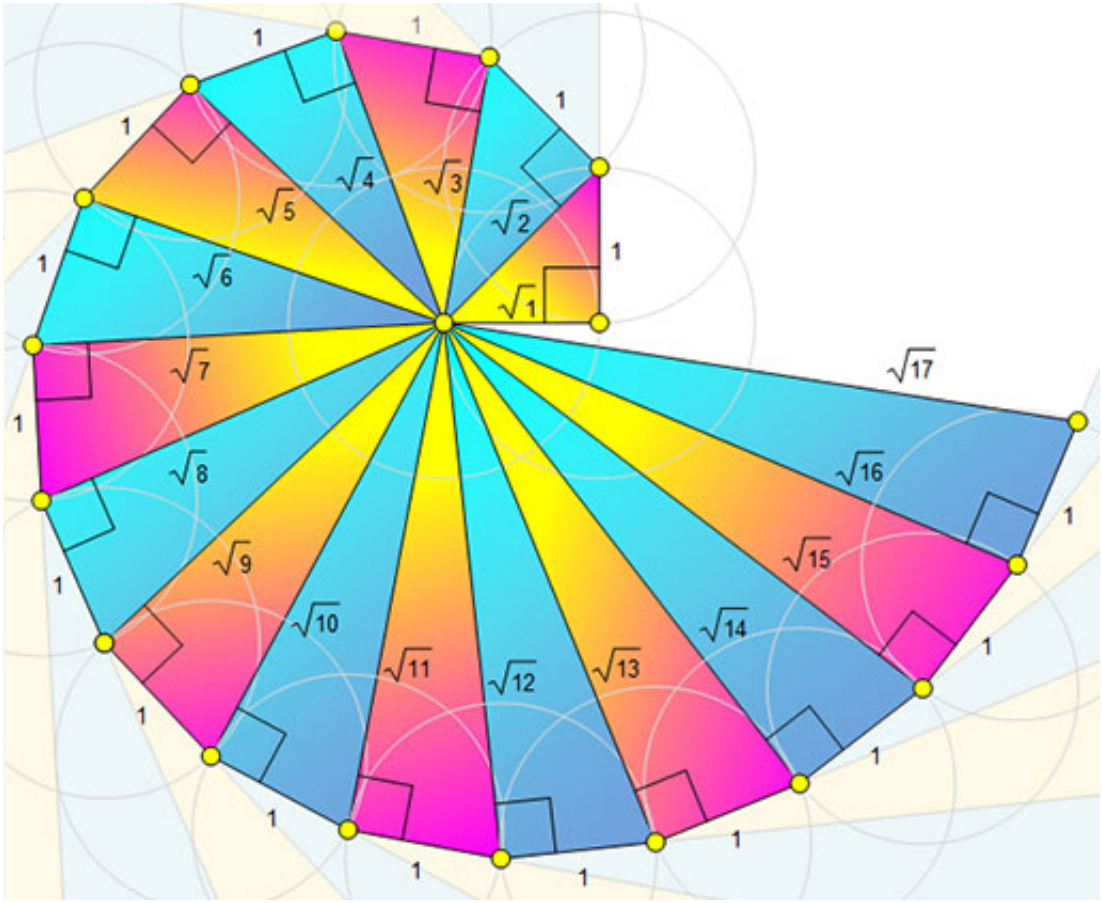
A-Level Mathematics
A-Level Further Mathematics
Pure Year 1

ALGEBRA

~ Surds and Indices II ~



Algebra

~ Surds and Indices ~

Lesson 1

A-Level Pure Mathematics : Year 1

Algebra of Surds and Indices II

1.1 Multiplication Involving Surds

Challenge #1

Find the exact value of,

$$\left(2 + 7\sqrt{2} - 5\sqrt{3}\right)\left(4 + 3\sqrt{2} + 2\sqrt{3}\right)$$

[6 marks]

Challenge #2

Find the exact value of,

$$\left(3\sqrt{2} + 4\sqrt{3} + 7\sqrt{6}\right)\left(5\sqrt{2} - 2\sqrt{3} + 3\sqrt{6}\right)$$

[6 marks]

1.2 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 50

Question 1

Expand and simplify, giving exact answers,

(i) $(5 + \sqrt{7})^2$

[2 marks]

(ii) $(2 + 5\sqrt{3})(4 - 2\sqrt{3})$

[3 marks]

(iii) $(3 + \sqrt{2})(3 - \sqrt{2})(2 + \sqrt{5})$

[4 marks]

(iv) $(6 + 2\sqrt{3} - \sqrt{5})(7 + \sqrt{3})$

[5 marks]

Question 2

A square-free number is an integer that is not divisible by any square number except 1.

(i) Which of the following are integers ?

$6 \quad \pi \quad \sqrt{8} \quad -4 \quad \sqrt{25} \quad 0.125 \quad \frac{4}{5} \quad \frac{81}{3}$

[1 mark]

(ii) Which of the following are square-free ?

7 121 50 12 6 99 65 147

[2 marks]

Question 3

Simplify each of the following, writing your answers in the form $a\sqrt{b}$ for integer values of a and b with b square free.

(i) $\sqrt{8}$

[1 mark]

(ii) $5\sqrt{20}$

[1 mark]

(iii) $\left(3\sqrt{2}\right)^3 - 4\sqrt{2}$

[2 marks]

(iv) $\sqrt{12} + 3\sqrt{48} + \sqrt{75}$

[2 marks]

Question 4

Show how to expand the brackets of

$$\left(1 + \sqrt{5}\right)\left(1 - \sqrt{5}\right)^3$$

to obtain an answer in the form $a + b\sqrt{5}$ where a and b are integers.

[3 marks]

Question 5

Find the exact value of,

$$\left(5 + 3\sqrt{2} + 6\sqrt{3}\right)\left(8 + 4\sqrt{2} - 2\sqrt{3}\right)$$

[6 marks]

Question 6

Which of the following numbers are square free ?

(i) $3 \times 5 \times 7$

(ii) $2 \times 3^2 \times 11$

(iii) 5×7^3

[2 marks]

Question 7

Explain why,

$$\sqrt{2 \times 3^3 \times 5^2} = 15\sqrt{6}$$

[2 marks]

Question 8

Simplify,

$$\sqrt{a^3 \times b^5}$$

where a and b are unknown integers.

Give your answer in the form $x\sqrt{y}$ where x and y are expressed in terms of a and b and y is square free.

[2 marks]

Question 9

The following number is too big for my calculator;

$$5^{52} \times 7 \times 13^{95}$$

Even so, square root this number, writing the answer in in the form $a\sqrt{p}$

where a & p are integers, that may be written in index form

and p is ☐ FREE.

[2 marks]

Question 10

Find the exact value of,

$$\left(7\sqrt{2} - 3\sqrt{5} + 4\sqrt{10}\right)\left(9\sqrt{2} + 2\sqrt{5} + 2\sqrt{10}\right)$$

[6 marks]

Question 11

Without using a calculator at any stage, devise a method that will determine the cube root of 1728

i.e. $\sqrt[3]{1728}$

[4 marks]

1.3 Answers

A-Level Pure Mathematics : Year 1
Algebra of Surds and Indices II

1.3.1 Solutions to Introductory Challenges

The key idea is to organise the multiplication process in some way.
I suggest a table.
Calculators will not do the whole calculation exactly, but will do the individual entries in the following tables exactly.
The individual entries can be done as mental arithmetic.

