11

$$\overrightarrow{AB} = b - a$$
$$= \begin{pmatrix} 8 \\ 5 \end{pmatrix} - \begin{pmatrix} 6 \\ -3 \end{pmatrix}$$
$$= \begin{pmatrix} 2 \\ 8 \end{pmatrix}$$

i.e. $2\boldsymbol{i} + 8\boldsymbol{j}$

Answer 12

$$y = 11 - 4x^{\frac{1}{2}}$$

$$\frac{dy}{dx} = -2x^{-\frac{1}{2}}$$

$$= -\frac{2}{\sqrt{x}}$$

Inserting $x = 4$ gives $\frac{dy}{dx} = -1$

$\therefore$ Normal's gradient $= 1$

Also $y = 11 - 4\sqrt{4}$

So $(4,\ 3)$ is on the normal

$$y = mx + c$$

$$3 = 1 \times 4 + c$$

$$c = -1$$

Normal is $y = x - 1$

Answer 13

$$(3 - 2x)^3 = 1 \times (3)^3 \times (-2x)^0$$
$$+ 3 \times (3)^2 \times (-2x)^1$$
$$+ 3 \times (3)^1 \times (-2x)^2$$
$$+ 1 \times (3)^0 \times (-2x)^3$$
$$= 27 - 54x + 36x^2 - 8x^3$$

Answer 14

Method 1

$$log_a b = c \quad \Leftrightarrow \quad a^c = b$$

This “jump out of logs” manoeuvre gives

$$log_3 x = -\frac{1}{2} \Leftrightarrow 3^{-\frac{1}{2}} = x$$

$$x = \frac{1}{\sqrt{3}}$$

Method 2

$$log_3 x = -\frac{1}{2}$$

Raise both sides to the power 3

$$3^{log_3 x} = 3^{-\frac{1}{2}}$$

$$x = -\frac{1}{\sqrt{3}}$$

Answer 15

$$log_7(y + 3) + log_7(2y + 1) = 1 \qquad y \in R,\ y > -\frac{1}{2}$$

$$log_7((y + 3)(2y + 1)) = 1$$

$$log_7\left(2y^2 + 7y + 3\right) = 1$$

$$2y^2 + 7y + 3 = 7^1$$

$$2y^2 + 7y - 4 = 0$$

$$(2y - 1)(y + 4) = 0$$

$$y = \frac{1}{2} \quad \text{or} \quad y = -4$$

But $y > -\frac{1}{2}$

so the only solution is $y = \frac{1}{2}$

Answer 16

$$9^x - 3^{x+1} - 18 = 0$$

$$\left(3^x\right)^2 - 3\left(3^x\right) - 18 = 0$$

$$\left(3^x - 6\right)\left(3^x + 3\right) = 0$$

$$3^x = 6 \quad or \quad 3^x = -3 \text{ (no solution)}$$

$$x = \frac{ln\,6}{ln\,3}$$

$$= 1.631$$

Answer 17

The circle will cross the x-axis when $y = 0$ in which case,

$$(x - 3)^2 + (-4)^2 = 25$$

$$(x - 3)^2 = 9$$

square root both sides

$$x - 3 = \pm 3$$

$$x = 0 \quad \text{or} \quad x = 6$$

$$Points\ :\ (0,\ 0),\ (6,\ 0)$$

Answer 18

Use the 'backwards' version of the cosine rule,

$$\cos\theta = \frac{5^2 + 5^2 - 6^2}{2 \times 5 \times 5}$$

$$= 0.28$$

$$\therefore \theta = \arccos 0.28$$

$$= 73.7°$$

Answer 19

$$12 + 4x > x^2$$

$$x^2 - 4x - 12 < 0$$

$$(x - 6)(x + 2) < 0$$

critical values are $x = -2$ and $x = 6$

Via a quick sketch graph,

$$-2 < x < 6$$

Answer 20

$$\int_1^2 24x^{-3}\,dx = \left[\frac{12}{x^2}\right]_1^2$$

$$= 9$$

Answer 21

$$2\cos^2\theta + \sin\theta - 1 = 0$$

$$2(1 - \sin^2\theta) + \sin\theta - 1 = 0$$

$$2\sin^2\theta - \sin\theta - 1 = 0$$

$$(\sin\theta - 1)(2\sin\theta + 1) = 0$$

either or

$$\sin\theta = 1 \qquad \sin\theta = -\frac{1}{2}$$

$$\theta = 90°,\ 210°,\ 330°$$

Twenty-One Today

NUMBER 4

Twenty-One short, sharp questions on all aspects of the Year 1 course
You may use a calculator

