10.3 Answers

Answer 1

$$e^y + y = x^2$$

Differentiate with respect to x

$$e^y \frac{dy}{dx} + \frac{dy}{dx} = 2x$$

$$\frac{dy}{dx}\left(e^y + 1\right) = 2x$$

$$\frac{dy}{dx} = \frac{2x}{e^y + 1}$$

$$\text{At } (1,\ 0),\ \frac{dy}{dx} = 1$$

$\therefore$ The normal will have gradient -1

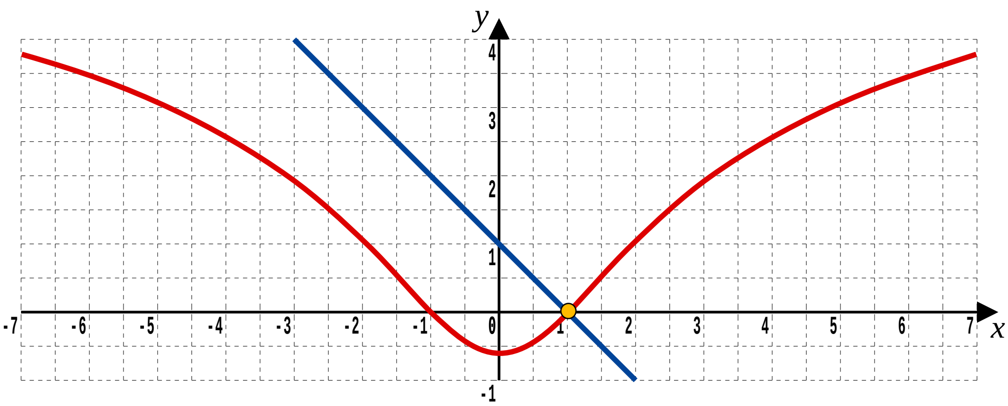
$$y = mx + c$$

$$0 = -1 \times 1 + c$$

$$c = 1$$

Normal at $(1,\ 0)$ has equation $y = -x + 1$

[6 marks]



[2 marks]

Answer 2

(a)
$$y\,e^{-2x} = 2x + y^2$$

Differentiate with respect to x

$$y(-2)\,e^{-2x} + \frac{dy}{dx}\,e^{-2x} = 2 + 2y\,\frac{dy}{dx}$$

$$\frac{dy}{dx}\left(e^{-2x} - 2y\right) = 2 + 2y\,e^{-2x}$$

$$\frac{dy}{dx} = \frac{2 + 2y\,e^{-2x}}{e^{-2x} - 2y}$$

[5 marks]

(b)
At $(0,\ 1)$, $\dfrac{dy}{dx} = \dfrac{2 + 2}{1 - 2}$

$$= -4$$

$\therefore$ The normal will have gradient $\dfrac{1}{4}$

$$y = mx + c$$

$$1 = \frac{1}{4} \times 0 + c$$

$$c = 1$$

Normal at $(0,\ 1)$ has equation $y = \dfrac{1}{4}x + 1$

That is, $x - 4y + 4 = 0$

[4 marks]

Answer 3

$$ln\,y = 2x\,ln\,x, \qquad x > 0, \qquad y > 0$$

Differentiate with respect to x

$$\frac{1}{y} \times \frac{dy}{dx} = 2x \times \frac{1}{x} + 2\,ln\,x$$

$$\frac{dy}{dx} = 2y + 2y\,ln\,x$$

$$= 2y\,(1 + ln\,x)$$

When $x = 2$, $ln\,y = 2 \times 2 \times ln\,2$

$$= 4\,ln\,2$$

$$= ln\,2^4$$

$$= ln\,16$$

$\therefore\ y = 16 \qquad\qquad \therefore\ \dfrac{dy}{dx} = 32\,(1 + ln\,2)$

[7 marks]

Answer 4

(a)

$$\cos 2x + \cos 3y = 1, \qquad -\frac{\pi}{4} < x < \frac{\pi}{4}, \qquad 0 \leqslant y \leqslant \frac{\pi}{6}$$

Differentiate with respect to x

$$(-2\sin 2x) + (-3\sin 3y)\frac{dy}{dx} = 0$$

$$(-3\sin 3y)\frac{dy}{dx} = 2\sin 2x$$

$$\frac{dy}{dx} = -\frac{2\sin 2x}{3\sin 3y}$$

[3 marks]

(b)

When $x = \frac{\pi}{6}$, $\cos\left(\frac{\pi}{3}\right) + \cos 3y = 1$

$$\cos 3y = 1 - \frac{1}{2}$$

$$= \frac{1}{2}$$

$$3y = \frac{\pi}{3}, \ \frac{5\pi}{3}, \ \ldots$$

$$y = \frac{\pi}{9}$$ only answer within constraints

[3 marks]

(c)

When $x = \frac{\pi}{6}$, $\frac{dy}{dx} = -\frac{2\sin\left(\frac{\pi}{3}\right)}{3\sin\left(\frac{\pi}{3}\right)}$

$$= -\frac{2}{3}$$

$$y = mx + c$$

$$\frac{\pi}{9} = -\frac{2}{3} \times \frac{\pi}{6} + c$$

$$c = \frac{2\pi}{9}$$

$$y = -\frac{2}{3}x + \frac{2\pi}{9}$$

That is, $6x + 9y - 2\pi = 0$

[3 marks]

Answer 5

(a)

$$\sin x + \cos y = 0.5$$

$$\cos x - \frac{dy}{dx}\sin y = 0$$

$$\frac{dy}{dx} = \frac{\cos x}{\sin y}$$

[2 marks]

(b)

$$\frac{dy}{dx} = 0 \text{ when } \cos x = 0$$

$$x = \pm\frac{\pi}{2}$$

$$\text{When } x = \frac{\pi}{2},\ \cos y = 0.5 - \sin\left(\frac{\pi}{2}\right)$$

$$= 0.5 - 1$$

$$y = \arccos(-0.5)$$

$$= \pm\frac{2\pi}{3}$$

$$\therefore \left(\frac{\pi}{2}, -\frac{2\pi}{3}\right), \left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$$

$$\text{When } x = -\frac{\pi}{2},\ \cos y = 0.5 - \sin\left(-\frac{\pi}{2}\right)$$

$$= 0.5 + 1$$

$$= 1.5 \quad \text{which has no solutions}$$

[5 marks]