Challenge #1

	2	$7\sqrt{2}$	$-5\sqrt{3}$
4	8	$28\sqrt{2}$	$-20\sqrt{3}$
$3\sqrt{2}$	$6\sqrt{2}$	42	$-15\sqrt{6}$
$2\sqrt{3}$	$4\sqrt{3}$	$14\sqrt{6}$	-30

$$\left(2 + 7\sqrt{2} - 5\sqrt{3}\right)\left(4 + 3\sqrt{2} + 2\sqrt{3}\right)$$
$$= 20 + 34\sqrt{2} - 16\sqrt{3} - \sqrt{6}$$

[6 marks]

Challenge #2
It will help to think of $\sqrt{6}$ as $\sqrt{2} \times \sqrt{3}$ if doing this by mental arithmetic.

	$3\sqrt{2}$	$4\sqrt{3}$	$7\sqrt{6}$
$5\sqrt{2}$	30	$20\sqrt{6}$	$70\sqrt{3}$
$-2\sqrt{3}$	$-6\sqrt{6}$	-24	$-42\sqrt{2}$
$3\sqrt{6}$	$18\sqrt{3}$	$36\sqrt{2}$	126

$$\left(3\sqrt{2} + 4\sqrt{3} + 7\sqrt{6}\right)\left(5\sqrt{2} - 2\sqrt{3} + 3\sqrt{6}\right)$$
$$= 132 - 6\sqrt{2} + 88\sqrt{3} + 14\sqrt{6}$$

[6 marks]

1.3.2 Solutions to 1.2 Exercise

Answer 1

(i) $$(5+\sqrt{7})^2 = 25 + 10\sqrt{7} + 7$$
$$= 32 + 10\sqrt{7}$$

[2 marks]

(ii) $$(2+5\sqrt{3})(4-2\sqrt{3}) = 8 - 4\sqrt{3} + 20\sqrt{3} - 30$$
$$= -22 + 16\sqrt{3}$$

[3 marks]

(iii) "difference of two squares"

$$(3+\sqrt{2})(3-\sqrt{2})(2+\sqrt{5}) = (7)(2+\sqrt{5})$$
$$= 14 + 7\sqrt{5}$$

[4 marks]

(iv) $$(6+2\sqrt{3}-\sqrt{5})(7+\sqrt{3})$$
$$= 42 + 6\sqrt{3} + 14\sqrt{3} + 6 - 7\sqrt{5} - \sqrt{15}$$
$$= 48 + 20\sqrt{3} - 7\sqrt{5} - \sqrt{15}$$

[5 marks]

Answer 2

(i) Integers are;

6 π $\sqrt{8}$ -4 $\sqrt{25}$ 0.125 $\frac{4}{5}$ $\frac{81}{3}$

[1 mark]

(ii) Square-free are;

7 121 50 12 6 99 65 147

[2 marks]

Answer 3

(i) $$\sqrt{8} = 2\sqrt{2}$$

[1 mark]

(ii) $$5\sqrt{20} = 10\sqrt{5}$$

[1 mark

(iii) $$(3\sqrt{2})^3 - 4\sqrt{2} = 27 \times 2\sqrt{2} - 4\sqrt{2}$$
$$= 50\sqrt{2}$$

[2 marks]

(iv) $$\sqrt{12} + 3\sqrt{48} + \sqrt{75} = 2\sqrt{3} + 12\sqrt{3} + 5\sqrt{3}$$
$$= 19\sqrt{3}$$

[2 marks]

Answer 4

$$(1+\sqrt{5})(1-\sqrt{5})^3$$
$$= (1+\sqrt{5})(1-\sqrt{5})(1-\sqrt{5})^2$$
$$= (-4)(1-\sqrt{5})^2$$
$$= (-4)(1-2\sqrt{5}+5)$$
$$= (-4)(6-2\sqrt{5})$$
$$= -24+8\sqrt{5}$$

[3 marks]

Answer 5

	5	$3\sqrt{2}$	$6\sqrt{3}$
8	40	$24\sqrt{2}$	$48\sqrt{3}$
$4\sqrt{2}$	$20\sqrt{2}$	24	$24\sqrt{6}$
$-2\sqrt{3}$	$-10\sqrt{3}$	$-6\sqrt{6}$	-36

$$(5+3\sqrt{2}+6\sqrt{3})(8+4\sqrt{2}-2\sqrt{3})$$
$$= 28+44\sqrt{2}+38\sqrt{3}+18\sqrt{6}$$

[6 marks]

Answer 6

Only (i) is square-free

[2 marks]

Answer 7

$$\text{LHS} = \sqrt{2\times 3^3\times 5^2}$$
$$= \sqrt{3^2\times 5^2\times 2\times 3}$$
$$= \sqrt{(3\times 5)^2\times(2\times 3)}$$
$$= \sqrt{15^2}\times\sqrt{6}$$
$$= 15\sqrt{6}$$
$$= \text{RHS} \qquad \square$$

[2 marks]

Answer 8

$$\sqrt{a^3\times b^5} = \sqrt{a^2\times b^4\times a\times b}$$
$$= a\,b^2\sqrt{ab}$$

i.e. $x = a\,b^2$ and $y = ab$

[2 marks]

Answer 9

$$\sqrt{5^{52} \times 7 \times 13^{95}} = \sqrt{5^{52} \times 13^{94} \times 7 \times 13}$$
$$= 5^{26} \times 13^{47} \times \sqrt{7 \times 13}$$

[2 marks]

Answer 10

It will help to think of $\sqrt{10}$ as $\sqrt{2} \times \sqrt{5}$ if doing this by mental arithmetic.

	$7\sqrt{2}$	$-3\sqrt{5}$	$4\sqrt{10}$
$9\sqrt{2}$	126	$-27\sqrt{10}$	$72\sqrt{5}$
$2\sqrt{5}$	$14\sqrt{10}$	-30	$40\sqrt{2}$
$2\sqrt{10}$	$28\sqrt{5}$	$-30\sqrt{2}$	80

$$\left(7\sqrt{2} - 3\sqrt{5} + 4\sqrt{10}\right)\left(9\sqrt{2} + 2\sqrt{5} + 2\sqrt{10}\right)$$
$$= 176 + 10\sqrt{2} + 100\sqrt{5} - 13\sqrt{10}$$

[6 marks]

Answer 11

A 'by hand' division ladder...

2	1728
2	864
2	432
2	216
2	108
2	54
3	27
3	9
3	3
	1

... in order to deduce (without using a calculators FACT button) that...

$$1728 = 2^6 \times 3^3$$
$$\therefore \sqrt[3]{1728} = 12$$

[4 marks]

Lesson 2

A-Level Pure Mathematics : Year 1

Algebra of Surds and Indices II

2.1 Rationalising the Denominator

Mathematicians' dislike fractions which have any occurrence of a square root in the denominator. There are standard techniques for manipulating such fractions to remove all occurrences of an offending square root from the denominator.
This may well result in square roots in the numerator, but this is considered fine !

2.2 Example (GCSE Level 8, 9)

Consider the function,

$$f(a) = \frac{a}{a - \sqrt{a}} \qquad a \in \mathbb{Z},\ a \geqslant 2$$

(i) Carefully explain the restriction upon the domain of the function.

[3 marks]

(ii) Given, furthermore, that a is a square free number, rationalise the denominator of the function.
Simplify your answer.

[3 marks]

2.3 Example (A-Level)

(i) Give a definition of a rational number.

