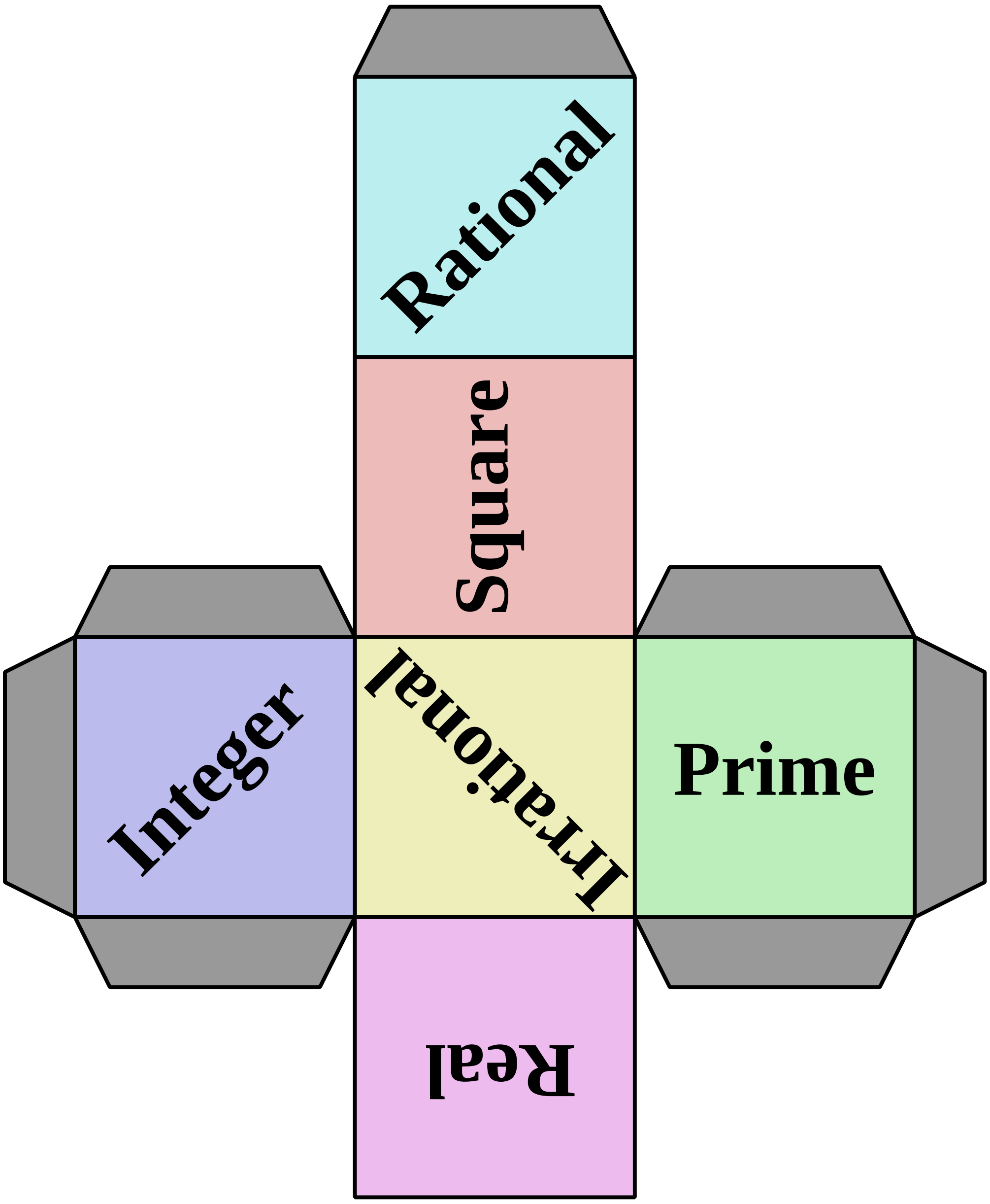
The Classification of Numbers



The Classification of Numbers

Chapter 1

1.1 The Integers

The integers are the numbers in the set,

$$\mathbb{Z} = [\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$$

A mathematician who's is primarily interested in integers is called a Number Theorist. This lesson is a reminder of some of the many pieces of GCSE mathematics that use only the integers. The statement, $n \in \mathbb{Z}$, states that "n is an integer".

1.2 Exercise

Do ***NOT*** use a calculator
Instead, use your mind !
Marks Available : 70

Question 1

List the integers which satisfy the following "sandwich" inequalities:

(i) $21 \leqslant n < 23$

[1 mark]

(ii) $3.2 < n \leqslant 6.8$

[1 mark]

(iii) $-2.13 < n < 0.75$

[1 mark]

(iv) $\sqrt{16} \leqslant n < \sqrt{81}$

[1 mark]

(v) $2\pi < n < 3\pi$

[1 mark]

(vi) $n^2 \leqslant 9$ (Hint: There are 7 integers that make this true)

[2 marks]

Question 2
The positive integers, with the exception of the number 1, are one of either "prime" or "composite".
For each of the following numbers state if the number is prime or composite;

(i)	7	**(ii)**	27	**(iii)**	1082
(iv)	43	**(v)**	91	**(vi)**	101
(vii)	234785	**(viii)**	23	**(ix)**	2

[9 marks]

Question 3
List the integers which satisfy the following "sandwich" inequalities:

(i) $32 < 3n + 8 \leqslant 41$

[2 marks]

(ii) $13 \leqslant 4n + 5 \leqslant 21$

[2 marks]

(iii) $4 < 2n - 1 \leqslant 10$

[2 marks]

(iv) $0.2 \leqslant 3n + 0.5 \leqslant 3.8$

[2 marks]

(v) $-3 < 7n - 3 < 25$

[2 marks]

Question 4
Demonstrate how to find the six digit value of $2^4 \times 3^3 \times 5^4$ without a calculator.

[3 marks]

Question 5

Prof Iama Duffer has a theory:

- Take any even number.
- Square it.
- Add one.

- The result is always a prime number.

Prof Iama Duffer is wrong.
Give an example that ***proves*** he is wrong.

[2 marks]

Question 6

Explain why the number 187 is NOT a prime number.

[2 marks]

Question 7

The number 60 can be written as a product of prime numbers like this:

$$60 = 2^2 \times 3 \times 5$$

Demonstrate how to write the number 588 as a product of prime numbers without the use of a calculator.

[3 marks]

Question 8

You are told the fact that $9 \times 9 \times 9 \times 9 = 6561$

Explain how to use this fact to find the square root of 6561 without using a calculator.

[2 marks]

Question 9

The Lowest Common Multiple (*lcm*) of 14 and 21 is the first number that is in the 14 times table and the 21 times table.
What is the *lcm* of 14 and 21 ?

[**2 marks]**

Question 10

Show how to find the *lcm* of 15 and 35 without using a calculator.

[3 marks]

Question 11

Complete the bottom two rows of the following arithmetical triangle.

1	$= 2^0$
$1 + 1$	$= 2^1$
$1 + 2 + 1$	$= 2^2$
$1 + 3 + 3 + 1$	$= 2^3$
$1 + 4 + 6 + 4 + 1$	$= 2^4$
...	= ...

This triangular array of numbers is world famous.
What is it's name ?

[1 mark]

Question 12

What is the Highest Common Factor (*hcf*) of 40 and 56 ?

[2 marks]

Question 13

Two numbers are ***coprime*** when they have no prime factors in common.
(In other words, when the *hcf* of the two numbers is 1)
For each of the following integer pairs state if they are coprime or not.

(i)	7 and 11	**(ii)**	15 and 21	**(iii)**	49 and 52
(iv)	10 and 11	**(v)**	13 and 39	**(vi)**	63 and 56

[6 marks]

Question 14

(i) Use a calculator with a "FACT" button to write 104 and 1375 as products of primes (The "FACT" is short for factorise).

[2 marks]

(ii) Hence, explain why 104 and 1375 are coprime.

[2 marks]

(iii) Find the *lcm* of 104 and 1375
(Hint: You may have noticed in question 13 that when two numbers are coprime their *lcm* is the product of the two numbers)

[2 marks]

Question 15
Without using a calculator determine the value of;

(i) 16^2 **(ii)** 16^0 **(iii)** 16^1

(iv) $16^{0.5}$ **(v)** 16^{-1} **(vi)** $16^{-0.25}$

[6 marks]

Question 16
(i) Without using a calculator, write 7056 as a product of primes.

[4 marks]

(ii) Use your part **(i)** answer to square root 7056

[2 marks]

* ***Did you know ?***
The number 2 is the ONLY even prime.
Which makes it rather odd. (Ha, ha !)