Question 1

Rationalise the denominator of, $\dfrac{\sqrt{2} + 3}{5 + \sqrt{2}}$

Question 2

Given that, $\int_2^a x^2 \, dx = 111\frac{2}{3}$

determine the value of the constant a

Question 3

$$f(x) = 5x^6 + 8x^5 - 13$$

Is $(x - 1)$ a factor of $f(x)$?
Justify your answer.

Question 4

What is the circumference of the circle with equation,

$$(x + 3)^2 + (y - 8)^2 = 64$$

Write your answer as a multiple of π.

Question 5

Find the coordinates where the line with equation $y = 4x - \dfrac{7}{2}$ meets the line with equation $y = -\dfrac{1}{4}x + 5$

Question 6

(i) What is the minimum value of the function $f(x) = (x + 7)^2 + 5$?

(ii) What is the maximum value of the function $g(x) = 8\,sin\,x + 13$?

(iii) Hence state the ***minimum*** value of,

$$\frac{f(x)}{g(x)}$$

Question 7

Write out Pascal's triangle up to the row that runs, 1, 6, 15, 20, 15, 6, 1

Question 8

Given that, in row 6 of Pascal's Triangle,

$$^{n}C_{r} = 20$$

State the value of *n* and the value of *r*

Question 9

Write without any brackets;

$$\left(\frac{3}{x} - \frac{x^2}{2}\right)^4$$

Question 10

A line segment has endpoints $(3,\ 5)$, $(8,\ -5)$

Find the equation of the perpendicular bisector of this line.

Question 11

Given that $y = 1.6\,x^4 + 0.9\,x^2$, what is $\int y\,dx$

Question 12

(i) Which part of $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ is termed the discriminant ?

(ii) Suppose that the discriminant is negative.

What can then be deduced about the graph of $y = a\,x^2 + bx + c$?

Question 13
Give the equation of the curve which passes through the point (1, 13) and which has a gradient equation given by;

$$\frac{dy}{dx} = 40\,x^3$$

Question 14
A circle has equation

$$x^2 + y^2 - 10x + 4y = 7$$

Determine the centre of the circle and its radius.

Question 15
Solve the inequality,

$$x^2 - 3 \geqslant 1$$

Question 16
Given that

$$128\sqrt{2} = 2^a$$

find the value of a

Question 17
Solve over the interval $0° \leqslant x \leqslant 360°$,

$$tan^2 x + cos^2 x = 1$$

Question 18
(i) Factorise completely,

$$f(x) = x^3 - x$$

(ii) Hence write down the coordinates of the three points where the graph of $f(x)$ crosses the x-axis

Question 19

With the help of your answer to question 18, or otherwise, sketch the graph of,

$$f(x) = x^3 - x$$

Question 20

Find the equation of the tangent to the curve $y = x^3 - x$ from the point $(1, 0)$

Question 21

Find the area bounded by $y = x^3 - x$, the x-axis, $x = -1$ and $x = 1$

Twenty-One Today
N U M B E R 4
Solutions

Answer 1

$$\frac{\sqrt{2}+3}{5+\sqrt{2}} = \frac{\sqrt{2}+3}{5+\sqrt{2}} \times \frac{5-\sqrt{2}}{5-\sqrt{2}}$$
$$= \frac{5\sqrt{2}-2+15-3\sqrt{2}}{25-5\sqrt{2}+5\sqrt{2}-2}$$
$$= \frac{13+2\sqrt{2}}{23}$$