[2 marks]

(ii) Prove that, $\frac{\sqrt{5} - \sqrt{13}}{3\sqrt{5} + \sqrt{13}} = \frac{7}{8} - \frac{1}{8}\sqrt{65}$

[4 marks]

2.4 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 50

Question 1

Prove that, $\frac{3\sqrt{7} - \sqrt{5}}{\sqrt{7} + \sqrt{5}} = 13 - 2\sqrt{35}$

[4 marks]

Question 2

Prove that, $\frac{1}{(2 + \sqrt{3})^2} = 7 - 4\sqrt{3}$

[4 marks]

Question 3

Show that, $\dfrac{5}{\sqrt{75} - \sqrt{50}}$

can be written in the form $\sqrt{a} + \sqrt{b}$ where a and b are integers

[4 marks]

Question 4

A right angled triangle has an exact area of 9 cm^2

One side, not the hypotenuse, has an exact length of $(5 + \sqrt{7})$ cm

(i) Showing your method clearly, determine the exact length of the other side, not the hypotenuse, of the triangle.

[3 marks]

(ii) By using the Theorem of Pythagoras, determine the exact perimeter of the right angled triangle.

[3 marks]

Question 5

By first rationalising the denominators, then getting a common denominator, show that

$$\frac{3}{\sqrt{2}} + \frac{5}{\sqrt{3}} = \frac{9\sqrt{2} + 10\sqrt{3}}{6}$$

[4 marks]

Question 6

Simplify

$$\frac{1}{\sqrt{a}} + \frac{4\sqrt{a}}{a}$$

where a is an integer.

Your final answer should have a denominator that is rational.

[5 marks]

Question 7

Solve the following equation, giving an exact answer,

$$8 - x\sqrt{12} = \frac{8x}{\sqrt{3}}$$

[5 marks]

Question 8

Given that $p = 3 - 2\sqrt{2}$ and $q = 2 - \sqrt{2}$ find the value of

$$\frac{p + q}{p - q}$$

Give your answer in the form $m + n\sqrt{2}$ where m and n are rational numbers that are to be found.

[5 marks]

Question 9

Showing clear working, evaluate the following expression giving an exact answer in as simple a form as possible,

$$\frac{1}{\sqrt{1}+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{4}}+\ldots+\frac{1}{\sqrt{2021}+\sqrt{2022}}$$

[7 marks]

Question 10

Write the following as a single fraction with a denominator that is rational,

$$\frac{7}{\sqrt{ab}} + \frac{b}{\sqrt{a}} + \frac{a}{\sqrt{b}}$$

where a and b are positive integers.

[6 marks]

2.5 Answers

A-Level Pure Mathematics : Year 1
Algebra of Surds and Indices II

2.5.1 Solutions to 2.2 Example (GCSE Level 8 or 9)

(i) No direct penalty for not knowing that $a \in \mathbb{Z}$ means that a is an integer.
1 mark for stating $a = 0$ and $a = 1$ are excluded to avoid division by zero.
1 mark for stating that division by zero is undefined in mathematics.
Not both of $a = 0$ and $a = 1$ must be mentioned, either one will suffice.
1 mark for explaining that all negative numbers have been excluded to avoid taking the square root of a negative number which would give a non-real (i.e. complex number) value to the function. Any answer that captures an aspect of this gets the mark which is more for realising that there are two separate reasons behind restricting the domain, one to do with division by zero and the other for the taking of negative square roots.

[3 marks]

(ii)

$$\frac{a}{a-\sqrt{a}} = \frac{a}{a-\sqrt{a}} \times \frac{a+\sqrt{a}}{a+\sqrt{a}} \qquad \text{Key step !}$$

$$= \frac{a\left(a+\sqrt{a}\right)}{a^2-a}$$

$$= \frac{a\left(a+\sqrt{a}\right)}{a\left(a-1\right)}$$

$$= \frac{a+\sqrt{a}}{a-1}$$

[3 marks]

2.5.2 Solution to 2.3 Example (A-Level)

(i) Rationals can be written $\frac{p}{q}$ for integer values of p and q and $q \neq 0$

[2 marks]

(ii)

$$\text{LHS} = \frac{\sqrt{5}-\sqrt{13}}{3\sqrt{5}+\sqrt{13}}$$

$$= \frac{\sqrt{5}-\sqrt{13}}{3\sqrt{5}+\sqrt{13}} \times \frac{3\sqrt{5}-\sqrt{13}}{3\sqrt{5}-\sqrt{13}}$$

$$= \frac{15-\sqrt{65}-3\sqrt{65}+13}{45-13}$$

$$= \frac{28-4\sqrt{65}}{32}$$

$$= \frac{7}{8} - \frac{1}{8}\sqrt{65}$$

$$= \text{RHS} \qquad \square$$

[4 marks]

2.5.3 Solutions to 2.4 Exercise

Answer 1

$$\begin{aligned}
\text{LHS} &= \frac{3\sqrt{7}-\sqrt{5}}{\sqrt{7}+\sqrt{5}} \\
&= \frac{3\sqrt{7}-\sqrt{5}}{\sqrt{7}+\sqrt{5}} \times \frac{\sqrt{7}-\sqrt{5}}{\sqrt{7}-\sqrt{5}} \\
&= \frac{21-3\sqrt{35}-\sqrt{35}+5}{7-5} \\
&= \frac{26-4\sqrt{35}}{2} \\
&= 13-2\sqrt{35} \\
&= \text{RHS} \qquad \square
\end{aligned}$$

[4 marks]

Answer 2

Method 1

$$\begin{aligned}
\text{LHS} &= \frac{1}{(2+\sqrt{3})^2} \\
&= \frac{1}{4+4\sqrt{3}+3} \\
&= \frac{1}{7+4\sqrt{3}} \times \frac{7-4\sqrt{3}}{7-4\sqrt{3}} \\
&= \frac{7-4\sqrt{3}}{49-48} \\
&= 7-4\sqrt{3} \\
&= \text{RHS} \qquad \square
\end{aligned}$$

Method 2 (Clever)

$$\begin{aligned}
\text{LHS} &= \frac{1}{(2+\sqrt{3})^2} \\
&= \frac{1}{(2+\sqrt{3})^2} \times \frac{(2-\sqrt{3})^2}{(2-\sqrt{3})^2} \\
&= \frac{(2-\sqrt{3})^2}{(4-3)(4-3)} \\
&= \frac{4-4\sqrt{3}+3}{1} \\
&= 7-4\sqrt{3} \\
&= \text{RHS} \qquad \square
\end{aligned}$$

[4 marks]

Answer 3

$$\begin{aligned}
\frac{5}{\sqrt{75}-\sqrt{50}} &= \frac{5}{5\sqrt{3}-5\sqrt{2}} \\
&= \frac{1}{\sqrt{3}-\sqrt{2}} \times \frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}+\sqrt{2}} \\
&= \frac{\sqrt{3}+\sqrt{2}}{3-2} \\
&= \sqrt{3}+\sqrt{2}
\end{aligned}$$

...which is of the required form

[4 marks]

Answer 4

(i)

$$\text{Area } \Delta = \frac{1}{2} \times \text{base} \times \text{height}$$

$$9 = \frac{1}{2} \times (5 + \sqrt{7}) \times \text{height}$$

$$\text{height} = \frac{18}{5 + \sqrt{7}}$$

$$= \frac{18}{5 + \sqrt{7}} \times \frac{5 - \sqrt{7}}{5 - \sqrt{7}}$$

$$= \frac{18\,(5 - \sqrt{7})}{25 - 7}$$

$$= \frac{18\,(5 - \sqrt{7})}{18}$$

$$= 5 - \sqrt{7}$$

[3 marks]

