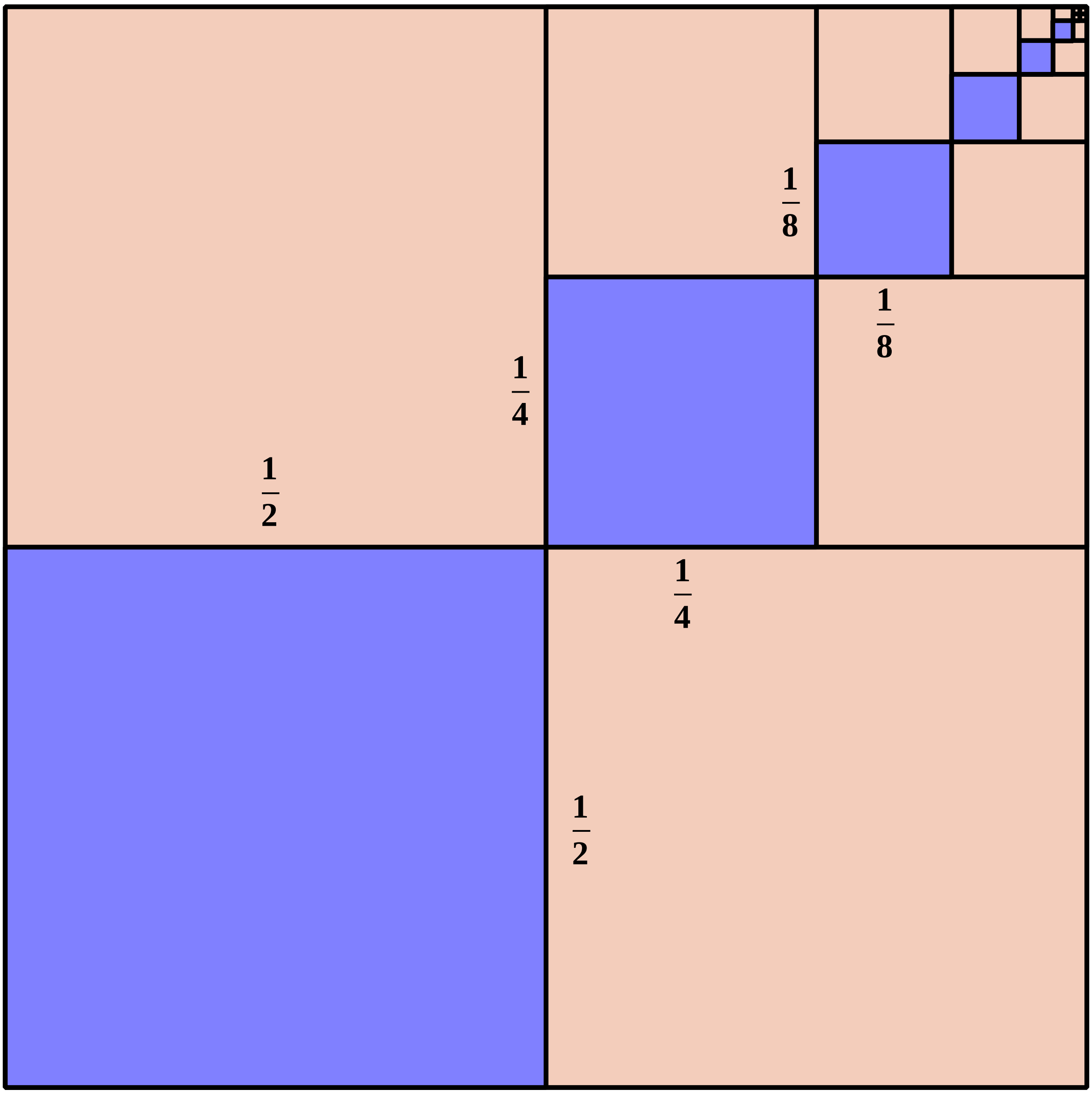
A-Level

~ Year 2 ~

Pure Mathematics

GEOMETRIC PROGRESSIONS



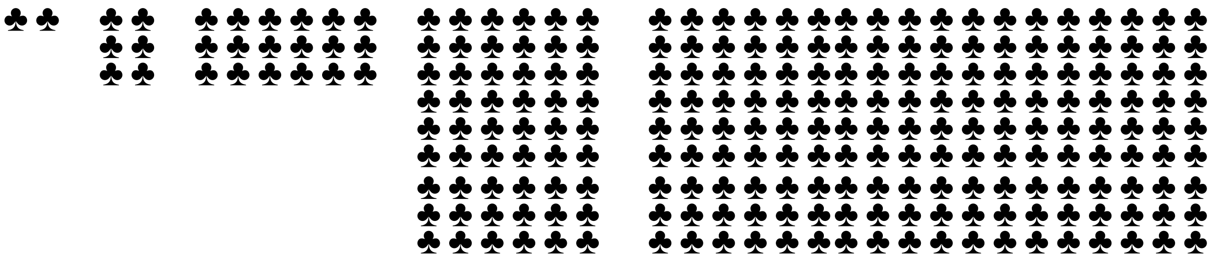
$$\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \frac{1}{1024} + \frac{1}{4096} + \ldots = \frac{1}{3}$$