Answer 2

$$\int_2^a x^2\,dx = 111\frac{2}{3}$$
$$\left[\frac{x^3}{3}\right]_2^a = 111\frac{2}{3}$$
$$\frac{a^3}{3} - \frac{2^3}{3} = 111\frac{2}{3}$$

multiply through by 3

$$a^3 - 2^3 = 335$$
$$a^3 = 343$$
$$a = 7$$

Answer 3

If $(x-1)$ is a factor then $f(1)$ will equal zero by the factor theorem.

$$f(1) = 5 \times 1^6 + 8 \times 1^5 - 13$$
$$= 0$$

$\therefore$ $(x-1)$ is a factor of $f(x)$

Answer 4

The radius of the circle is 8

$$C = 2\pi r$$
$$= 2\pi 8$$
$$= 16\pi$$

Answer 5

$$4x - \frac{7}{2} = -\frac{x}{4} + 5$$

multiply through by 4

$$16x - 14 = -x + 20$$

$$17x = 34$$

$$x = 2$$

The point of intersection is $(2,\ 4.5)$

Answer 6

(i) 5 **(ii)** 21 **(iii)** $\frac{5}{21}$

Answer 7

$$\begin{array}{ccccccccccccc} &&&&&&1&&&&&& \\ &&&&&1&&1&&&&& \\ &&&&1&&2&&1&&&& \\ &&&1&&3&&3&&1&&& \\ &&1&&4&&6&&4&&1&& \\ &1&&5&&10&&10&&5&&1& \\ 1&&6&&15&&20&&15&&6&&1 \end{array}$$

Answer 8

$$n = 6 \qquad r = 3 \qquad {}^{6}C_{3} = 20$$

Answer 9

$$\begin{aligned} \left(\frac{3}{x} - \frac{x^2}{2} \right)^4 &= 1 \times \left(\frac{3}{x} \right)^4 \times \left(-\frac{x^2}{2} \right)^0 \\ &+ 4 \times \left(\frac{3}{x} \right)^3 \times \left(-\frac{x^2}{2} \right)^1 \\ &+ 6 \times \left(\frac{3}{x} \right)^2 \times \left(-\frac{x^2}{2} \right)^2 \\ &+ 4 \times \left(\frac{3}{x} \right)^1 \times \left(-\frac{x^2}{2} \right)^3 \\ &+ 1 \times \left(\frac{3}{x} \right)^0 \times \left(-\frac{x^2}{2} \right)^4 \\ &= \frac{81}{x^4} - \frac{54}{x} + \frac{27x^2}{2} - \frac{3x^5}{2} + \frac{x^8}{16} \end{aligned}$$

Answer 10

To obtain the midpoint,

$$\left(\frac{3 + 8}{2}, \frac{5 + (-5)}{2} \right) = (5.5, \ 0)$$

Gradient of line between the given points,

$$m = \frac{\Delta Y}{\Delta X}$$

$$= \frac{5 - (-5)}{3 - 8}$$

$$= -2$$

The gradient of the perpendicular bisector will be the sign changed reciprocal,

$$m_p = \frac{1}{2}$$

To obtain the equation of the perpendicular bisector through (5.5, 0), gradient 0.5,

$$y = mx + c$$

$$0 = \frac{1}{2} \times 5.5 + c$$

$$\therefore \ c = -2.75$$

The equation of the line is thus,

$$y = \frac{1}{2} x - 2.75$$

Answer 11

$$\int y \, dx = \int 1.6 x^4 + 0.9 x^2 \, dx$$

$$= 0.32 x^5 + 0.3 x^3 + c$$

Answer 12

(i) The discriminant, D, is the piece under the square root.

i.e. $D = b^2 - 4ac$

(ii) If $D < 0$ then the graph of the quadratic does not cross or touch the x-axis because it has no roots.