(ii)

$$\text{Hypotenuse}^2 = (5 + \sqrt{7})^2 + (5 - \sqrt{7})^2$$

$$= 25 + 10\sqrt{7} + 7 + 25 - 10\sqrt{3} + 7$$

$$= 64$$

$$\text{Hypotenuse} = 8$$

$$\text{Perimeter} = 8 + 5 + \sqrt{7} + 5 - \sqrt{7}$$

$$= 18 \text{ cm}$$

[3 marks]

Answer 5

$$\text{LHS} = \frac{3}{\sqrt{2}} + \frac{5}{\sqrt{3}}$$

$$= \frac{3}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} + \frac{5}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$

$$= \frac{3\sqrt{2}}{2} + \frac{5\sqrt{3}}{3}$$

$$= \frac{3\sqrt{2}}{2} \times \frac{3}{3} + \frac{5\sqrt{3}}{3} \times \frac{2}{2}$$

$$= \frac{9\sqrt{2}}{6} + \frac{10\sqrt{3}}{6}$$

$$= \frac{9\sqrt{2} + 10\sqrt{3}}{6}$$

$$= \text{RHS} \qquad \square$$

[4 marks]

Answer 6

$$\frac{1}{\sqrt{a}} + \frac{4\sqrt{a}}{a} = \frac{1}{\sqrt{a}} \times \frac{\sqrt{a}}{\sqrt{a}} + \frac{4\sqrt{a}}{a}$$

$$= \frac{\sqrt{a} + 4\sqrt{a}}{a}$$

$$= \frac{5\sqrt{a}}{a}$$

[5 marks]

Answer 7

$$8 - x\sqrt{12} = \frac{8x}{\sqrt{3}}$$

Multiply through by $\sqrt{3}$

$$8\sqrt{3} - x\sqrt{3} \times \sqrt{12} = 8x$$

$$8\sqrt{3} - x\sqrt{36} = 8x$$

$$8\sqrt{3} - 6x = 8x$$

$$14x = 8\sqrt{3}$$

$$x = \frac{4\sqrt{3}}{7}$$

[5 marks]

Answers 8

$$\frac{p+q}{p-q} = \frac{3 - 2\sqrt{2} + 2 - \sqrt{2}}{3 - 2\sqrt{2} - 2 + \sqrt{2}}$$

$$= \frac{5 - 3\sqrt{2}}{1 - \sqrt{2}}$$

$$= \frac{5 - 3\sqrt{2}}{1 - \sqrt{2}} \times \frac{1 + \sqrt{2}}{1 + \sqrt{2}}$$

$$= \frac{5 + 5\sqrt{2} - 3\sqrt{2} - 6}{1 - 2}$$

$$= \frac{-1 + 2\sqrt{2}}{-1}$$

$$= 1 - 2\sqrt{2}$$

i.e. $m = 1$ and $n = -2$

[5 marks]

Answer 9

The initial step is to rationalise the denominators.

The clever way to do this is to note that they are all of the form,

$$\frac{1}{\sqrt{n}\ +\sqrt{n+1}}$$

So,

$$\begin{aligned}\frac{1}{\sqrt{n}\ +\sqrt{n+1}} &= \frac{1}{\sqrt{n}\ +\sqrt{n+1}} \times \frac{\sqrt{n}\ -\sqrt{n+1}}{\sqrt{n}\ -\sqrt{n+1}} \\ &= \frac{\sqrt{n}\ -\sqrt{n+1}}{n-(n+1)} \\ &= \frac{\sqrt{n}\ -\sqrt{n+1}}{-1} \\ &= \sqrt{n+1}\ -\sqrt{n}\end{aligned}$$

Now to make use of this in the given expression,

$$\begin{aligned}&\frac{1}{\sqrt{1}\ +\sqrt{2}} + \frac{1}{\sqrt{2}\ +\sqrt{3}} + \frac{1}{\sqrt{3}\ +\sqrt{4}} + \ldots + \frac{1}{\sqrt{2021}\ +\sqrt{2022}} \\ &= \sqrt{2}\ -\sqrt{1}\ +\sqrt{3}\ -\sqrt{2}\ +\sqrt{4}\ -\sqrt{3}\ +\ldots+\sqrt{2022}\ -\sqrt{2021} \\ &= \sqrt{2022}\ -\sqrt{1} \\ &= \sqrt{2022}\ -1\end{aligned}$$

[7 marks]

Answer 10

$$\begin{aligned}\frac{7}{\sqrt{ab}} + \frac{b}{\sqrt{a}} + \frac{a}{\sqrt{b}} &= \frac{7}{\sqrt{ab}} + \frac{b}{\sqrt{a}} \times \frac{\sqrt{b}}{\sqrt{b}} + \frac{a}{\sqrt{b}} \times \frac{\sqrt{a}}{\sqrt{a}} \\ &= \frac{7 + b\sqrt{b}\ + a\sqrt{a}}{\sqrt{ab}} \\ &= \frac{7 + b\sqrt{b}\ + a\sqrt{a}}{\sqrt{ab}} \times \frac{\sqrt{ab}}{\sqrt{ab}} \\ &= \frac{7\sqrt{ab}\ + b^2\sqrt{a}\ + a^2\sqrt{b}}{ab}\end{aligned}$$

[6 marks]

Lesson 3

A-Level Pure Mathematics : Year 1
Algebra of Surds and Indices II

3.1 Equations Involving Surds & Indices

Example #1 : Index Matching
Given that

$$81\sqrt{3} = 3^a$$

find the value of a.

[2 marks]

Example #2 : Index Matching
Given that

$$\left(27x^{12}\right)^{\frac{5}{3}} = 3^a x^b$$

find the value of a and the value of b.

[2 marks]

Example #3 : Surd Coefficient Matching
Given that,

$$\left(3+\sqrt{c}\right)\left(2\sqrt{c}-3\right) = 1 + k\sqrt{c}$$

where c and k are prime numbers, find the value of c and the value of k

[3 marks]

Example #4 : Double Square Root Surds

Given that,

$$\sqrt{10 + 2\sqrt{21}} = \sqrt{a} + \sqrt{b} \qquad \text{with } a < b$$

find the value of a and the value of b, both of which are integers.

[3 marks]

3.2 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 55

Question 1

Without using a calculator, write down the value of each of the following.
Answers should be written as exact simplified rational numbers without indices.

(i) $25^{\frac{1}{2}}$ **(ii)** $27^{\frac{1}{3}}$ **(iii)** 3^{-2} **(iv)** $4^{\frac{3}{2}}$ **(v)** $9^{-\frac{1}{2}}$

(vi) $100^{-\frac{3}{2}}$ **(vii)** $81^{\frac{1}{4}}$ **(viii)** $81^{\frac{3}{4}}$ **(ix)** $81^{-\frac{1}{2}}$ **(x)** 81^{0}

[5 marks]

Question 2

Given that a, b and c are distinct prime numbers and that

$$y = 2^5 \times 3^4 \times 5^3$$

determine the value of $\sqrt{y}$

Write your answer in the form $a\sqrt{b}$ where a and b are integers and b is $\square$ free.

[2 marks]

Question 3

Without using a calculator, write down the value of each of the following. Answers should be written as simplified exact numbers without indices and with denominators that are rational.

(i) $\left(\frac{4}{7}\right)^{2}$ **(ii)** $\left(\frac{3}{2}\right)^{-2}$ **(iii)** $\left(\frac{22}{77}\right)^{0}$

(iv) $\left(\frac{3}{8}\right)^{-1}$ **(v)** $\left(\frac{50}{32}\right)^{\frac{1}{2}}$

[5 marks]

Question 4

Without using a calculator, write down the value of each of the following. Answers should be written as simplified exact numbers without indices and with denominators that are rational.