Geometric Progressions

Lesson 1

A-Level Pure Mathematics, Year 2
Geometric Progressions

1.1 How To Spot A Geometric Progression



Consider the sum

$$2 + 6 + 18 + 54 + 162 + \ldots$$

Explain why this series not an Arithmetic Progression

☞

[1 mark]

Teaching Video : http://www.NumberWonder.co.uk/v9077/1.mp4

Observe that the terms are linked; each is three times the previous.
This is the hallmark of a Geometric Progression.
In this case it is said that the common ratio is 3

Expressed algebraically, a Geometric Progression is of the form

$$a, \quad ar, \quad a\,r^2, \quad a\,r^3, \quad a\,r^4, \quad \ldots$$

where a is the first term
and r is the common ratio

Write down a formula for the n^{th} term, G_n of a Geometric Progression

☞

[1 mark]

1.2 Example

The 5^{th} term if a Geometric Progression is 567 and the 2^{nd} term is 21

(i) What is the common ratio ?

[3 marks]

(ii) Write out the first 6 terms of the Geometric Progression.

[2 marks]

(iii) Determine the exact value of the 20^{th} term.

[2 marks]

1.3 Exercise

Marks Available: 60

Question 1

Write out the first five terms of the Geometric Progression with first term 8 and common ratio 1.5

[2 marks]

Question 2

Write out the first five terms of the Geometric Progression with first term 8 and common ratio 0.5

[2 marks]

Question 3

Write out the first five terms of the Geometric Progression with first term 3 and common ratio -2

[2 marks]

Question 4

Consider the following Geometric Progression;

$$0.3, \quad 0.03, \quad 0.003, \quad 0.0003, \ \ldots$$

(i) State the value of the first term, a, and the value of the common ratio, r

[2 marks]

The sum of this Geometric Progression has an infinite number of terms

$$0.3 + 0.03 + 0.003 + 0.0003 + \ldots$$

This infinite sum has a finite answer.

(ii) Give the exact value of this “sum to infinity”

[2 marks]

Question 5

What is the exact value of the 20^{th} term of the following Geometric Progression ?

$$5, \quad 15, \quad 45, \quad 135, \ \ldots$$

[3 marks]

Question 6

The 5^{th} term of a Geometric Progression is 3750 and the 2^{nd} term is 30

(i) What is the common ratio ?

[3 marks]

(ii) Write out the first 6 terms of the Geometric Progression.

[2 marks]

(iii) Determine the exact value of the 12^{th} term.

[2 marks]

Question 7

The 6^{th} term of a Geometric Progression is 0.375 and the 3^{rd} term is – 3

(i) What is the common ratio ?

[3 marks]

(ii) Write out the first 6 terms of the Geometric Progression.

[2 marks]

(iii) Determine the exact value of the 20^{th} term.

Write your answer as a $\frac{p}{q}$ fraction, for integer p and q

[2 marks]

Question 8

For each of the following series state if the terms are in

- Arithmetic Progression
- Geometric Progression
- Neither Arithmetic nor Geometric Progression

(i) $7 + 3 - 1 - 5 - \ldots$

(ii) $1 + 8 + 27 + 64 + \ldots$

(iii) $0.1^3 + 0.1^5 + 0.1^7 + 0.1^9 + \ldots$

(iv) $3 - 1 + \frac{1}{3} - \frac{1}{9} + \ldots$

(v) $1 - 1 + 1 - 1 + 1 - 1 + \ldots$

[5 marks]

Question 9
Determine the value of this series which is in Geometric Progression, and expressed in sigma notation

$$\sum_{1}^{4} 3^n$$

[3 marks]

Question 10
Determine the value of this series which is in Geometric Progression, and expressed in sigma notation

$$\sum_{1}^{5} 3 \times 2^n$$

[3 marks]

Question 11
If 3, x and 9 are the first three terms of a sequence in Geometric Progression, find
(i) the possible exact values of x

[3 marks]

(ii) the possible exact values of the 4^{th} term.