1.3 Solutions

Answer 1

(i)	21, 22
(ii)	4, 5, 6
(iii)	– 2, – 1, 0
(iv)	4, 5, 6, 7, 8
(v)	7, 8, 9
(vi)	– 3, –2, – 1, 0, 1, 2, 3

[1, 1, 1, 1, 1, 2 marks]

Answer 2

(i)	prime	**(ii)**	composite	**(iii)**	composite
(iv)	prime	**(v)**	composite	**(vi)**	prime
(vii)	composite	**(viii)**	prime	**(ix)**	prime

[9 marks]

Answer 3

(i)	$8 < n \leqslant 11$	9, 10, 11
(ii)	$2 \leqslant n \leqslant 4$	2, 3, 4
(iii)	$2.5 < n \leqslant 5.5$	3, 4, 5
(iv)	$-0.1 \leqslant n \leqslant 1.1$	0, 1
(v)	$0 < n < 4$	1, 2, 3

[2 marks each = 10 in total]

Answer 4

$$\begin{aligned} 2^4 \times 3^3 \times 5^4 &= (2 \times 5) \times (2 \times 5) \times (2 \times 5) \times (2 \times 5) \times 3^3 \\ &= 10 \times 10 \times 10 \times 10 \times 27 \\ &= 270\,000 \end{aligned}$$

[3 marks]

Answer 5

The possible counterexamples to the Prof's conjecture up to 30 are;
(Only one counterexample is asked for)

$8^2 + 1 = 65$ which is NOT prime, as required.

$12^2 + 1 = 145$ which is NOT prime, as required.

$18^2 + 1 = 325$ which is NOT prime, as required.

$22^2 + 1 = 285$ which is NOT prime, as required.

$28^2 + 1 = 785$ which is NOT prime, as required.

$30^2 + 1 = 901$ which is NOT prime, as required.

[2 marks]

Answer 6
A prime is only divisible by itself and 1
187 is divisible by other numbers, such as 11 or 17
$\therefore$ 187 is not a prime number.

[2 marks]

Answer 7
The division ladder method...

$$\begin{array}{r|l} 2 & 588 \\ \hline 2 & 294 \\ \hline 3 & 147 \\ \hline 7 & 49 \\ \hline 7 & 7 \\ \hline & 1 \\ \hline \end{array}$$

$$\therefore \; 588 = 2^2 \times 3 \times 7^2$$

[3 marks]

Answer 8

$$6561 = (9 \times 9) \times (9 \times 9)$$
$$= 81 \times 81$$
$$\therefore \sqrt{6561} = 81$$

[2 marks]

Answer 9

Multiples of 14:	14,	28,	42,	...
Multiples of 21:	21,	42,	...	

$\therefore$ $lcm\{14, 21\} = 42$

[2 marks]

Answer 10

Multiples of 15:	15,	30,	45,	60,	75,	90,	105, ...
Multiples of 35:	35,	70,	105,	...			

$\therefore$ $lcm\{15,35\} = 105$

[3 marks]

Answer 11
Pascal's Triangle

[1 mark]

Answer 12
$hcf\,\{40,56\} = 8$

[2 marks]

Answer 13

(i)	coprime	**(ii)**	3 in common	**(iii)**	coprime
(iv)	coprime	**(v)**	13 in common	**(vi)**	7 in common

[6 marks]

Answer 14

(i) $104 = 2 \times 2 \times 2 \times 13$

$1375 = 5 \times 5 \times 5 \times 11$

[2 marks]

(ii) 104 and 1375 are coprime because they have no factors in common.

hcf {104, 1375 } = 1

[2 marks]

(iii) $104 \times 1375 = 2 \times 5 \times 2 \times 5 \times 2 \times 5 \times 13 \times 11$

$= 10 \times 10 \times 10 \times 13 \times 11$

$= 1000 \times 143$

$= 143\,000$

[2 marks]

Answer 15

(i)	256	**(ii)**	1	**(iii)**	16
(iv)	4	**(v)**	1/16 0.0625	**(vi)**	0.5

[6 marks]

Answer 16

(i) $7056 = 2^4 \times 3^2 \times 7^2$

[4 marks]

(ii) $\sqrt{7056} = 2^2 \times 3 \times 7$

$= 4 \times 21$

$= 84$

[2 marks]

Chapter 2

2.1 The Rationals

A ***rational*** number is a number than can be written in the form

$$\frac{p}{q}$$

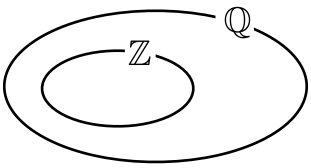
where p and q are integers and $q \neq 0$.

Like the integers, the rationals can be methodically listed,

$$\mathbb{Q} = \left\{\dots, -\frac{2}{3}, -\frac{3}{2}, -\frac{4}{1}, -\frac{3}{1}, -\frac{1}{3}, -\frac{1}{2}, -\frac{2}{1}, -\frac{1}{1}, \frac{0}{1}, \frac{1}{1}, \frac{2}{1}, \frac{1}{2}, \frac{1}{3}, \frac{3}{1}, \frac{4}{1}, \frac{3}{2}, \frac{2}{3}, \dots\right\}$$

The statement, $n \in \mathbb{Q}$, states that "n is a rational number".

In the list of rationals, some entries are highlighted in red.
Those entries were the integer values.
This illustrates the fact that all numbers that are integers, are also rationals.



The GCSE examination often has a questions that gives candidates a number written as a decimal fraction and the task is to show that it is a rational number by writing it in the form,

$$\frac{p}{q} \qquad p, q \in \mathbb{Z}, \ q \neq 0$$

2.2 Terminating Decimal Example

Show that $0.125 \in \mathbb{Q}$, giving your answer in as simple a form as possible.

[2 marks]

2.3 Repeating Pattern Decimal Example

Show that 0.63 63 63 63 63 63... is rational.
Give your answer in as simple a form as possible.

[3 marks]

2.4 Exercise

Do ***NOT*** use a calculator
Marks Available : 50

Question 1

Show that 0.225 is rational by writing it in the form $\frac{p}{q}$ for integer p and q with $q \neq 0$.

Simplify your answer if it is possible to do so.

[2 marks]

Question 2

Consider the number, $0.\dot{3}\ddot{6}$

Show that this number is rational by writing it in the form $\frac{p}{q}$ for integer p and q and with $q \neq 0$. Simplify your answer if it is possible to do so.

[3 marks]

Question 3

Consider the number, $0.\dot{7}\ddot{5}$

Show this is rational by writing it in the form $\dfrac{p}{q}$ for $p, q \in \mathbb{Z}$, $q \neq 0$.

Simplify your answer if it is possible to do so.

[3 marks]

Question 4

Show that the following number is rational;

$$0.\dot{3}5\ddot{1}$$

Simplify your answer if it is possible to do so.

[3 marks]

Question 5

(i) Write 0.875 in the form $\dfrac{p}{q}$ for integer p and q with $q \neq 0$.

[2 marks]

(ii) What one of the following is true ?

- ☐ 0.875 is a natural number, $\mathbb{N}$
- ☐ 0.875 is an integer, $\mathbb{Z}$
- ☐ 0.875 is a positive integer, $\mathbb{Z}^+$
- ☐ 0.875 is a rational number, $\mathbb{Q}$

[1 mark]

Question 6

Write 2.52 in the form $\dfrac{p}{q}$ for integer p and q with $q \neq 0$.

Simplify your answer if it is possible to do so.

[2 marks]

Question 7

(i) Write the integer 23 in the form $\dfrac{p}{q}$ for integer p and q with $q \neq 0$.

[1 mark]

(ii) Explain why all integers are also rational numbers.

[2 marks]

Question 8

Show that the infinitely repeating decimal $0.\dot{1}463\ddot{4}$ is a member of the set of rational numbers, $\mathbb{Q}$, by writing it in the form $\dfrac{p}{q}$ where $p, q \in \mathbb{Z}$ and $q \neq 0$.

Give your answer in simplified form.

[5 marks]

Question 9

(i) Show that $0.44444444\ldots \in \mathbb{Q}$

[2 marks]

(ii) Show that any decimal of the form $0.aaaaaaaaa\ldots$ is rational.

[5 marks]

Question 10

(i) Show that the decimal $0.1222222222\ldots$ is rational

[2 marks]

(ii) Show that any decimal of the form $0.1aaaaaaaaa\ldots$ is rational

[5 marks]

Question 11
Show that any decimal of the form 0.*ab ab ab ab ab ab* ... is rational

[5 marks]

Question 12
Prove that the decimal;

$$0.\dot{0}2197\dot{8}$$

is a rational number.