(i) $\left(-\frac{16}{54}\right)^{-\frac{1}{3}}$ **(ii)** $\left(\frac{9}{6\sqrt{2}}\right)^{3}$

(iii) $\left(\frac{25}{8}\right)^{\frac{1}{2}}$ **(iv)** $\left(\frac{7+\sqrt{5}}{8}\right)^{-1}$

[4 marks]

Question 5

Given that $8\sqrt{2} = 2^{a}$ find the value of a

[2 marks]

Question 6

Given that,

$$\sqrt{3 + 2\sqrt{2}} = \sqrt{a} + \sqrt{b} \qquad \text{with } a > b$$

find the value of a and the value of b, showing clear reasoning to justify your answer.

[3 marks]

Question 7

Given that,

$$\left(7 - \sqrt{c}\right)\left(4 + 2\sqrt{c}\right) = 6 + k\sqrt{c}$$

where c and k are integers and c is square free, find the value of c and the value of k

[3 marks]

Question 8

Showing all steps in your reasoning, work out the exact value of n, given that

$$\frac{1}{\sqrt[3]{9^4}} = 3^n$$

[2 marks]

Question 9

Given that

$$\left(a + \sqrt{5}\right)\left(3 + 2\sqrt{5}\right) = 31 + b\sqrt{5}$$

find the value of a and the value of b both of which are integers.

[3 marks]

Question 10

Given that,

$$x = \sqrt{6 + 2\sqrt{5}} - \sqrt{6 - 2\sqrt{5}}$$

prove that x has the value 2, exactly.

[4 marks]

Question 11

Given that for some prime, p,

$$p^m = \frac{1}{p \times \sqrt[3]{p^2}}$$

Find the value of m

[2 marks]

Question 12

Two composite numbers, g and h have prime number decompositions

$$g = a^3 \times b \times c^2 \qquad h = a \times b \times c^3$$

where a, b and c are distinct prime numbers.

(a) Express gh as a product of powers of its prime factors
Simplify your answer

[2 marks]

(b) Find the value of x, the value of y and the value of z given that,

$$\frac{g}{h} = a^x \times b^y \times c^z$$

[3 marks]

Question 13

Given that,

$$\sqrt{8 - 4\sqrt{3}} = \sqrt{a} - \sqrt{b} \qquad \text{with } a > b$$

find the value of a and the value of b, both of which are integers

[3 marks]

Question 14

GCSE Examination Question, January 2017, Paper 3H(R), Q18

Given that p is a prime number, rationalise the denominator of

$$\frac{7\sqrt{p} - p^2}{\sqrt{p^3}}$$

Simplify your answer

[3 marks]

Question 15

A-Level Examination Question, June 2019, Paper 2, Q1

Given

$$2^x \times 4^y = \frac{1}{2\sqrt{2}}$$

express y as a function of x.

[3 marks]

Question 16

Without using a calculator, and making your method clear, find the square root of

$$2^7 \times 3 \times 5^4$$

Writing your answer in the form $a\sqrt{b}$ where a and b are integers and b is $\square$ free.

[2 marks]

Question 17

Carefully showing your working, rationalise the denominator of,

$$\frac{1}{1 + \sqrt{2} + \sqrt{3}}$$

[4 marks]

3.3 Answers

A-Level Pure Mathematics : Year 1
Algebra of Surds and Indices II

3.3.1 Solutions to 3.1 Introductory Examples

Example #1 Answer

$$81\sqrt{3} = 3^a$$
$$3^4 \times 3^{\frac{1}{2}} = 3^a$$
$$3^{4.5} = 3^a$$
$$\therefore\ a = 4.5$$

[2 marks]

Example #2 Answer

$$\left(27x^{12}\right)^{\frac{5}{3}} = 3^a x^b$$
$$\left(3x^4\right)^5 = 3^a x^b$$
$$3^5 x^{20} = 3^a x^b$$
$$\therefore\ a = 5 \text{ and } b = 20$$

[2 marks]

Example #3 Answer

$$\left(3+\sqrt{c}\right)\left(2\sqrt{c}-3\right) = 1 + k\sqrt{c}$$
$$6\sqrt{c} - 9 + 2c - 3\sqrt{c} = 1 + k\sqrt{c}$$
$$\left(2c-9\right) + 3\sqrt{c} = 1 + k\sqrt{c}$$
$$\therefore\ k = 3 \text{ and } 2c - 9 = 1$$
$$2c = 10$$
$$c = 5$$

[3 marks]

Example #4 Answer

$$\sqrt{10 + 2\sqrt{21}} = \sqrt{10 + 2\sqrt{3}\sqrt{7}}$$
$$= \sqrt{3 + 2\sqrt{3}\sqrt{7} + 7}$$
$$= \sqrt{\left(\sqrt{3}\right)^2 + 2\sqrt{3}\sqrt{7} + \left(\sqrt{7}\right)^2}$$

Recall that : $a^2 + 2ab + b^2 = (a+b)^2$

$$= \sqrt{\left(\sqrt{3}+\sqrt{7}\right)^2}$$
$$= \sqrt{3} + \sqrt{7}$$
$$\therefore\ a = 3 \text{ and } b = 7 \qquad \text{as told } a < b$$

[3 marks]

3.3.2 Solutions (3.2 Exercise)

Answer 1

(i) 5 **(ii)** 3 **(iii)** $\frac{1}{9}$ **(iv)** 8 **(v)** $\frac{1}{3}$

(vi) $\frac{1}{1000}$ **(vii)** 3 **(viii)** 27 **(ix)** $\frac{1}{9}$ **(x)** 1

[5 marks]

Answer 2

$$\begin{aligned}\sqrt{y} &= \sqrt{2^5 \times 3^4 \times 5^3} \\ &= \sqrt{2^4 \times 3^4 \times 5^2} \times \sqrt{2 \times 5} \\ &= 2^2 \times 3^2 \times 5 \times \sqrt{10} \\ &= 4 \times 9 \times 5 \times \sqrt{10} \\ &= 180\sqrt{10}\end{aligned}$$

[2 marks]

Answer 3

(i) $\frac{16}{49}$ **(ii)** $\frac{4}{9}$ **(iii)** 1 **(iv)** $\frac{8}{3}$ **(v)** $\frac{5}{4}$

[5 marks]

Answer 4

(i) $-\frac{3}{2}$ **(ii)** $\frac{27\sqrt{2}}{32}$ **(iii)** $\frac{5\sqrt{2}}{4}$ **(iv)** $\frac{14 - 2\sqrt{5}}{11}$

[4 marks]

Answer 5

$$\begin{aligned}8\sqrt{2} &= 2^a \\ 2^3 \times 2^{\frac{1}{2}} &= 2^a \\ 2^{3.5} &= 2^a \Rightarrow a = 3.5\end{aligned}$$

[2 marks]

Answer 6

$$\begin{aligned}\sqrt{3 + 2\sqrt{2}} &= \sqrt{2 + 2\sqrt{2}\sqrt{1} + 1} \\ &= \sqrt{(\sqrt{2})^2 + 2\sqrt{2}\sqrt{1} + (\sqrt{1})^2}\end{aligned}$$

Recall that : $a^2 + 2ab + b^2 = (a + b)^2$

$$\begin{aligned}&= \sqrt{(\sqrt{2} + \sqrt{1})^2} \\ &= \sqrt{2} + \sqrt{1}\end{aligned}$$