[2 marks]

Question 12

The 7th term of a Geometric Progression is exactly 1.9487171

and the 3rd term is exactly 1.331

(i) What is the common ratio ?

[3 marks]

(ii) Write out the first 6 terms of the Geometric Progression.

[2 marks]

(iii) Express the sum of first 40 terms of this Geometric Progression in sigma notation.

[2 marks]

Question 13

A geometric sequence has first term 4 and third term 1

Find the two possible values of the 6th term.

[4 marks]

Question 14

The first three terms of a geometric sequence are given by

$$8 - x, \qquad 2x, \qquad x^2$$

respectively where $x > 0$

(i) Show that $x^3 - 4x^2 = 0$

[2 marks]

(ii) Find the value of the 20^{th} term.

[3 marks]

(iii) State, with a reason, whether 4096 is a term in the sequence.

[1 mark]

1.4 Solutions

☞ The difference between successive terms is not constant

[1 mark]

☞

$$G_n = a\,r^{n-1}$$

[1 mark]

1.4.1 Solution to 1.2 Example (Teaching Video)

(i)

$$\frac{a\,r^4}{a\,r} = \frac{567}{21}$$

$$r^3 = 27$$

$$r = 3$$

[3 marks]

(ii) 7, 21, 63, 189, 567, 1701, ...

[2 marks]

(iii)

$$a\,r^{19} = 7 \times 3^{19}$$

$$= 8\,135\,830\,269$$

[2 marks]

1.4.2 Solutions to Exercise 1.3

Answer 1
8, 12, 18, 27, 40.5, ...

[2 marks]

Answer 2
8, 4, 2, 1, 0.5, ...

[2 marks]

Answer 3

3, – 6, 12, – 24, 48, ...

[2 marks]

Answer 4

(i) $a = 0.3, \quad r = 0.1$ **(ii)** $\frac{1}{3}$

[2, 2 marks]

Answer 5
5 811 307 335

[3 marks]

Answer 6
(i) 5
(ii) 6, 30, 150, 750, 3750, 18750, ...
(iii) 292 968 750

[3, 2, 2 marks]

Answer 7

(i) $-\ 0.5$

(ii) $-12,\ 6,\ -3,\ 1.5,\ -0.75,\ 0.375$

(iii) $\dfrac{3}{131072}$

[3, 2, 2 marks]

Answer 8

(i)	AP	**(ii)**	neither	**(iii)**	GP
(iv)	GP	**(v)**	GP		

[5 marks]

Answer 9

120

Answer 10

186

[3, 3 marks]

Answer 11

(i) $\pm 3\sqrt{3}$ **(ii)** $\pm 9\sqrt{3}$

[3, 2 marks]

Answer 12

(i) 1.1

(ii) 1.1, 1.21, 1.331, 1.4641, 1.61051, 1.771561

(iii) $\displaystyle\sum_{1}^{40} 1.1^{n}$

[3, 2, 2 marks]

Answer 13

± 0.125

[4 marks]

Answer 14

(i)

$$\frac{x^2}{2x} = \frac{2x}{8 - x}$$

$$x^2\,(8 - x) = 2x \times 2x$$

$$8x^2 - x^3 = 4x^2$$

$$0 = x^3 - 4x^2 \qquad \square$$

$$x^2(x - 4) = 0$$

$$\therefore x = 4, \text{ as told } x > 0$$

(ii) 2 097 152

(iii) Yes, as 4096 is the 11th term.

[2, 3, 1 marks]

2.1 The Sum Of A Geometric Progression

It is not too difficult to use a calculator or even simply mentally sum the following series which is in Geometric Progression,

$$4 + 20 + 100 + 500 + 2500 + 12500 + 62500$$

For a Geometric Progressions with more terms the direct method of simply writing out all the terms and then adding then up becomes impractical.

Before tackling this "find the sum" problem as an algebraic entity, it's helpful to look at a numerical method to sum the above Geometric Progression but that can be generalised to derive an algebraic formula.

The key idea is to call the sum of the series, S, then multiply S by 5.
Where did that 5 come from ?
Well, it is the common ratio of this particular series.

$$\begin{array}{rrcccccc}
 & 5 \times S = & & 20 & + \ 100 & + \ \ldots & + \ 62500 & + \ 312500 \\
\text{SUBTRACT} & S = & 4 & + \ 20 & + \ 100 & + \ \ldots & + \ 62500 & \\
\hline
 & 4\,S = & -\,4 & & & & & + \ 312500
\end{array}$$

$$\begin{aligned}
4\,S &= 312496 \\
S &= \frac{312496}{4} \\
S &= 78124
\end{aligned}$$

2.2 Formula To Sum A Geometric Progression

The proof of the formula to sum the terms of a Geometric Progression is occasionally asked for in examinations.