[4 marks]

Question 13

Prove that $0.999999999999... = 1$

[3 marks]

2.5 Answers

2.5.1 Solution to 2.2 Terminating Decimal Example

$$0.125 = \frac{125}{1000} = \frac{5}{40} = \frac{1}{8}$$ which is in the form of a rational number. □

[2 marks]

2.5.2 Solution to 2.3 Repeating Pattern Decimal Example

$$\begin{aligned} 100\,x &= 63.63\;63\;63\;63\;... \\ SUBTRACT \qquad x &= 0.63\;63\;63\;63\;... \\ \hline 99x &= 63 \\ x &= \frac{63}{99} \\ x &= \frac{7}{11} \end{aligned}$$

which is in the form of a rational number. □

[3 marks]

2.5.3 Solution to 2.4 Exercise

Answer 1

$$0.225 = \frac{225}{1000} = \frac{45}{200} = \frac{9}{40}$$ which is in the form of a rational number. □

[2 marks]

Answer 2

$$\begin{aligned} 100\,x &= 36.36\;36\;36\;36\;... \\ SUBTRACT \qquad x &= 0.36\;36\;36\;36\;... \\ \hline 99x &= 36 \\ x &= \frac{36}{99} \\ x &= \frac{4}{11} \end{aligned}$$

which is in the form of a rational number. □

[3 marks]

Answer 3

$$\begin{aligned} 100x &= 75.75\,75\,75\,75\ldots \\ \textit{SUBTRACT} \qquad x &= 0.75\,75\,75\,75\ldots \\ \hline 99x &= 75 \\ x &= \frac{75}{99} \\ x &= \frac{25}{33} \end{aligned}$$

which is in the form of a rational number. ☐

[3 marks]

Answer 4

$$\begin{aligned} 1000x &= 351.351\,351\,351\,351\ldots \\ \textit{SUBTRACT} \qquad x &= 0.351\,351\,351\,351\ldots \\ \hline 999x &= 351 \\ x &= \frac{351}{999} \\ x &= \frac{13}{37} \end{aligned}$$

which is in the form of a rational number. ☐

[3 marks]

Answer 5

(i) $0.875 = \frac{875}{1000} = \frac{175}{200} = \frac{35}{40} = \frac{7}{8}$

which is in the form $\frac{p}{q}$ for integer p and q with $q \neq 0$.

[2 marks]

(ii) ☐ 0.875 is a rational number, $\mathbb{Q}$

[1 mark]

Answer 6

$2.52 = \frac{252}{100} = \frac{126}{50} = \frac{63}{25}$ which is in the form $\frac{p}{q}$ for integer p and q with $q \neq 0$.

[2 marks]

Answer 7

(i) $23 = \frac{23}{1}$ which is in the form $\frac{p}{q}$ for integer p and q with $q \neq 0$.

[1 mark]

(ii) For any integer n, $n = \frac{n}{1}$ which is in the form $\frac{p}{q}$ for integer p and q with $q \neq 0$ which is the form of a rational number.

$\therefore$ All integers are rational. ☐

[2 marks]

Answer 8

$$\begin{array}{lrcl} & 100000\,x & = & 14634.14634\ 14634\ 14634\ 14634\ \ldots \\ \textit{SUBTRACT} & x & = & 0.14634\ 14634\ 14634\ 14634\ \ldots \\ \hline & 99999x & = & 14634 \\ & x & = & \dfrac{14634}{99999} \\ & x & = & \dfrac{6}{41} \end{array}$$

which is in the form of a rational number. ☐

[5 marks]

Answer 9

(i)

$$\begin{array}{lrcl} & 10\,x & = & 4.4\ 4\ 4\ 4\ 4\ 4\ \ldots \\ \textit{SUBTRACT} & x & = & 0.4\ 4\ 4\ 4\ 4\ 4\ \ldots \\ \hline & 9x & = & 4 \\ & x & = & \dfrac{4}{9} \end{array}$$

which is in the form of a rational number. ☐

[2 marks]

(ii)

$$\begin{array}{lrcl} & 10\,x & = & a.a\ a\ a\ a\ a\ a\ \ldots \\ \textit{SUBTRACT} & x & = & 0.a\ a\ a\ a\ a\ a\ \ldots \\ \hline & 9x & = & a \\ & x & = & \dfrac{a}{9} \end{array}$$

which is in the form of a rational number. ☐

[5 marks]

Answer 10

(i)

$$\begin{array}{lrcl} & 100\,x & = & 12.2\ 2\ 2\ 2\ 2\ 2\ \ldots \\ \textit{SUBTRACT} & 10x & = & 1.2\ 2\ 2\ 2\ 2\ 2\ \ldots \\ \hline & 90x & = & 11 \\ & x & = & \dfrac{11}{90} \end{array}$$

which is in the form of a rational number. ☐

[2 marks]

(ii)

$$\begin{array}{lrcl} & 100\,x & = & 1a.a\ a\ a\ a\ a\ a\ \ldots \\ \textit{SUBTRACT} & 10\,x & = & 0.a\ a\ a\ a\ a\ a\ \ldots \\ \hline & 90x & = & 10 + a \\ & x & = & \dfrac{10 + a}{90} \end{array}$$

which is in the form of a rational number. ☐

[5 marks]

Answer 11

$$\begin{array}{rrcl} & 100\,x & = & ab.ab\;ab\;ab\;ab\;... \\ SUBTRACT & x & = & 0.ab\;ab\;ab\;ab\;... \\ \hline & 99x & = & ab \\ & x & = & \dfrac{10a\,+\,b}{99} \end{array}$$

which is in the form of a rational number. ☐

[5 marks]

Answer 12

$$0.\dot{0}2197\dot{8}$$

$$\begin{array}{rrcl} & 1000000\,x & = & 021978.021978\;021978\;021978\;... \\ SUBTRACT & x & = & 0.021978\;021978\;021978\;... \\ \hline & 999999x & = & 021978 \\ & x & = & \dfrac{21978}{999999} \\ & x & = & \dfrac{2}{91} \end{array}$$

which is in the form of a rational number. ☐

[4 marks]

Answer 13

Take the obvious equivalence $0.333333333... = \dfrac{1}{3}$

and multiply both sides by 3 $\quad 0.999999999... = 1$ ☐

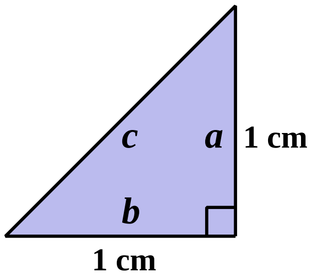
[3 marks]

3.1 Discovery Of Irrationals

Pythagoras was an Ancient Greek philosopher who was born around 570 BC. He founded a political and religious movement, The Pythagoreans, who's motto was “number is all”. They believed that through an understanding of mathematics they would have mastery of all workings of the Universe. They loved integers, and embraced rational numbers which they thought of as ratios of integers.

They viewed the fraction $\frac{p}{q}$ as the ratio $p : q$ and believed (mistakenly) that all numbers could be written in this way. Although “The Theorem of Pythagoras” was actually known well before Pythagoras was born by the Babylonians and the Indians it is attributed to Pythagoras, partly because it presented The Pythagoreans with a major problem that challenged a fundamental belief; that all numbers were rational.

The problem that profoundly challenged their view of mathematics was the following; Determine the length of the hypotenuse, c, of a right angled triangle in which each of the shorter sides, a and b, were of length 1.



$$\begin{aligned} c^2 &= a^2 + b^2 \quad \text{(By the theorem of Pythagoras)} \\ &= 1^2 + 1^2 \\ &= 2 \\ \therefore\ c &= \sqrt{2} \end{aligned}$$

Try as they might, none of the Pythagorean brotherhood could write this troublesome answer as an exact ratio of integers. Initially, they assumed they had just not been clever enough to work out the ratio but one of them, Hippasus, found a logical argument that proved that no matter how long they searched, $\sqrt{2}$ could never be written as a ratio of integers. He showed that $\sqrt{2}$ was a fundamentally different type of number. Hippasus was murdered, drowned at sea, in a vain attempt to keep his discovery secret for it undermined the Pythagorean's core belief in a precise and orderly Universe.

Hippasus is now thought by scholars to have been the person who discovered a new type of number, the ***irrationals***.

3.2 How To Suspect A Number Is Irrational

An irrational person is someone who is not predictable. When an irrational number starts to be written out as a decimal, it's not obvious what digit will be in the next decimal place.

$$\sqrt{2} = 1.\,?$$
$$\sqrt{2} = 1.4\,?$$
$$\sqrt{2} = 1.41\,?$$
$$\sqrt{2} = 1.414\,?$$
$$\sqrt{2} = 1.4142\,?$$
$$\ldots$$
$$\sqrt{2} = 1.414213562\,?$$
$$\ldots$$
$$\sqrt{2} = 1.41421356237 30\,?$$

So, when an irrational number is written out as a decimal, the decimal never terminates and there is no pattern that continuously repeats. This is the hallmark of an irrational number, the fingerprint that gives it away.

3.3 Example

Consider the following sequence of square rooted integers;

$\sqrt{1}$ $\sqrt{2}$ $\sqrt{3}$ $\sqrt{4}$ $\sqrt{5}$ $\sqrt{6}$ $\sqrt{7}$ $\sqrt{8}$

$\sqrt{9}$ $\sqrt{10}$ $\sqrt{11}$ $\sqrt{12}$ $\sqrt{13}$ $\sqrt{14}$ $\sqrt{15}$ $\sqrt{16}$

Under each member of this sequence write down $\mathbb{Q}$ if the member is *rational*, and $\mathbb{P}$ if the member is *irrational*.

3.4 A Useful Table Of squares

So that the next exercise can be done without a calculator, here is a table of integers and their squares.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	4	9	16	25	36	49	64	81	100	121	144	169	196	225

16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
256	289	324	361	400	441	484	529	576	625	676	729	784	841	900

3.5 Exercise

Do ***NOT*** use a calculator

Marks Available : 64

Question 1

Under each expression write $\mathbb{Q}$ if it's *rational*, or $\mathbb{P}$ if it's *irrational*.