$\therefore$ $a = 2$ and $b = 1$ as told $a > b$

[3 marks]

Answer 7

$$\begin{aligned}
\left(7-\sqrt{c}\right)\left(4+2\sqrt{c}\right) &= 6+k\sqrt{c} \\
28+14\sqrt{c}-4\sqrt{c}-2c &= 6+k\sqrt{c} \\
\left(28-2c\right)+10\sqrt{c} &= 6+k\sqrt{c} \\
\therefore\ k = 10 \text{ and } 28-2c &= 6 \\
2c &= 22 \\
c &= 11
\end{aligned}$$

[3 marks]

Answer 8

$$\begin{aligned}
\frac{1}{\sqrt[3]{9^4}} &= 3^n \\
\left(9^4\right)^{-\frac{1}{3}} &= 3^n \\
\left(3^8\right)^{-\frac{1}{3}} &= 3^n \\
3^{-\frac{8}{3}} &= 3^n \\
n &= -\frac{8}{3}
\end{aligned}$$

[2 marks]

Answer 9

$$\begin{aligned}
\left(a+\sqrt{5}\right)\left(3+2\sqrt{5}\right) &= 31+b\sqrt{5} \\
3a+3\sqrt{5}+2a\sqrt{5}+10 &= 31+b\sqrt{5} \\
3a+10+\left(3+2a\right)\sqrt{5} &= 31+b\sqrt{5} \\
\therefore\ 3a+10 = 31 \quad &\text{and} \quad 3+2a = b \\
3a &= 21 \\
a &= 7 \quad \text{and so } b = 17
\end{aligned}$$

[3 marks]

Answer 10

Method 1

$$\begin{aligned}
x &= \sqrt{6+2\sqrt{5}}-\sqrt{6-2\sqrt{5}} \\
&= \sqrt{5+2\sqrt{5}\sqrt{1}+1}-\sqrt{5-2\sqrt{5}\sqrt{1}+1} \\
&= \sqrt{\left(\sqrt{5}\right)^2+2\sqrt{5}\sqrt{1}+\left(\sqrt{1}\right)^2}-\sqrt{\left(\sqrt{5}\right)^2-2\sqrt{5}\sqrt{1}+\left(\sqrt{1}\right)^2} \\
&= \sqrt{\left(\sqrt{5}+\sqrt{1}\right)^2}-\sqrt{\left(\sqrt{5}-\sqrt{1}\right)^2} \\
&= \sqrt{5}+\sqrt{1}-\sqrt{5}+\sqrt{1}
\end{aligned}$$

$\therefore\ x = 2 \quad \square$

Method 2

$$x = \sqrt{6 + 2\sqrt{5}} - \sqrt{6 - 2\sqrt{5}}$$

$$x^2 = 6 + 2\sqrt{5} - 2\sqrt{6 + 2\sqrt{5}} \times \sqrt{6 - 2\sqrt{5}} + 6 - 2\sqrt{5}$$

$$= 12 - 2\sqrt{(6 + 2\sqrt{5})(6 - 2\sqrt{5})}$$

$$= 12 - 2\sqrt{36 + 12\sqrt{5} - 12\sqrt{5} - 20}$$

$$= 12 - 2\sqrt{36 - 20}$$

$$= 12 - 2\sqrt{16}$$

$$= 4$$

$$\therefore x = 2 \quad \square$$

[4 marks]

Answer 11

$$p^m = \frac{1}{p \times \sqrt[3]{p^2}}$$

$$= \frac{1}{p} \times \frac{1}{\sqrt[3]{p^2}}$$

$$= p^{-1} \times p^{-\frac{2}{3}}$$

$$= p^{-\frac{5}{3}}$$

$$\therefore m = -\frac{5}{3}$$

[2 marks]

Answer 12

(a) $gh = a^4 \times b^2 \times c^5$

[2 marks]

(b) $\frac{g}{h} = a^2 \times b^0 \times c^{-1}$

$x = 2, \quad y = 0, \quad z = -1$

[3 marks]

Answer 13

$$\sqrt{8 - 4\sqrt{3}} = \sqrt{8 - 2\sqrt{12}}$$

$$= \sqrt{6 - 2\sqrt{12} + 2}$$

$$= \sqrt{\left(\sqrt{6}\right)^2 - 2\sqrt{6}\sqrt{2} + \left(\sqrt{2}\right)^2}$$

Recall that : $a^2 - 2ab + b^2 = (a - b)^2$

$$= \sqrt{\left(\sqrt{6} - \sqrt{2}\right)^2}$$

$$= \sqrt{6} - \sqrt{2}$$

$\therefore\ a = 6$ and $b = 2$ as told $a > b$

[3 marks]

Answer 14

If you were going to differentiate you'd do this,

$$\frac{7\sqrt{p} - p^2}{\sqrt{p^3}} = \frac{7p^{\frac{1}{2}}}{p^{\frac{3}{2}}} - \frac{p^2}{p^{\frac{3}{2}}}$$

$$= 7p^{-1} - p^{\frac{1}{2}}$$

$$= \frac{7}{p} - \sqrt{p}$$

Which is by far the most elegant answer,
Other equivalent answers are likely, such as,

$$\frac{7 - p\sqrt{p}}{p}$$

[3 marks]

Answer 15

$$2^x \times 4^y = \frac{1}{2\sqrt{2}}$$

$$2^x \times \left(2^2\right)^y = 2^{-\frac{3}{2}}$$

$$x + 2y = -\frac{3}{2}$$

$$2y = -\frac{3}{2} - x$$

$$y = -\frac{3}{4} - \frac{x}{2}$$

[3 marks]

Answer 16

$$2^7 \times 3 \times 5^4 = 2^6 \times 5^4 \times 2 \times 3$$

$$\sqrt{2^7 \times 3 \times 5^4} = \sqrt{2^6 \times 5^4} \times \sqrt{2 \times 3}$$

$$= 2^3 \times 5^2 \times \sqrt{6}$$

$$= 8 \times 25 \times \sqrt{6}$$

$$= 200\sqrt{6}$$

[2 marks]

Answer 17

$$\frac{1}{1+\sqrt{2}+\sqrt{3}} = \frac{1}{(1+\sqrt{2})+\sqrt{3}}$$

$$= \frac{1}{(1+\sqrt{2})+\sqrt{3}} \times \frac{(1+\sqrt{2})-\sqrt{3}}{(1+\sqrt{2})-\sqrt{3}}$$

$$= \frac{(1+\sqrt{2})-\sqrt{3}}{(1+\sqrt{2})^2-3}$$

$$= \frac{1+\sqrt{2}-\sqrt{3}}{1+2\sqrt{2}+2-3}$$

$$= \frac{1+\sqrt{2}-\sqrt{3}}{2\sqrt{2}}$$

$$= \frac{1+\sqrt{2}-\sqrt{3}}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{\sqrt{2}+2-\sqrt{6}}{4}$$

[4 marks]

Lesson 4

A-Level Pure Mathematics : Year 1
Algebra of Surds and Indices II

4.1 TEST on Surds & Indices

There are 75 marks available
Solutions based entirely on graphical or numerical methods are not acceptable
Target Time : 80 minutes

Question 1
A-level C1 Examination Question, May 2013, Q1
Simplify

$$\frac{7 + \sqrt{5}}{\sqrt{5} - 1}$$

giving your answer in the form $a + b\sqrt{5}$, where a and b are integers.
Show full working.