What follows is a video from Exam Solutions of the proof.
Over the page is a written out version.

Teaching Video : http://www.NumberWonder.co.uk/v9077/2.mp4

Algebraically, a Geometric Progression is a number sequence of the form;

$$a, \quad a\,r, \quad a\,r^2, \quad a\,r^3, \quad \ldots., \quad a\,r^{n-1}$$

where a is the initial term

r is the common ratio

n is number of terms

Notice that the power of the last term is not n but $n - 1$

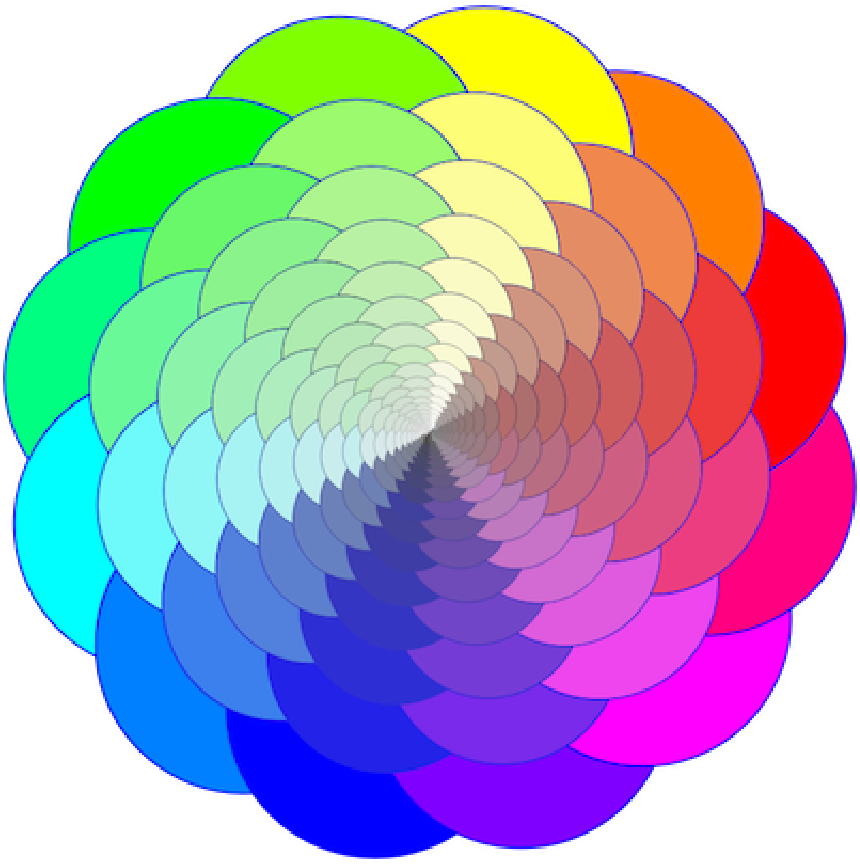
That is, $G_n = a\,r^{n-1}$

$$
\begin{array}{lrcl}
 & rS & = & \quad ar + a\,r^2 + \ldots + a\,r^{n-1} + a\,r^n \\
\text{SUBTRACT} & S & = & a + ar + a\,r^2 + \ldots + a\,r^{n-1} \\
\hline
 & rS - S & = & -a \qquad\qquad\qquad\qquad\qquad\qquad + a\,r^n \\
 & (r - 1)\,S & = & a\,(r^n - 1) \\
 & S & = & \dfrac{a\,(r^n - 1)}{(r - 1)} \qquad r \neq 1
\end{array}
$$

The above version of the "Sum of a GP" formula is most useful if $r > 1$

If $r < 1$ this alternative, equivalent, version may be easier to work with;

$$S = \frac{a\,(1 - r^n)}{(1 - r)} \qquad r \neq 1$$



A Geometric Series in Bloom by Nikki Geib

2.3 Exercise

Marks Available: 40

Question 1

Find the sum of the first 10 terms of the following Geometric Progression;

$$3 + 12 + 48 + 192 + \ \ldots$$

[2 marks]

Question 2

Find the sum of the first 10 terms of the following Geometric Progression;

$$59049 + 19683 + 6561 + 2187 + \ \ldots$$

[2 marks]

Question 3

(i) Write out the first four terms of the Geometric Progression described by;

$$\sum_{1}^{12} 0.8^{n}$$

[2 marks]

(ii) Determine the exact value of

$$\sum_{1}^{12} 0.8^{n}$$

[2 marks]

Question 4

A geometric series has first four terms,

$$5 - 2 + 0.8 - 0.32 + \ldots$$

Give the exact value of the sum of the first 10 terms

[3 marks]

Question 5

A geometric series has first term *a* and common ratio 2.