(i) $\sqrt{8}$ **(ii)** $\sqrt{144}$ **(iii)** $\sqrt{14}$

(iv) $\sqrt{225}$ **(v)** $\sqrt{324}$ **(vi)** $\sqrt{37}$

(vii) $\sqrt{361}$ **(viii)** $\sqrt{305}$ **(ix)** $\sqrt{\sqrt{625}}$

[9 marks]

Question 2

Under each expression write $\mathbb{Q}$ if it's *rational*, or $\mathbb{P}$ if it's *irrational*.

(i) $\sqrt{3}$ **(ii)** $\dfrac{\sqrt{3}}{\sqrt{3}}$ **(iii)** $\sqrt{3} \times \sqrt{3}$

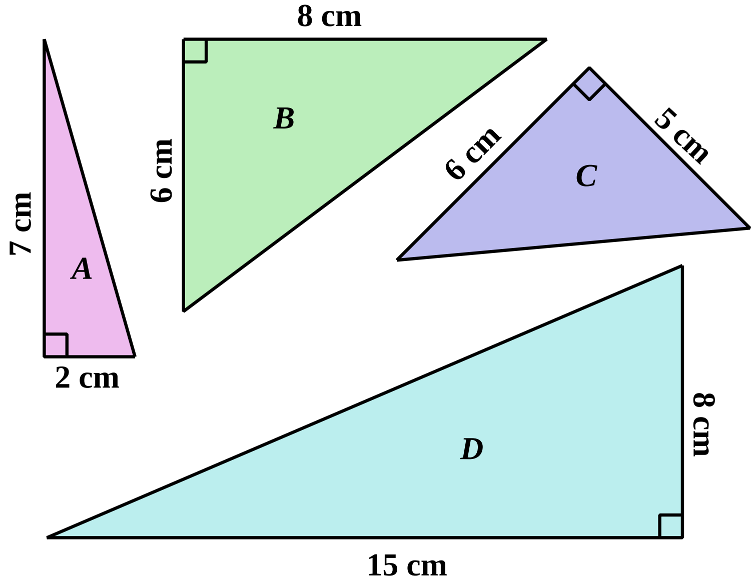
(iv) $(\sqrt{3})^0$ **(v)** $\sqrt{3} \times \sqrt{12}$ **(vi)** $\dfrac{\sqrt{12}}{\sqrt{3}}$

[6 marks]

Question 3

Work out the hypotenuse of each of the following triangles.

In each case, state if the answer is ***Rational,*** $\mathbb{Q}$ or ***Irrational,*** $\mathbb{P}$



[8 marks]

Question 4

Show that each of the following is rational, $\mathbb{Q}$, by writing them in the form $\dfrac{p}{q}$

where p and q are integers and $q \neq 0$.

(i) $\dfrac{3}{7} \times \dfrac{5}{2}$ **(ii)** $4\dfrac{1}{5}$ **(iii)** $\dfrac{2}{3} + \dfrac{5}{7}$

(iv) $\left(\dfrac{5}{6}\right)^2$ **(v)** $\left(\dfrac{1}{9}\right)^2$ **(vi)** $\left(\dfrac{3}{2}\right)^3$

(vii) $\left(\dfrac{100}{121}\right)^{\frac{1}{2}}$ **(viii)** 13^2 **(ix)** $\left(2 + \dfrac{1}{2}\right)^2$

(x) $\sqrt{\dfrac{9}{16}}$ **(xi)** $\sqrt{\dfrac{3^2 + 4^2}{4}}$ **(xii)** $8^{\frac{1}{3}}$

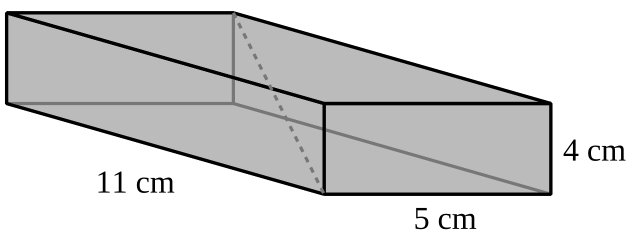
(xiii) $\sqrt[3]{\dfrac{1}{27}}$ **(xiv)** $\sqrt{\dfrac{169}{196}}$ **(xv)** $\sqrt{\left(6 + \dfrac{1}{4}\right)}$

[15 marks]

Question 5

In a cuboid with sides of lengths a, b and c, the longest diagonal in the box, d, is given by a three dimensional version of Pythagoras' Theorem.

$$d^2 = a^2 + b^2 + c^2$$



(i) Work out the longest diagonal's length in an 11 cm by 5 cm by 4 cm cuboid.

[2 marks]

(ii) Is your part **(i)** answer a rational number, $\mathbb{Q}$, or an irrational number, $\mathbb{P}$?

[1 mark]

Question 6

Under each expression write $\mathbb{Q}$ if it's *rational*, or $\mathbb{P}$ if it's *irrational.*

(i) 0.425 **(ii)** $0.\dot{3}4\ddot{8}$ **(iii)** 3.14159

(iv) $\left(\sqrt{5}\right)^3$ **(v)** 72 **(vi)** 3^{-2}

(vii) $0.1762222222...$ **(viii)** $1^{\sqrt{2}}$ **(ix)** $\dfrac{1}{\sqrt{2}}$

[9 marks]

Question 7

(i) Write down a ***Rational*** number that lies between 1 and 2

[1 mark]

(ii) Write down an ***Irrational*** number that lies between the integers 1 and 2

[1 mark]

Question 8

Show that each of the following is a member of the set of rational numbers, $\mathbb{Q}$.

(i) $\sqrt{2\frac{1}{4}}$ **(ii)** $\sqrt{36}$ **(iii)** $\sqrt{7\frac{1}{9}}$

(iv) $\sqrt{3600}$ **(v)** $\sqrt{14\,\dfrac{1}{16}}$ **(vi)** $\sqrt{0.36}$

[12 marks]

3.6 Solutions

Answer 1

(i)	$\mathbb{P}$ irrational	**(ii)**	$\mathbb{Q}$ rational	**(iii)**	$\mathbb{P}$ irrational
(iv)	$\mathbb{Q}$ rational	**(v)**	$\mathbb{Q}$ rational	**(vi)**	$\mathbb{P}$ irrational
(vii)	$\mathbb{Q}$ rational	**(viii)**	$\mathbb{P}$ irrational	**(ix)**	$\mathbb{Q}$ rational

[9 marks]

Answer 2

(i)	$\mathbb{P}$ irrational	**(ii)**	$\mathbb{Q}$ rational	**(iii)**	$\mathbb{Q}$ rational
(iv)	$\mathbb{Q}$ rational	**(v)**	$\mathbb{Q}$ rational	**(vi)**	$\mathbb{Q}$ rational

[6 marks]

Answer 3

(A)	$\sqrt{53}$	$\mathbb{P}$ irrational
(B)	10	$\mathbb{Q}$ rational
(C)	$\sqrt{61}$	$\mathbb{P}$ irrational
(D)	17	$\mathbb{Q}$ rational

[8 marks]

Answer 4

(i)	$\frac{15}{14}$	**(ii)**	$\frac{21}{5}$	**(iii)**	$\frac{29}{21}$
(iv)	$\frac{25}{36}$	**(v)**	$\frac{1}{81}$	**(vi)**	$\frac{27}{8}$
(vii)	$\frac{10}{11}$	**(viii)**	$\frac{169}{1}$	**(ix)**	$\frac{25}{4}$
(x)	$\frac{3}{4}$	**(xi)**	$\frac{5}{2}$	**(xii)**	$\frac{2}{1}$
(xiii)	$\frac{1}{3}$	**(xiv)**	$\frac{13}{14}$	**(xv)**	$\frac{5}{2}$

[15 marks]

Answer 5

(i) $\sqrt{162} = 9\sqrt{2}$ **(ii)** $\mathbb{P}$ irrational

[2, 1 marks]

Answer 6

(i)	$\mathbb{Q}$ rational	**(ii)**	$\mathbb{Q}$ rational	**(iii)**	$\mathbb{Q}$ rational
(iv)	$\mathbb{P}$ irrational	**(v)**	$\mathbb{Q}$ rational	**(vi)**	$\mathbb{Q}$ rational
(vii)	$\mathbb{Q}$ rational	**(viii)**	$\mathbb{Q}$ rational	**(ix)**	$\mathbb{P}$ irrational

[9 marks]

Answer 7

(i) $\frac{3}{2}$ for example

(ii) $\sqrt{2}$ for example

[2 marks]

Answer 8

(i)	$\frac{3}{2}$	**(ii)**	$\frac{6}{1}$	**(iii)**	$\frac{8}{3}$
(iv)	$\frac{60}{1}$	**(v)**	$\frac{15}{4}$	**(vi)**	$\frac{3}{5}$

[12 marks]