Answer 13

$$\frac{dy}{dx} = 40x^3$$

Integrate both sides wrt x

$$y = 10x^4 + c$$

Now to use the fact that this curve passes through $(1, \; 13)$

$$13 = 10 \times 1^4 + c$$

$$c = 3$$

$$\therefore \; y = 10x^4 + 3$$

Answer 14

$$x^2 + y^2 - 10x + 4y = 7$$

$$\left[x^2 - 10x \right] + \left[y^2 + 4y \right] = 7$$

$$\left[(x - 5)^2 - 25 \right] + \left[(y + 2)^2 - 4 \right] = 7$$

$$(x - 5)^2 + (y + 2)^2 = 36$$

$\therefore$ centre $(5, -2)$, radius 6

Answer 15

$$x^2 - 3 \geqslant 1$$

$$x^2 - 2^2 \geqslant 0$$

$$(x + 2)(x - 2) \geqslant 0$$

critical values at ± 2

and so, from a sketch graph,

$x \leqslant -2$ or $x \geqslant 2$

Answer 16

$$128\sqrt{2} = 2^7 \times 2^{0.5}$$

$$= 2^{7.5}$$

Answer 17
The crucial aspect of this question is to avoid a division by *sin x* because this would be dividing by zero, which is not defined in mathematics.

Method 1

$$\tan^2 x + \cos^2 x = 1$$

$$\frac{\sin^2 x}{\cos^2 x} + \cos^2 x = 1$$

multiply through by $\cos^2 x$

$$\sin^2 x + \cos^4 x = \cos^2 x$$

$$(1 - \cos^2 x) + \cos^4 x = \cos^2 x$$

$$\cos^4 x - 2\cos^2 x + 1 = 0$$

$$(\cos^2 x - 1)^2 = 0$$

square root both sides

$$\cos^2 x - 1 = 0$$

$$\cos^2 x = 1$$

square root both sides

$$\cos x = \pm 1$$

$$\therefore x = 0°,\ 180°,\ 360°$$

Method 2

$$\tan^2 x + \cos^2 x = 1$$

$$\frac{\sin^2 x}{\cos^2 x} = \sin^2 x$$

$$\sin^2 x = \sin^2 x \cos^2 x$$

$$\sin^2 x - \sin^2 x \cos^2 x = 0$$

$$\sin^2 x (1 - \cos^2 x) = 0$$

$$\sin^2 x \sin^2 x = 0$$

$$\sin^4 x = 0$$

$$\sin x = 0$$

$$\therefore x = 0°,\ 180°,\ 360°$$

Answer 18

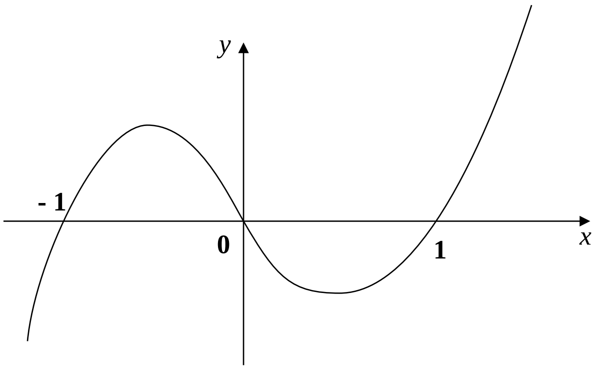
(i)

$$f(x) = x^3 - x$$
$$= x(x^2 - 1)$$
$$= x(x+1)(x-1)$$

(ii)

$$(1,\ 0),\ (0,\ 0),\ (-1,\ 0)$$

Answer 19



Answer 20

$$y = x^3 - x$$
$$\frac{dy}{dx} = 3x^2 - 1$$

from which it's apparent that the gradient when $x = 1$ is 2

$$y = mx + c$$
$$0 = 2 \times 1 + c$$
$$c = -2$$
$$\therefore \text{ Tangent } : \ y = 2x - 2$$

Answer 21

Integration will see any area underneath the *x*-axis as negative, so the previous questions have been raising awareness that this curve has this issue.

In my solution I've gone for the left hand part of the curve, which will have a positive area of 0.25, and then used the symmetry to get the area requested by doubling that answer.

$$
\begin{aligned}
Area &= 2 \times \int_{-1}^{0} x^3 - x \; dx \\
&= 2\left[\frac{x^4}{4} - \frac{x^2}{2} \right]_{-1}^{0} \\
&= \left[\frac{x^4}{2} - x^2 \right]_{-1}^{0} \\
&= [\,0\,] - \left[\frac{1}{2} - 1 \right] \\
&= \frac{1}{2}
\end{aligned}
$$