A different geometric series has first term *b* and common ratio 3.

Given that the sum of the first 4 terms of both series is the same, show that

$$\frac{a}{b} = \frac{8}{3}$$

[3 marks]

Question 6

A number of plastic bottles being washed up on a beach is estimated to be growing at a rate of 15% a year. In 2020, 400 plastic bottles were washed up.

(i) How many plastic bottles are projected to be washed up in;

(a) 2021

[1 mark]

(b) 2022

[1 mark]

(c) 2023

[1 mark]

(ii) In total how many bottles are projected to be washed up over the ten year period from 2021 - 2029 (inclusive) ?

[3 marks]

Question 7

The first three terms of a geometric series are

$$(k - 6), \quad k, \quad (2k + 5)$$

where k is a positive constant.

(i) Show that $k^2 - 7k - 30 = 0$

[3 marks]

(ii) Hence find the value of k

[2 marks]

(iii) Find the common ratio of this series

[1 mark]

(iv) Find the sum of the first 10 terms of this series, giving your answer to the nearest whole number.

[2 marks]

Question 8

Determine the exact value of

$$\sum_{1}^{12} 7 \times 3^n$$

[3 marks]

Question 9
Find the difference between the sums to ten terms of the arithmetic series and the geometric series whose first two terms are -2 and 4

[4 marks]

Question 10

(i) Show that there are two possible geometric series in each of which the first term is 8, and the sum of the first three terms is 14.

[3 marks]

(ii) Write out the first 6 terms of each series.

[2 marks]

2.4 Solutions to 2.3 Exercise

Answer 1
1 048 575

Answer 2
88 572

Answer 3
(i) 0.8, 0.64, 0.512, 0.4096
(ii) 3.725 122 093

[2, 2, 4 marks]

Answer 4
3.571 054 08

Answer 5

$$a + 2a + 4a + 8a = b + 3b + 9b + 27b$$

$$15a = 40b$$

$$\frac{a}{b} = \frac{8}{3} \qquad \square$$

[3, 3 marks]

Answer 6
(i) **(a)** 460
(b) 529
(c) 608.35

[1, 1, 1 marks]

(ii) 8121

[3 marks]

Answer 7
(i)

$$\frac{k}{k - 6} = \frac{2k + 5}{k}$$

$$k^2 = (2k + 5)(k - 6)$$

$$k^2 = 2k^2 - 7k - 30$$

$$k^2 - 7k - 30 = 0 \qquad \square$$

[3 marks]

(ii) $\therefore (k - 10)(k + 3) = 0 \quad \Rightarrow \quad k = 10$

[2 marks]

(iii) The series is 4, 10, 25, ... and so $r = 2.5$

[1 mark]

(iv) 25 429

[2 marks]

Answer 8
5 580 120

[3 marks]

Answer 9

For the Arithmetic Progression...

$$\begin{aligned} S_n &= \frac{n}{2}\left\{ 2a + (n - 1)d \right\} \\ &= \frac{10}{2}\left\{ 2 \times (-2) + (10 - 1)6 \right\} \\ &= 5\left\{-4 + 54\right\} \\ &= 250 \end{aligned}$$

For the Geometric Progression...

$$\begin{aligned} S_n &= \frac{a(r^n - 1)}{(r - 1)} \\ &= \frac{(-2)\left((-2)^{10} - 1\right)}{((-2) - 1)} \\ &= \frac{(-2)(1024 - 1)}{-3} \\ &= 682 \end{aligned}$$

So the difference between the two is 432

[4 marks]

Answer 10

(i) **METHOD 1**

$$8 + 8r + 8r^2 = 14$$

$$4r^2 + 4r - 3 = 0$$

$$(2r + 3)(2r - 1) = 0$$

$$r = -\frac{3}{2} \text{ or } \frac{1}{2}$$

METHOD 2 (Out of interest - not recommended !)