4.1 π

This world famous number is defined as being the ratio of ***any*** circle's circumference, C, to its diameter, d;

$$C \;:\; d$$

In other words,

$$\pi = \frac{C}{d}$$

No matter what circle is having its circumference and diameter measured, when the first is divided by the second, the same number, π, is the result.
Like $\sqrt{2}$, π is an irrational number, $\mathbb{P}$, and so it can not be written ***exactly*** as a ratio of integers, or exactly as a decimal.

$$\pi = 3.14\ldots \;(approximately)$$

The value of π is stored to more decimal places on your calculator. A practical use of knowing π, and of knowing that it is the same for all circles, is that a couple of useful formulae now allow us to measure a particular circle's radius, (which is easy from a practical point of view) and calculate its area, A, and its circumference, C, (which are awkward to do by other means).

$$A = \pi r^2$$

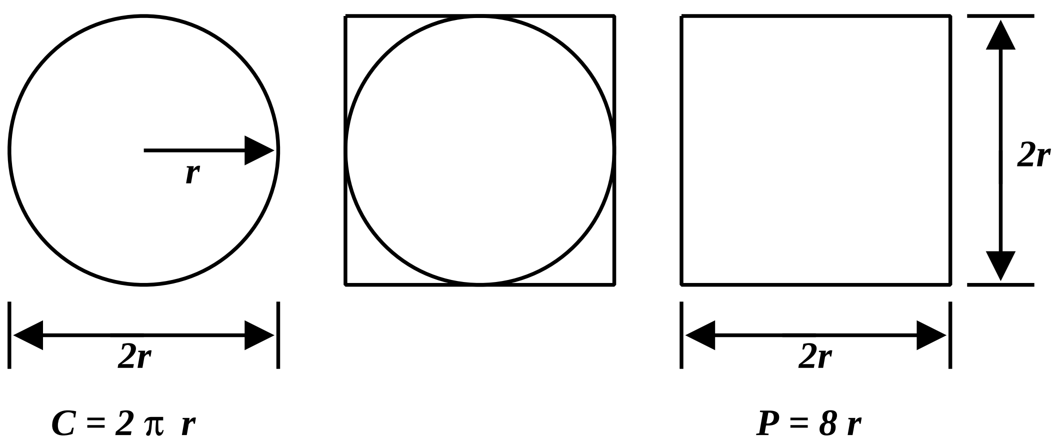
and

$$C = 2\pi r$$

4.2 Proof that π is not an integer

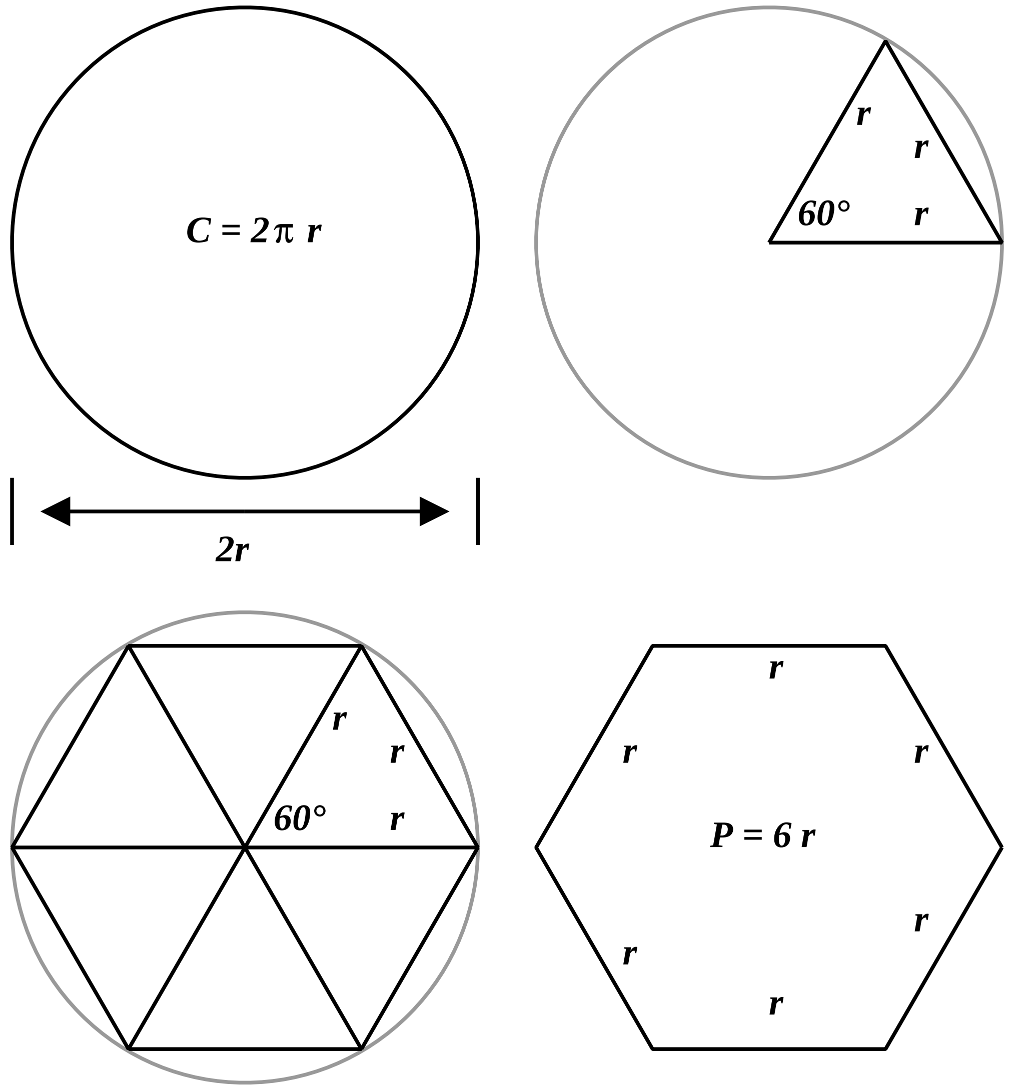
It is difficult to prove that π is irrational, but we can easily prove it's not an integer. The proof is in two parts.

Firstly, consider the following where a circle is placed inside a square,



$$C < 8r$$

Secondly, consider the following where a hexagon is placed inside a circle,



$$C \;>\; 6r$$

So, we have a sandwich inequality,

$$6r \;<\; C \;<\; 8r$$

$$6r \;<\; 2\pi r \;<\; 8r$$

$$6 \;<\; 2\pi \;<\; 8 \qquad \text{(Dividing through by } r\text{)}$$

$$3 \;<\; \pi \;<\; 4 \qquad \text{(Dividing through by 2)}$$

As there are no integers between 3 and 4 and we have a mathematical proof (a watertight argument) that π can not be an integer, □

4.3 Exercise

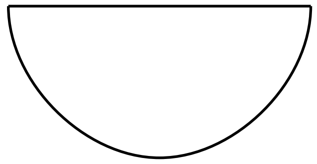
You may use a calculator

Marks Available : 60

Question 1

A bathroom towel holder is made from a length of wire in the shape of a semi-circle of diameter 20 cm.

20 cm



Calculate the length of the wire used.
Give your answer correct to two decimal places.

[3 marks]

Question 2

Which of the following is the closest approximation to π ?

$\frac{25}{8}$ $\qquad$ $\frac{3}{1}$ $\qquad$ $\sqrt{10}$

[2 marks]

Question 3

Which of the following is the closest approximation to $\sqrt{2}$?

$\frac{3}{2}$ $\qquad$ $\frac{\pi}{2}$ $\qquad$ $\frac{7}{5}$

[2 marks]

Question 4

Under each expression write $\mathbb{Q}$ if it's *rational*, or $\mathbb{P}$ if it's *irrational*.

(i) π^2 **(ii)** $\dfrac{8\pi}{3\pi}$

(iii) $\sqrt{\pi}$ **(iv)** π^0

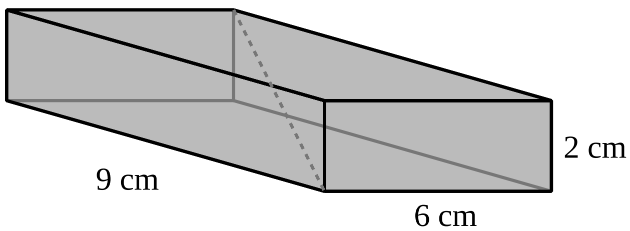
(v) $\dfrac{1}{\pi}$ **(vi)** $(\pi + 5)(\pi - 5) - \pi^2$

[6 marks]

Question 5

In a cuboid with sides of lengths *a*, *b* and *c*, the longest diagonal in the box, *d*, is given by a three dimensional version of Pythagoras' Theorem.

$$d^2 = a^2 + b^2 + c^2$$



(i) Work out the length of the longest diagonal of a cuboid that measures 9 cm by 6 cm by 2 cm

[2 marks]

(ii) Is your part **(i)** answer a rational number or an irrational number ?

[1 mark]

Question 6

Although $\sqrt{12}$ is an irrational number, and $\sqrt{3}$ is also an irrational number, the expression $\dfrac{7\sqrt{12}}{2\sqrt{3}}$ is rational. Explain why this is so.