[4 marks]

Question 2
A-Level C1 Examination Question, January 2008, Q2
(a) Write down the value of

$$16^{\frac{1}{4}}$$

[1 mark]

(b) Simplify

$$\left(16x^{12}\right)^{\frac{3}{4}}$$

[2 marks]

Question 3

Without using a calculator, simplify each of the following.
Answers should be written with denominators that are rational.

(i)

$$\left(\frac{18\,x^5\,y^2}{128\,x} \right)^{-\frac{1}{2}}$$

[3 marks]

(ii)

$$\left(\frac{8\,z^2}{12\sqrt{2}} \right)^{3}$$

[3 marks]

(iii)

$$\left(\frac{49\,w^6}{27} \right)^{\frac{1}{2}}$$

[2 marks]

(iv)

$$\left(\frac{1\,+\,\sqrt{2}}{\sqrt{3}} \right)^{-2}$$

[4 marks]

Question 4

Let a, b, c and d be positive integers.

Given that,

$$\left(a + b\sqrt{c}\right)^2 = 6 + d\sqrt{c}$$

find the two possible sets of values of a, b, c, and d

[3 marks]

Question 5

Find the exact value of

$$\left(5\sqrt{2} + 6\sqrt{3} - 2\sqrt{6}\right)\left(4\sqrt{2} - 2\sqrt{3} + 3\sqrt{6}\right)$$

[5 marks]

Question 6

Given that

$$64\sqrt{2} = 2^{a}$$

find the value of a.

[2 marks]

Question 7

Solve the equation,

$$12 - x\sqrt{24} = \frac{6x}{\sqrt{2}}$$

Give an exact answer in as simple a form as possible.

[5 marks]

Question 8

Consider the function,

$$f(x) = \frac{x}{\sqrt{x} - x^2} \qquad x \in \mathbb{R},\ x > 1$$

(i) Carefully explain the restriction upon the domain of the function.

[3 marks]

(ii) Given, furthermore, that x is a square free number, rationalise the denominator of the function.
Simplify your answer.

[3 marks]

Question 9

Given that,

$$\sqrt{18 + 2\sqrt{77}} = \sqrt{a} + \sqrt{b} \qquad \text{with } a > b$$

find the value of a and the value of b, both of which are integers.

[3 marks]

Question 10

Without using a calculator, and making your reasoning clear, find the square root of

$$2^7 \times 3^3 \times 5^2$$

Writing your answer in the form $a\sqrt{b}$ where a and b are integers and b is $\square$ free.

[2 marks]

Question 11

A-Level C1 Examination Question, May 2016, Q3 (edited)

(a) Showing the reasons why, simplify,

$$\sqrt{50} - \sqrt{18}$$

giving your answer in the form $a\sqrt{2}$ where a is an integer

[2 marks]

(b) Hence, or otherwise, simplify

$$\frac{12\sqrt{3}}{\sqrt{50} - \sqrt{18}}$$

giving your answer in the form $b\sqrt{c}$, where b and c are integers and $b \neq 1$. Show full working.

[3 marks]

Question 12

GCSE Examination Question, January 2016, Paper 4H, Q25

Given that,

$$y = 16 \times 10^{8k}$$

where k is an integer, find an expression, in terms of k for

$$y^{\frac{5}{4}}$$

Give your answer in standard form.

[3 marks]

Question 13

Given that,

$$\left(4 + \sqrt{p}\right)\left(3\sqrt{p} - 1\right) = 17 + k\sqrt{p}$$

where p and k are prime numbers, find the value of p and the value of k

[3 marks]

Question 14

Given

$$\frac{9^x}{27^y} = \frac{1}{3\sqrt{3}}$$

express y as a function of x.

[4 marks]

Question 15

P1, January 2002, Q1

Given that

$$2^x = \frac{1}{\sqrt{2}} \quad \text{and} \quad 2^y = 4\sqrt{2}$$

(a) find the exact value of x and the exact value of y

[3 marks]

(b) calculate the exact value of

$$2^{y-x}$$

[2 marks]

Question 16

Carefully showing your working, rationalise the denominator of,

$$\frac{12\sqrt{3}}{\sqrt{2}+\sqrt{3}+\sqrt{5}}$$

[5 marks]

Question 17

Given that

$$s = \sqrt{6 + \sqrt{6 + \sqrt{6 + \sqrt{6 + \sqrt{6 + \sqrt{6 + \ldots}}}}}}$$

determine the value of s.

Provide detailed reasoning to support your answer.

[5 marks]

4.2 Answers

A-Level Pure Mathematics : Year 1
Algebra of Surds and Indices II

Answer 1

$$\frac{7+\sqrt{5}}{\sqrt{5}-1} \times \frac{\sqrt{5}+1}{\sqrt{5}+1} = \frac{7\sqrt{5}+5+7+\sqrt{5}}{5-\sqrt{5}+\sqrt{5}-1}$$

$$= \frac{12+8\sqrt{5}}{4}$$

$$= \frac{4\,(3+2\sqrt{5}\,)}{4}$$

$$= 3+2\sqrt{5}$$

[4 marks]

Answer 2

(a) 2 **(b)** $8x^9$

[1, 2 marks]

Answer 3

(i)

$$\left(\frac{18x^5y^2}{128x}\right)^{-\frac{1}{2}} = \left(\frac{9x^4y^2}{64}\right)^{-\frac{1}{2}}$$

$$= \sqrt{\left(\frac{64}{9x^4y^2}\right)}$$

$$= \frac{8}{3x^2y}$$

[3 marks]

(ii)

$$\left(\frac{8z^2}{12\sqrt{2}}\right)^3 = \left(\frac{2z^2}{3\sqrt{2}}\right)^3$$

$$= \frac{8z^6}{27 \times 2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{2\sqrt{2}\,z^6}{27}$$

[3 marks]

(iii)

$$\left(\frac{49w^6}{27}\right)^{\frac{1}{2}} = \frac{7w^3}{3\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$

$$= \frac{7\sqrt{3}\,w^3}{9}$$

[2 marks]

(iv)

$$\left(\frac{1+\sqrt{2}}{\sqrt{3}}\right)^{-2} = \left(\frac{\sqrt{3}}{1+\sqrt{2}}\right)^{2}$$

$$= \frac{3}{1+2\sqrt{2}+2}$$

$$= \frac{3}{3+2\sqrt{2}} \times \frac{3-2\sqrt{2}}{3-2\sqrt{2}}$$

$$= \frac{3(3-2\sqrt{2})}{9-8}$$

$$= 9-6\sqrt{2}$$

[4 marks]

Answer 4

$$\left(a+b\sqrt{c}\right)^2 = 6 + d\sqrt{c}$$

$$a^2 + 2ab\sqrt{c} + b^2 c = 6 + d\sqrt{c}$$

$$\therefore\ a^2 + b^2 c = 6 \quad \text{and} \quad 2ab = d$$

As a, b, c and d are positive integers

If $a = 2 \Rightarrow b = 1 \Rightarrow c = 2 \Rightarrow d = 4$

If $a = 1 \Rightarrow b = 1 \Rightarrow c = 5 \Rightarrow d = 2$

[3 marks]

Answer 5

It will help to think of $\sqrt{6}$ as $\sqrt{2} \times \sqrt{3}$ if doing this by mental arithmetic.