$$S = \frac{a(r^n - 1)}{(r - 1)} \qquad r \neq 1$$

$$14 = \frac{8(r^3 - 1)}{(r - 1)}$$

$$14r - 14 = 8r^3 - 8$$

$$8r^3 - 14r + 6 = 0$$

Divide through by 2

$$4r^3 - 7r + 3 = 0$$

Clearly $r = 1$ is a solution, $\therefore (r - 1)$ is a factor

$$\begin{array}{r|lllllll}
 & & & 4r^2 & + & 4r & - & 3 \\
\hline
r - 1 & 4x^3 & + & 0x^2 & - & 7r & + & 3 \\
 & 4x^3 & - & 4r^2 & & & & \\
\cline{2-4}
 & & & 4r^2 & - & 7r & & \\
 & & & 4r^2 & - & 4r & & \\
\cline{3-6}
 & & & & - & 3r & + & 3 \\
 & & & & - & 3r & + & 3 \\
\cline{5-8}
 & & & & & & & 0
\end{array}$$

$$(r - 1)\left(4r^2 + 4r - 3\right) = 0$$

$$(r - 1)(2r - 1)(2r + 3) = 0$$

Either $r = 1$ (reject as $r \neq 1$) or $r = \frac{1}{2}$ or $r = -\frac{3}{2}$

[3 marks]

(ii) 8, 4, 2, 1, 0.5, 0.25 ... or 8, -12, 18, -27, 40.5, -60.75

[2 marks]

Lesson 3

A-Level Pure Mathematics, Year 2
Geometric Progressions

3.1 The Sum To Infinity

Teaching Video : http://www.NumberWonder.co.uk/v9077/3a.mp4 (Part 1)
http://www.NumberWonder.co.uk/v9077/3b.mp4 (Part 2)

<= Part 1 Part 2 =>

For a typical Arithmetic Progression...

$$7 = 7$$
$$7 + 11 = 18$$
$$7 + 11 + 15 = 33$$
$$7 + 11 + 15 + 19 = 52$$

Observation:

☞

For a typical Geometric Progression with either $r > 1$ or $r < 1$...

$$4 = 4$$
$$4 + 20 = 24$$
$$4 + 20 + 100 = 124$$
$$4 + 20 + 100 + 500 = 624$$

Observation:

☞

For a typical Geometric Progression with either $-1 < r < 1$

$$64 = 64$$
$$64 + 32 = 96$$
$$64 + 32 + 16 = 112$$
$$64 + 32 + 16 + 8 = 120$$
$$64 + 32 + 16 + 8 + 4 = 124$$

Question Time !
As more terms are added, will this ever sum to more than

(i) 500 ? ☞ **(ii)** 200 ? ☞ **(iii)** 130 ? ☞

(iv) 128 ? ☞ **(v)** 126 ? ☞

$$64 = 64$$
$$64 + 32 = 96$$
$$64 + 32 + 16 = 112$$
$$64 + 32 + 16 + 8 = 120$$
$$64 + 32 + 16 + 8 + 4 = 124$$
$$64 + 32 + 16 + 8 + 4 + 2 = 126$$
$$64 + 32 + 16 + 8 + 4 + 2 + 1 = 127$$
$$64 + 32 + 16 + 8 + 4 + 2 + 1 + \frac{1}{2} = 127\frac{1}{2}$$
$$64 + 32 + 16 + 8 + 4 + 2 + 1 + \frac{1}{2} + \frac{1}{4} = 127\frac{3}{4}$$
$$64 + 32 + 16 + 8 + 4 + 2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = 127\frac{7}{8}$$

The series is approaching a limit of 128 but never quite gets there.
This series would be described as having a sum to infinity of 128, which is the upper bound of the series, and is the smallest number this series can not sum to.

For a Geometric Progression with $-1 < r < 1$ it makes sense to talk about a sum to infinity because such a series is convergent on a fixed number.

3.2 The Sum To Infinity Formula For A Geometric Progression

$$Sum_{\infty} = \frac{a}{1 - r} \qquad -1 < r < 1$$

3.3 Example

Show that 128 is the sum to infinity of the geometric series,

$$64 + 32 + 16 + \ldots$$

[2 marks]

3.4 Exercise

Marks Available : 46

Question 1

Find S_∞ of the geometric series,

$$12 - 6 + 3 - 1.5 + \ldots$$

[2 marks]

Question 2

A geometric series has first term -5 and sum to infinity -3

Find the common ratio.

[3 marks]

Question 3

For the geometric series with $S_3 = 9$ and $S_\infty = 8$, find the value of the common ratio and also the value of the first term.

[4 marks]

Question 4

C2 Examination question from June 2009, Q5.