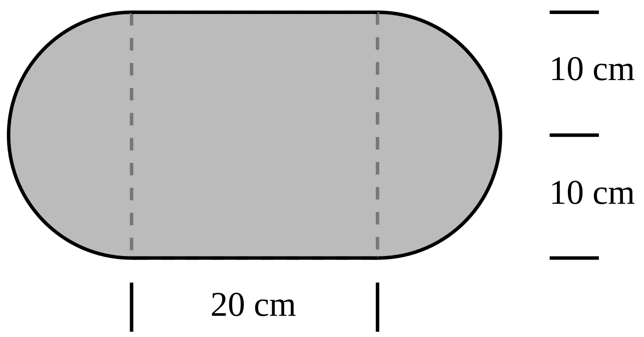
[3 marks]

Question 7

(i) Find the AREA of a circle of radius 10 cm.
Give your answer correct to the nearest integer.

[2 marks]

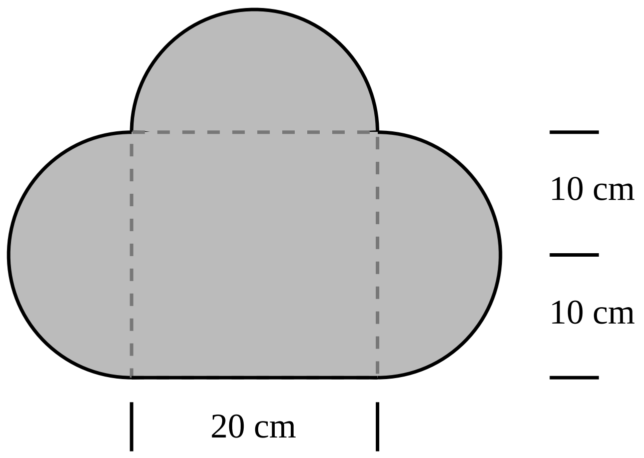
(ii)



Find the AREA of the shape.
Give your answer to the nearest integer.

[2 marks]

(iii)



Find the AREA of the shape.
Give your answer to the nearest integer.

[2 marks]

Question 8

GCSE Examination Question from June 1995, Q11 (Edexcel)

$$x = \sqrt{a^2 + b^2}$$

State whether x is ***rational*** or ***irrational*** in each of the following cases, and show sufficient working to justify each answer.

(a) $a = 5$ and $b = 12$

[2 marks]

(b) $a = 5$ and $b = 6$

[2 marks]

(c) $a = \sqrt{2}$ and $b = \sqrt{7}$

[2 marks]

(d) $a = \frac{3}{7}$ and $b = \frac{4}{7}$

[2 marks]

Question 9
GCSE Examination Question from June 1996, Q8 (Edexcel)
Here are some ***irrational*** numbers.

$$\sqrt{3} \qquad \sqrt{5} \qquad \sqrt{8} \qquad \pi \qquad \sqrt{12} \qquad \sqrt{50}$$

(a) Use two of these numbers to show that, if two ***irrational*** numbers are multiplied together, the result can be a ***rational*** number.

[2 marks]

(b) Use two of these numbers to show that, if two ***irrational*** numbers are divided, the result can be a ***rational*** number.

[2 marks]

Question 10
GCSE Examination Question from June 1994, Q7 (Edexcel)

Which of the following numbers are ***rational*** and which ***irrational*** ?

$$\sqrt{4\frac{1}{4}} \qquad \sqrt{6\frac{1}{4}} \qquad \frac{1}{3} + \sqrt{3} \qquad \left(\frac{1}{3}\sqrt{3}\right)^2$$

Express each of the ***rational*** numbers in the form $\frac{p}{q}$ where p and q are integers, $q \neq 0$.

[5 marks]

Question 11
Prove that each of the following statements is false by giving a counter example.

(i) The square of ***any number*** is always greater than the original number.

[2 marks]

(ii) For every pair of ***integers***, n and m, if $n > m$ then $n^2 > m^2$

[2 marks]

(iii) The product of two ***irrational*** numbers is always irrational.

[2 marks]

Question 12
Under each expression write $\mathbb{Q}$ if it's *rational*, or $\mathbb{P}$ if it's *irrational.*

(i) $\sqrt{2}$ **(ii)** $\sqrt{2} \times \sqrt{8}$ **(iii)** $8 \times \sqrt{2}$

(iv) $\sqrt{8 + 2}$ **(v)** $\dfrac{\sqrt{8}}{\sqrt{2}}$ **(vi)** $\sqrt{8}$

[6 marks]

Question 13

If $a = 1 + \sqrt{2}$ and $b = 1 - \sqrt{2}$ find the ***exact*** value of the following;
(You will have to leave irrational numbers written as square roots)
In each case state if your answer is ***Rational*** or ***Irrational***.

(i) $a + b$ **(ii)** $a - b$ **(iii)** $2a + b$

(iv) $\sqrt{2}\,a$ **(v)** a^2 **(vi)** ab

[6 marks]

4.4 Solutions

Answer 1

$10\pi + 20 = 51.42$ cm to two decimal places

[3 marks]

Answer 2

$$\frac{25}{8}$$

[2 marks]

Answer 3

$$\frac{7}{5}$$

[2 marks]

Answer 4

(i)	$\mathbb{P}$ irrational	**(ii)**	$\mathbb{Q}$ rational	**(iii)**	$\mathbb{P}$ irrational
(iv)	$\mathbb{Q}$ rational	**(v)**	$\mathbb{P}$ irrational	**(vi)**	$\mathbb{Q}$ rational

[6 marks]

Answer 5

(i) 11 cm **(ii)** $\mathbb{Q}$ rational

[2, 1 marks]

Answer 6

$$\frac{7\sqrt{12}}{2\sqrt{3}} = \frac{7\sqrt{4 \times 3}}{2\sqrt{3}} = \frac{7 \times 2\sqrt{3}}{2\sqrt{3}} = \frac{7}{1}$$ which is rational

[3 marks]

Answer 7

(i) 314 cm^2 **(ii)** 714 cm^2 **(iii)** 871 cm^2

[2, 2, 2 marks]

Answer 8

(a) 13 $\mathbb{Q}$ rational

(b) $\sqrt{61}$ $\mathbb{P}$ irrational

(c) 3 $\mathbb{Q}$ rational

(d) $\frac{5}{7}$ $\mathbb{Q}$ rational

[8 marks]

Answer 9

(a) One of $\sqrt{3} \times \sqrt{12} = \frac{6}{1}$ or $\sqrt{8} \times \sqrt{50} = \frac{20}{1}$

[2 marks]

(b) One of $\frac{\sqrt{12}}{\sqrt{3}} = \frac{2}{1}$ or $\frac{\sqrt{50}}{\sqrt{8}} = \frac{5}{2}$

[2 marks]

Answer 10

$\sqrt{4\,\frac{1}{4}} = \frac{\sqrt{17}}{\sqrt{4}} = \frac{\sqrt{17}}{2}$ $\mathbb{P}$ irrational

$\sqrt{6\,\frac{1}{4}} = \sqrt{\frac{25}{4}} = \frac{5}{2}$ $\mathbb{Q}$ rational

$\frac{1}{3} + \sqrt{3}$ $\mathbb{P}$ irrational

$\left(\frac{1}{3}\sqrt{3}\right)^2 = \frac{1}{3}\sqrt{3} \times \frac{1}{3}\sqrt{3} = \frac{1}{3}$ $\mathbb{Q}$ rational

[5 marks]

Answer 11

Specific examples are expected, but here are the generalised requirements.

(i) Any number, n, where $-1 < n < 1$, is made less by being squared.

(ii) For any two numbers, m and n, where $m < 0$ and $n > |\, m\,|$ have the property that $n > m$ and $n^2 > m^2$

(iii) Any two irrational numbers, m and n of the form $\sqrt{a\,b^{2d}}$ and $\sqrt{a\,c^{2e}}$ have the property that

$$\sqrt{a\,b^{2d}} \times \sqrt{a\,c^{2e}} = a \times b^d \times c^e$$

which is rational, where a, b, c, d and e are positive integers.

[6 marks]

Answer 12

(i)	$\mathbb{P}$ irrational	**(ii)**	$\mathbb{Q}$ rational	**(iii)**	$\mathbb{P}$ irrational
(iv)	$\mathbb{P}$ irrational	**(v)**	$\mathbb{Q}$ rational	**(vi)**	$\mathbb{P}$ irrational

[6 marks]

Answer 13

(i)	2	$\mathbb{Q}$ rational
(ii)	$2\sqrt{2}$	$\mathbb{P}$ irrational
(iii)	$3 + \sqrt{2}$	$\mathbb{P}$ irrational
(iv)	$2 + \sqrt{2}$	$\mathbb{P}$ irrational
(v)	$3 + 2\sqrt{2}$	$\mathbb{P}$ irrational
(vi)	-1	$\mathbb{Q}$ rational

[6 marks]

5.1 Revision

Make sure you know the difference between,

- ***Integers*** $\mathbb{Z}$
- ***Rationals*** $\mathbb{Q}$
- ***Irrationals*** $\mathbb{P}$

If all of the rational numbers, $\mathbb{Q}$, are merged with all of the irrational numbers, $\mathbb{P}$, the result is the real numbers, $\mathbb{R}$, which (for GCSE) are "all the numbers you know about" (At A-Level another type of number is encountered).