	$5\sqrt{2}$	$6\sqrt{3}$	$-2\sqrt{6}$
$4\sqrt{2}$	40	$24\sqrt{6}$	$-16\sqrt{3}$
$-2\sqrt{3}$	$-10\sqrt{6}$	-36	$12\sqrt{2}$
$3\sqrt{6}$	$30\sqrt{3}$	$54\sqrt{2}$	-36

$$\left(5\sqrt{2} + 6\sqrt{3} - 2\sqrt{6}\right)\left(4\sqrt{2} - 2\sqrt{3} + 3\sqrt{6}\right)$$

$$= -32 + 66\sqrt{2} + 14\sqrt{3} + 14\sqrt{6}$$

[5 marks]

Answer 6

$$64\sqrt{2} = 2^a$$

$$2^6 \times 2^{\frac{1}{2}} = 2^a$$

$$2^{6.5} = 2^a$$

$$a = 6.5$$

[2 marks]

Answer 7

$$12 - x\sqrt{24} = \frac{6x}{\sqrt{2}}$$

Multiply through by $\sqrt{2}$

$$12\sqrt{2} - x\sqrt{48} = 6x$$

$$6x + x\sqrt{16 \times 3} = 12\sqrt{2}$$

$$6x + 4x\sqrt{3} = 12\sqrt{2}$$

Divide through by 2

$$3x + 2x\sqrt{3} = 6\sqrt{2}$$

$$x\left(3 + 2\sqrt{3}\right) = 6\sqrt{2}$$

$$x = \frac{6\sqrt{2}}{3 + 2\sqrt{3}} \times \frac{3 - 2\sqrt{3}}{3 - 2\sqrt{3}}$$

$$= \frac{18\sqrt{2} - 12\sqrt{6}}{9 - 12}$$

$$= \frac{18\sqrt{2} - 12\sqrt{6}}{(-3)}$$

$$= 4\sqrt{6} - 6\sqrt{2}$$

or $2\sqrt{2}\left(2\sqrt{3} - 3\right)$

[5 marks]

Answer 8

(i) The $x \in R$ suggests that the initial intention was to allow all real numbers into the function, but then restrictions have been imposed. Firstly the taking of negative square roots has been avoided, which are not real. Secondly, the values of $x = 0$ and $x = 1$ have been excluded as these would cause a division by zero which is not defined.

[3 marks]

(ii)

$$\frac{x}{\sqrt{x} - x^2} = \frac{x}{\sqrt{x} - x^2} \times \frac{\sqrt{x} + x^2}{\sqrt{x} + x^2}$$ Key step !

$$= \frac{x\left(\sqrt{x} + x^2\right)}{x - x^4}$$

$$= \frac{x\left(\sqrt{x} + x^2\right)}{x\left(1 - x^3\right)}$$

$$= \frac{\sqrt{x} + x^2}{1 - x^3}$$

[3 marks]

Answer 9

$$\sqrt{18 + 2\sqrt{77}} = \sqrt{11 + 2\sqrt{11}\sqrt{7} + 7}$$

$$= \sqrt{\left(\sqrt{11}\right)^2 + 2\sqrt{11}\sqrt{7} + \left(\sqrt{7}\right)^2}$$

Recall that : $a^2 + 2ab + b^2 = (a + b)^2$

$$= \sqrt{\left(\sqrt{11} + \sqrt{7}\right)^2}$$

$$= \sqrt{11} + \sqrt{7}$$

$\therefore \; a = 11$ and $b = 7$ as told $a > b$

[3 marks]

Answer 10

Must make non-calculator method clear...

$$\sqrt{2^7 \times 3^3 \times 5^2} = \sqrt{2^6 \times 2 \times 3^2 \times 3 \times 5^2}$$

$$= 2^3 \times \sqrt{2} \times 3 \times \sqrt{3} \times 5$$

$$= 120\sqrt{6}$$

[2 marks]

Answer 11

(a)

$$\sqrt{50} - \sqrt{18} = \sqrt{25 \times 2} - \sqrt{9 \times 2}$$

$$= 5\sqrt{2} - 3\sqrt{2}$$

$$= 2\sqrt{2}$$

$\therefore \; a = 2$

[2 marks]

(b)

$$\frac{12\sqrt{3}}{\sqrt{50} - \sqrt{18}} = \frac{12\sqrt{3}}{2\sqrt{2}}$$

$$= \frac{12\sqrt{3}}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{12\sqrt{6}}{4}$$

$$= 3\sqrt{6}$$

[3 marks]

Answer 12

$3.2 \times 10^{10k+1}$

[3 marks]

Answer 13

$$\left(4+\sqrt{p}\right)\left(3\sqrt{p}-1\right) = 17 + k\sqrt{p}$$
$$12\sqrt{p} - 4 + 3p - \sqrt{p} = 17 + k\sqrt{p}$$
$$3p - 4 + 11\sqrt{p} = 17 + k\sqrt{p}$$
$$\therefore\ k = 11 \text{ and } 3p - 4 = 17$$
$$3p = 21$$
$$p = 7$$

[3 marks]

Answer 14

$$\frac{9^x}{27^y} = \frac{1}{3\sqrt{3}}$$
$$\frac{\left(3^2\right)^x}{\left(3^3\right)^y} = 3^{-1.5}$$
$$3^{2x-3y} = 3^{-1.5}$$
$$\therefore 2x - 3y = -1.5$$
$$3y = 2x + \frac{3}{2}$$
$$y = \frac{2}{3}x + \frac{1}{2}$$

[4 marks]

Answer 15

(a)

$$2^x = \frac{1}{\sqrt{2}} \quad \text{and} \quad 2^y = 4\sqrt{2}$$
$$x = -\frac{1}{2} \quad \text{and} \quad y = \frac{5}{2}$$

[3 marks]

(b)

$$2^{y-x} = 2^3 = 8$$

[2 marks]

Answer 16

$$\frac{12\sqrt{3}}{\sqrt{2}+\sqrt{3}+\sqrt{5}} = \frac{12\sqrt{3}}{(\sqrt{2}+\sqrt{3})+\sqrt{5}} \times \frac{(\sqrt{2}+\sqrt{3})-\sqrt{5}}{(\sqrt{2}+\sqrt{3})-\sqrt{5}}$$

$$= \frac{12\sqrt{3}\,(\sqrt{2}+\sqrt{3}-\sqrt{5})}{(\sqrt{2}+\sqrt{3})^2-5}$$

$$= \frac{12\sqrt{3}\,(\sqrt{2}+\sqrt{3}-\sqrt{5})}{2+2\sqrt{2}\sqrt{3}+3-5}$$

$$= \frac{12\sqrt{3}\,(\sqrt{2}+\sqrt{3}-\sqrt{5})}{2\sqrt{2}\sqrt{3}}$$

$$= \frac{6\,(\sqrt{2}+\sqrt{3}-\sqrt{5})}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{6\,(\sqrt{2}+\sqrt{3}-\sqrt{5})\sqrt{2}}{2}$$

$$= 3\sqrt{2}\,(\sqrt{2}+\sqrt{3}-\sqrt{5})$$

$$= 6+3\sqrt{6}-3\sqrt{10}$$

[5 marks]

Answer 17

$$s = \sqrt{6+\sqrt{6+\sqrt{6+\sqrt{6+\sqrt{6+\sqrt{6+\ldots}}}}}}$$

$$s^2 = 6+\sqrt{6+\sqrt{6+\sqrt{6+\sqrt{6+\sqrt{6+\ldots}}}}}$$

$$s^2 = 6+s$$

$$s^2-s-6 = 0$$

$$(s-3)(s+2) = 0$$

$$\therefore\ s = 3 \qquad \text{(Reject } s = -2 \text{ as not possible)}$$

[5 marks]