The third term of a geometric sequence is 324 and the sixth term is 96

(a) Show that the common ratio of the sequence is $\frac{2}{3}$

[2 marks]

(b) Find the first term of the sequence

[2 marks]

(c) Find the sum of the first 15 terms of the sequence

[3 marks]

(d) Find the sum to infinity of the sequence

[2 marks]

Question 5

C2 Examination question from January 2007, Q10

A geometric series is

$$a + ar + ar^2 + \ldots$$

(a) Prove that the sum of the first n terms of this series is given by

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

[4 marks]

(b) Find $\sum_{k=1}^{10} 100(2^k)$

[3 marks]

(c) Find the sum to infinity of the geometric series

$$\frac{5}{6} + \frac{5}{18} + \frac{5}{54} + \ldots$$

[3 marks]

(d) State the condition for an infinite geometric series with common ratio r to be convergent

[1 mark]

Question 6

C2 Examination question from January 2009, Q9

The first three terms of a geometric series are

$$(k + 4), \quad k, \quad (2k - 15)$$

where k is a positive constant.

(i) Show that $k^2 - 7k - 60 = 0$

[4 marks]

(ii) Hence show that $k = 12$

[2 marks]

(iii) Find the common ratio of this series

[1 mark]

(iv) Find the sum to infinity of this series.

[2 marks]

Question 7

C2 Examination question from January 2005, Q6

The second and fourth terms of a geometric series are 7.2 and 5.832 respectively.

The common ratio of the series is positive.

For this series, find

(a) the common ratio

[2 marks]

(b) the first term

[2 marks]

(c) the sum of the first 50 terms, giving your answer to 3 decimal places

[2 marks]

(d) the difference between the sum to infinity and the sum of the first 50 terms, giving your answer to 3 decimal places

[2 marks]

3.5 Solutions to 3.4 Exercise

Answer 1 8

Answer 2 $-\dfrac{2}{3}$

Answer 3 $r = -\dfrac{1}{2} \quad a = 12$

[2, 3, 4 marks]

Answer 4

(a) $$\frac{a r^5}{a r^2} = \frac{96}{324}$$
$$r^3 = \frac{8}{27}$$
$$r = \frac{2}{3}$$

(b) 729 **(c)** 2182 **(d)** 2187

[2, 2, 3, 2 marks]

Answer 5

(a) See Lesson 2 notes & video

(b) 204 600 **(c)** $\dfrac{5}{4}$ **(d)** $-1 < r < 1$

[4, 3, 3, 1 marks]

Answer 6

(i) $$\frac{k}{k+4} = \frac{2k-15}{k}$$
$$k^2 = (2k-15)(k+4)$$
$$k^2 = 2k^2 + 8k - 15k - 60$$
$$k^2 - 7k - 60 = 0 \quad \square$$

(ii) $$(k-12)(k+5) = 0$$
$\therefore k = 12 \quad \square$ Reject $k = -5$ as told k is a positive constant

(iii) 0.75

(iv) 64

[4, 2, 1, 2 marks]

Answer 7

(a) 0.9

(b) 8

(c) 79.582

(d) 0.412

[2, 2, 2, 2 marks]

Lesson 4

A-Level Pure Mathematics, Year 2
Geometric Progressions

4.1 A Logarithm Surprise

There is a situation that routinely arise in questions about Geometric Progressions that requires an ability to use logarithms.

4.2 Example

Sum the following series which is in geometric progression;

$$3 + 6 + 12 + 24 + \ldots + 49152$$

Teaching Video : http://www.NumberWonder.co.uk/v9077/4.mp4

[4 marks]

4.3 Exercise

Marks Available: 40

Question 1

What is the first term in the following geometric progression to exceed 1 million ?

$$2, \; 6, \; 18, \; 54, \; 162, \; \ldots$$

HINT :

This is about solving $a\,r^{n-1} > 1\,000\,000$

[4 marks]

Question 2

What is the first term in the following geometric progression to exceed 200 ?

$$0.4, \; 0.6, \; 0.9, \; 1.35, \; 2.025, \; \ldots$$

[4 marks]

Question 3

Sum the following series which is in geometric progression;

$$19683 + 6561 + \dots + 1$$

[4 marks]

Question 4

A population of rabbits is increasing at a rate of 35% per annum on a large and uninhabited island with lush vegetation.

At the start of 2011 there were 40 rabbits.

In what year will the rabbit population first exceed 1000 rabbits ?

HINT : Be careful about exactly what this question is asking.