The Real Numbers → **Irrational Numbers**

Numbers which can't be written in the form $\frac{p}{q}$ where p and q are integers and $q \neq 0$

Examples π $\sqrt{\pi}$ π^2 $\frac{1}{\pi}$ $\sqrt{2}$ $\frac{1}{\sqrt{3}}$ $\sqrt{\textit{any prime number}}$

$\sqrt{\textit{any number not a perfect square}}$

The Real Numbers ↓ **Rational Numbers**

Numbers which can be written in the form $\frac{p}{q}$ where p and q are integers and $q \neq 0$

Integers: $\mathbb{Z}$ Numbers from the set { ... -3, -2, -1, 0, 1, 2, 3, ... }

If it helps, think of these as the "not fractions"

$\left(\text{But keep in mind that, for example, } 7 = \frac{7}{1}\right)$

Rationals: $\mathbb{Q}$ Numbers that can be written in the form $\frac{p}{q}$ for integer p and q.

Note that $q \neq 0$, as division by zero has no mathematical meaning.

When written as a decimal expansion the digits of the expansion

either * terminate

or * form a repeating pattern without end

Irrationals: $\mathbb{P}$ Numbers like π and $\sqrt{2}$

Such numbers CANNOT be written as $\frac{p}{q}$ for integer p and q.

When written as a decimal expansion the digits of the expansion

* never terminate

and * form no repeating pattern

Exam questions often try to write a rational number in a way that makes it look like an irrational number,

$$(\sqrt{\pi})^0$$

or make an irrational number look like it is rational,

$$\frac{\pi + 1}{\pi}$$

5.2 Exercise

Do NOT use a calculator

Marks Available : 50

Question 1

$\sqrt{7} = 2.645751311$ ...

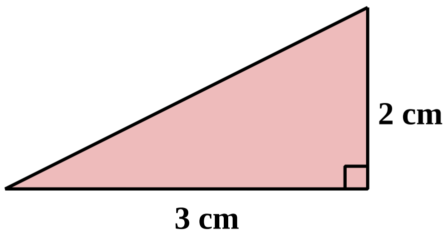
List all of the integers, n, which satisfy

$$-\sqrt{7} < n < \sqrt{7}$$

[2 marks]

Question 2

Calculate the ***exact*** length of this triangle's hypotenuse.



[3 marks]

Question 3

(i) Leaving your answer written in terms of π, calculate the ***exact*** area of a circle which has a radius of 12 cm.

[3 marks]

(ii) Give your part (i) answer as a decimal accurate to one decimal place.

[1 mark]

Question 4

A helpful table of squares:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	4	9	16	25	36	49	64	81	100	121	144	169	196	225	256	289	324	361	400

By using the helpful table of squares, or otherwise, determine whether each of the following numbers is ***Rational***, $\mathbb{Q}$, ***Irrational***, $\mathbb{P}$.

(i) $\sqrt{256}$

(ii) $\sqrt{291}$

(ii) $\sqrt{10^2 + 5^2 + 10^2}$

[3 marks]

Question 5

Show that each of the following is rational by writing them in the form

$$\frac{p}{q}$$

where p and q are integers, $q \neq 0$.

(i) $\left(\frac{7}{9}\right)^2$

[1 mark]

(ii) $\sqrt{\frac{100}{289}}$

[1 mark]

(iii) $\left(3 + \frac{1}{4}\right)^2$

[2 marks]

Question 6

$\sqrt{5}\ =\ 2.236067977\ ...$

$\sqrt{6}\ =\ 2.449489743\ ...$

Write down a rational number that lies between $\sqrt{5}$ and $\sqrt{6}$

[2 marks]

Question 7

For each expression state if it is ***Rational***, $\mathbb{Q}$, or ***Irrational***, $\mathbb{P}$.
Show working where necessary.

(i) $\left(\sqrt{3}\right)^2$ **(ii)** $\dfrac{\sqrt{20}}{\sqrt{4}}$

(iii) 1^{π} **(iv)** $\left(\sqrt{7}\right)^0$

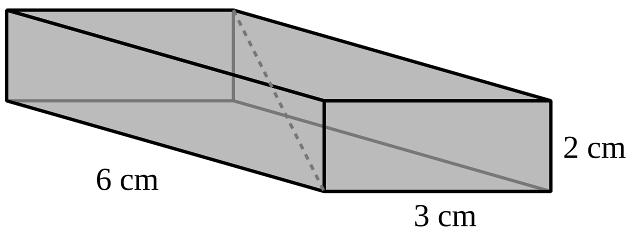
(v) $5^{0.5}$ **(vi)** $\left(\sqrt{3}+\sqrt{2}\right)\left(\sqrt{3}-\sqrt{2}\right)$

[6 marks]

Question 8

In a cuboid with sides of lengths *a*, *b* and *c*, the longest diagonal in the box, *d*, is given by a three dimensional version of Pythagoras' Theorem.

$$d^2\ =\ a^2\ +\ b^2\ +\ c^2$$



(i) Work out the length of the longest diagonal of a cuboid that measures 6 cm by 3 cm by 2 cm

[2 marks]

(ii) Is your part **(i)** answer a rational number, $\mathbb{Q}$, or an irrational number, $\mathbb{P}$?

[1 mark]

Question 9

Show that 0.625 can be written in the form $\dfrac{p}{q}$ for $p, q \in \mathbb{Z}$ and $q \neq 0$.

Simplify your answer if it is possible to do so.

[3 marks]

Question 10

Victor says that as $\pi = \dfrac{22}{7}$, π must be a rational number.

Explain his error.

[2 marks]

Question 11

List the integers, if any, which satisfy

$$29 \leqslant 3n + 14 < 38$$

[3 marks]

Question 12

Consider the number $0.\dot{4}\ddot{5}$

That is, 0.454 545 454 545 ...

Show how this number can be rewritten in the form $\dfrac{p}{q}$ for integer p and q with $q \neq 0$.

Simplify your answer if it is possible to do so.

[3 marks]

Question 13

Show that each of the following is rational by writing them in the form

$\dfrac{p}{q}$ where p and q are integers and $q \neq 0$.

(i) $\left(\dfrac{1}{2} + \dfrac{1}{3}\right)^2$ **(ii)** $\sqrt{\dfrac{3}{5} + \dfrac{1}{25}}$

[4 marks]

Question 14

Use the theorem of Pythagoras to show that a right angled triangle with shorter sides of lengths $\dfrac{6}{5}$ cm and $\dfrac{8}{5}$ cm has a hypotenuse of integer length.

[3 marks]

Question 15

Consider the number

$$0.041\dot{6}$$

Let $x = 0.0416666666666666.....$

(i) Work out the value of $10\,000\,x$

[1 mark]

(ii) Work out the value of $1\,000\,x$

[1 mark]

(iii) By subtracting answer **(b)** from answer **(a)** work out $9\,000\,x$

[1 mark]

(iv) Hence write $0.041\dot{6}$ in the form of a rational number.

That is, in the form $\frac{p}{q}$ for integer p and q with $q \neq 0$

Simplify your answer if it is possible to do so.

[2 marks]

5.2 Solutions

Answer 1

$n \in \{-2, -1, 0, 1, 2\}$

[2 marks]

Answer 2

$\sqrt{13}$ one mark lost if written as a decimal approximation, 3.605551275...

[3 marks]

Answer 3

(i) 144π

[3 marks]

(ii) 452.4

[1 mark]

Answer 4

(i) $\mathbb{Q}$ rational **(ii)** $\mathbb{P}$ irrational **(iii)** $\mathbb{Q}$ rational

[3 marks]

Answer 5

(i) $\frac{49}{81}$ **(ii)** $\frac{10}{17}$ **(iii)** $\frac{169}{16}$

[1, 1, 2 marks]

Answer 6

For example, 2.3

[2 marks]

Answer 7

(i) $\mathbb{Q}$ rational **(ii)** $\mathbb{P}$ irrational **(iii)** $\mathbb{Q}$ rational
(iv) $\mathbb{Q}$ rational **(v)** $\mathbb{P}$ irrational **(vi)** $\mathbb{Q}$ rational

[6 marks]

Answer 8

(i) 7 cm **(ii)** $\mathbb{Q}$ rational

[2, 1 marks]

Answer 9

$$0.625 = \frac{625}{1000} = \frac{5}{8}$$

[3 marks]

Answer 10

$\pi \neq \frac{22}{7}$ although $\frac{22}{7}$ is a well known, good, rational approximation to π

[2 marks]

Answer 11

$n \in \{ 5, 6, 7 \}$

[3 marks]

Answer 12

Let $x = 0.454\,545\,454\,545\,454\ ...$

$$\begin{array}{rrcl} & 100\,x & = & 45.454\,545\,454\,545\ ... \\ SUBTRACT & x & = & 0.454\,545\,454\,545\ ... \\ \hline & 99\,x & = & 45 \\ & x & = & \frac{45}{99} \\ & x & = & \frac{5}{11} \end{array}$$