[4 marks]

Question 5

Sum the following series which is in geometric progression;

$$1 - 2 + 4 - 8 + 16 - 32 + \ldots + 1073741824$$

HINT : To avoid *ln* (– 2) and a 'math error'...

$$(-2)^{n-1} = 1073741824$$

$$(-1)^{n-1}\left(2^{n-1}\right) = 1073741824$$

$$(-1)^{n-1} \textit{ must equal } 1 \textit{ and n must be odd}$$

$$\therefore 2^{n-1} = 1073741824$$

[4 marks]

Question 6

C2 Examination Question, May 2006, Q9

A geometric series has first term a and common ratio r

The second term of the series is 4 and the sum to infinity of the series is 25

(a) Show that

$$25r^2 - 25r + 4 = 0$$

[4 marks]

(b) Find the two possible values of r

[2 marks]

(c) Find the corresponding two possible values of a

[2 marks]

(d) Show that the sum, S_n , of the first n terms of the series is given by

$$S_n = 25(1 - r^n)$$

[1 mark]

Given that r takes the larger of its two possible values,

(e) find the smallest value of n for which S_n exceeds 24

[2 marks]

Question 7

C2 Examination Question, June 2008, Q6

A geometric series has first term 5 and common ratio $\frac{4}{5}$

Calculate

(a) the 20th term of the series, to 3 decimal places

[2 marks]

(b) the sum to infinity of the series

[2 marks]

Given that the sum to k terms of the series is greater than 24.95

(c) show that

$$k > \frac{log\,0.002}{log\,0.8}$$

[4 marks]

(d) find the smallest possible value of k

[1 mark]

4.4 Answers

4.4.1 Solution to 4.2 Example

The problem is that to use the formula to work out the sum of the series, the number of terms, *n*, needs to be found first.

$$\begin{aligned} a\,r^{n-1} &= 49152 \\ 3 \times 2^{n-1} &= 49152 \\ 2^{n-1} &= 16384 \end{aligned}$$

take the natural log of both sides

$$\begin{aligned} \ln 2^{n-1} &= \ln 16384 \\ (n - 1)\ln 2 &= \ln 16384 \\ n - 1 &= \frac{\ln 16384}{\ln 2} \\ n - 1 &= 14 \\ n &= 15 \end{aligned}$$

$$\begin{aligned} S_{15} &= \frac{3(2^{15} - 1)}{(2 - 1)} \\ &= 98301 \end{aligned}$$

[4 marks]

4.4.2 Solutions to 4.3 Exercise

Answer 1

$$a\,r^{n-1} > 1\,000\,000$$

$$2 \times 3^{n-1} > 1\,000\,000$$

$$3^{n-1} > 500\,000$$

take the natural log of both sides

$$ln\,3^{n-1} > ln\,500\,000$$

$$(n - 1)\,ln\,3 > ln\,500\,000$$

$$n - 1 > \frac{ln\,500\,000}{ln\,3}$$

$$n - 1 > 11.94$$

$$n > 12.94$$

$$\text{i.e. } n = 13$$

The 13^{th} term is the first to exceed one million

2, 6, 18, 54, 162, 486, 1458, 4374, 13122, 39366, 118098, 354294, 1062882 ...

[4 marks]

Answer 2

The 17^{th} term

[4 marks]

Answer 3

n = 10, Sum = 29524

[4 marks]

Answer 4

n > 11.7 $\therefore$ First exceeds 1000 in the 12th year of 2023 (Not 11th !)

$n > 11.7$ $\therefore$ First exceeds 1000 in the 12^{th} year of 2023 (Not the 11^{th} !)

[4 marks]

Answer 5

n = 31, Sum = 715 827 883

[4 marks]

Answer 6

(a) Told $ar = 4$ and $\dfrac{a}{1 - r} = 25$

$$a = 25\,(1 - r)$$

$$\frac{4}{r} = 25\,(1 - r)$$

$$4 = 25r - 25\,r^2$$

$$25\,r^2 - 25r + 4 = 0 \qquad \square$$

(b) 0.2 or 0.8

(c) 20 or 5

(d) $S_n = \dfrac{a\,(1 - r^n)}{(1 - r)}$ but $\dfrac{a}{1 - r} = 25$

so $S_n = 25\left(1 - r^n\right)$ $\square$

(e) $n > \dfrac{\log 0.04}{\log 0.8}$ $n = 15$

Note : Division of an inequality by a negative quantity reverses the inequality.
$\ln x$ is negative for $0 < x < 1$

[4, 2, 2, 1, 2 marks]

Answer 7

(a) 0.072

(b) 25

(c) $\dfrac{5\left(1 - 0.8^k\right)}{1 - 0.8} > 24.95$

$$k > \frac{\log 0.002}{\log 0.8}$$

Note : Division of an inequality by a negative quantity reverses the inequality.
$\ln x$ is negative for $0 < x < 1$

(d) 28

[2, 2, 4,1]