[3 marks]

Answer 13

(i) $\frac{25}{36}$ **(ii)** $\frac{4}{5}$

[2, 2 marks]

Answer 14

$$\begin{aligned} \sqrt{\left(\frac{6}{5}\right)^2 + \left(\frac{8}{5}\right)^2} &= \sqrt{\frac{36}{25} + \frac{64}{25}} \\ &= \sqrt{\frac{100}{25}} \\ &= \sqrt{4} \\ &= 2 \quad \text{which is an integer} \quad \square \end{aligned}$$

[3 marks]

Answer 15

(i) $10\,000\,x = 416.666\,666\,666\,66...$

(ii) $1\,000\,x = 41.666\,666\,666\,66...$

(iii) $9\,000\,x = 375$

(iv) $x = \frac{1}{24}$

[1, 1, 1, 2 marks]

6.1 Test

Do NOT use a calculator
Marks Available : 50

Question 1

$\sqrt{8} = 2.828427125 \ldots$

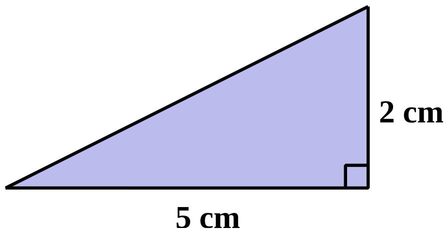
List all of the integers, n, which satisfy

$$-\sqrt{8} < n < \sqrt{8}$$

[2 marks]

Question 2

Calculate the ***exact*** length of this triangle's hypotenuse.



[3 marks]

Question 3

(i) Leaving your answer written in terms of π, calculate the ***exact*** area of a circle which has a radius of 3 cm.

[3 marks]

(ii) Give your part (i) answer as a decimal accurate to one decimal place. Use $\pi = 3.14$ in your calculation.

[1 mark]

Question 4

A helpful table of squares:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	4	9	16	25	36	49	64	81	100	121	144	169	196	225	256	289	324	361	400

By using the helpful table of squares, or otherwise, determine whether each of the following numbers is ***Rational***, $\mathbb{Q}$, ***Irrational***, $\mathbb{P}$.

(i) $\sqrt{225}$

(ii) $\sqrt{281}$

(iii) $\sqrt{12^2 + 1^2 + 12^2}$

[3 marks]

Question 5

Show that each of the following is rational by writing them in the form

$$\frac{p}{q}$$

where p and q are integers, $q \neq 0$.

(i) $\left(\frac{7}{8}\right)^2$

[1 mark]

(ii) $\sqrt{\frac{324}{361}}$

[1 mark]

(iii) $\left(6 + \frac{1}{4}\right)^{0.5}$

[2 marks]

Question 6

$\sqrt{6} = 2.449489743 \ldots$
$\sqrt{7} = 2.645751311 \ldots$

Write down a rational number that lies between $\sqrt{6}$ and $\sqrt{7}$

[2 marks]

Question 7

For each expression state if it is ***Rational***, $\mathbb{Q}$, or ***Irrational***, $\mathbb{P}$.
Show working where necessary.

(i) $\left(\sqrt{\pi}\right)^2$ **(ii)** $\dfrac{7\pi}{3\pi}$

(iii) $1^{\sqrt{2}}$ **(iv)** $\left(\sqrt{\pi}\right)^0$

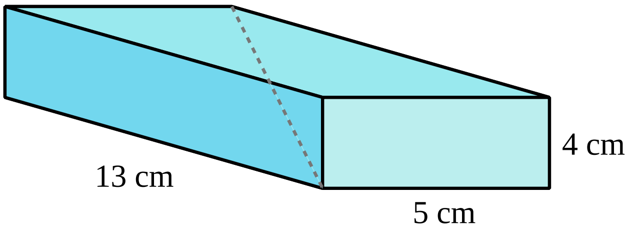
(v) $3^{0.5}$ **(vi)** $\left(\sqrt{3} + 1\right)^2$

[6 marks]

Question 8

In a cuboid with sides of lengths a, b and c, the longest diagonal in the box, d, is given by a three dimensional version of Pythagoras' Theorem.

$$d^2 = a^2 + b^2 + c^2$$



(i) Work out the length of the longest diagonal of a cuboid that measures 13 cm by 5 cm by 4 cm

[2 marks]

(ii) Is your part **(i)** answer a rational number, $\mathbb{Q}$, or an irrational number, $\mathbb{P}$?

[1 mark]

Question 9

Show that 0.875 can be written in the form $\frac{p}{q}$ for $p, q \in \mathbb{Z}$ and $q \neq 0$.

Simplify your answer if it is possible to do so.

[3 marks]

Question 10

Viola says that as $\pi = 3.14$, which terminates, π must be a rational number.

Explain her error.

[2 marks]

Question 11

List the integers, if any, which satisfy

$$32 \leqslant 5n + 17 < 52$$

[3 marks]

Question 12

Consider the number, $0.\dot{1}\dot{8}$

That is, 0.181 818 181 818 181 818 ....

Show how this number can be rewritten in the form $\frac{p}{q}$ for integer p and q with $q \neq 0$.

Simplify your answer if it is possible to do so.

[3 marks]

Question 13

Show that each of the following is rational by writing them in the form

$\dfrac{p}{q}$ where p and q are integers and $q \neq 0$.

(i) $\left(\dfrac{1}{2} + \dfrac{1}{4}\right)^2$ **(ii)** $\sqrt{\dfrac{2}{5} + \dfrac{6}{25}}$

[4 marks]

Question 14

Use the theorem of Pythagoras to show that a right angled triangle with shorter sides of lengths $\dfrac{12}{5}$ cm and $\dfrac{16}{5}$ cm has a hypotenuse of integer length.

[3 marks]

Question 15

Consider the number

$$0.208\dot{3}$$

Let $x = 0.20833333333333333.....$

(i) Work out the value of $10\ 000\,x$

[1 mark]

(ii) Work out the value of $1\ 000\,x$

[1 mark]

(iii) By subtracting answer **(b)** from answer **(a)** work out $9\ 000\,x$

[1 mark]

(iv) Hence write $0.208\dot{3}$ in the form of a rational number.

That is, in the form $\dfrac{p}{q}$ for integer p and q with $q \neq 0$

Simplify your answer if it is possible to do so.

[2 marks]

6.2 Solutions

Answer 1

$n \in \{-2, -1, 0, 1, 2\}$

[2 marks]

Answer 2

$\sqrt{29}$ cm (one mark lost if written as a decimal approximation, 5.385164807...)

[3 marks]

Answer 3

(i) 9π

[3 marks]

(ii) 28.3 cm^2

[1 mark]

Answer 4

(i) $\mathbb{Q}$ rational **(ii)** $\mathbb{P}$ irrational **(iii)** $\mathbb{Q}$ rational

[3 marks]

Answer 5

(i) $\frac{49}{64}$ **(ii)** $\frac{18}{19}$ **(iii)** $\frac{5}{2}$

[1, 1, 2 marks]

Answer 6

For example, 2.5

[2 marks]

Answer 7

(i) $\mathbb{P}$ irrational **(ii)** $\mathbb{Q}$ rational **(iii)** $\mathbb{Q}$ rational

(iv) $\mathbb{Q}$ rational **(v)** $\mathbb{P}$ irrational **(vi)** $\mathbb{P}$ irrational

[6 marks]

Answer 8

(i) 15 cm **(ii)** $\mathbb{Q}$ rational

[2, 1 marks]

Answer 9

$$0.875 = \frac{875}{1000} = \frac{7}{8}$$

[3 marks]

Answer 10

$\pi \neq 3.14$ although 3.14 is a good rational approximation to π

[2 marks]

Answer 11

$n \in \{ 3, 4, 5, 6 \}$

[3 marks]

Answer 12

Let $x = 0.181\,818\,181\,818\ \ldots$

$$\begin{array}{rrl} & 100\,x & = 18.181\,818\,181\,818\,181\ \ldots \\ SUBTRACT & x & = 0.181\,818\,181\,818\,181\ \ldots \\ \hline & 99\,x & = 18 \\ & x & = \dfrac{18}{99} \\ & x & = \dfrac{2}{11} \end{array}$$

[3 marks]

Answer 13

(i) $\dfrac{9}{16}$ **(ii)** $\dfrac{4}{5}$

[2, 2 marks]

Answer 14

$$\begin{aligned} \sqrt{\left(\frac{12}{5}\right)^2 + \left(\frac{16}{5}\right)^2} &= \sqrt{\frac{144}{25} + \frac{256}{25}} \\ &= \sqrt{\frac{400}{25}} \\ &= \sqrt{16} \\ &= 4 \qquad \text{which is an integer} \qquad \square \end{aligned}$$

[3 marks]

Answer 15

(i) $10\,000\,x = 2083.333\,333\,333\,33...$

(ii) $1\,000\,x = 208.333\,333\,333\,33...$

(iii) $9\,000\,x = 1875$

(iv) $x = \dfrac{5}{24}$

[1, 1, 1, 2 marks]