3.4 Answers

A-Level Pure Mathematics : Year 2
Differentiation IV

3.4.1 Solution to the Example (Teaching Video)

(i) The graph is a circular path that starts at 12 O'clock when $\theta = 0$. It heads off clockwise, is at 3 O'clock when $\theta = \frac{\pi}{2}$ and stops at 6 O'clock when $\theta = \pi$.

[2 marks]

(ii)
$$x = \sin\theta \Rightarrow \frac{dx}{d\theta} = \cos\theta \Rightarrow \frac{d\theta}{dx} = \frac{1}{\cos\theta}$$
$$y = \cos\theta \Rightarrow \frac{dy}{d\theta} = -\sin\theta$$
$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$
$$= -\sin\theta \times \frac{1}{\cos\theta}$$
$$= -\tan\theta$$

[3 marks]

(iii) 1, as is also obvious from the part (i) sketch

[1 mark]

3.4.2 Solutions to 3.3 Exercise

Answer 1

$$x = 5t^4 \Rightarrow \frac{dx}{dt} = 20t^3 \Rightarrow \frac{dt}{dx} = \frac{1}{20t^3}$$
$$y = 3t^5 \Rightarrow \frac{dy}{dt} = 15t^4$$
$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$
$$= 15t^4 \times \frac{1}{20t^3}$$
$$= \frac{3}{4}t \qquad \square$$

[3 marks]

Answer 2

$$x = t^2 + t \Rightarrow \frac{dx}{dt} = 2t + 1 \Rightarrow \frac{dt}{dx} = \frac{1}{2t + 1}$$

$$y = 2t - t^2 \Rightarrow \frac{dy}{dt} = 2 - 2t$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$\frac{dy}{dx} = \frac{2 - 2t}{2t + 1}$$

[3 marks]

Answer 3

$$x = (4t + 2)^3 \Rightarrow \frac{dx}{dt} = 3(4t + 2)^2 \times 4 \Rightarrow \frac{dt}{dx} = \frac{1}{3 \times 4(4t + 2)^2}$$

$$y = (3t + 4)^2 \Rightarrow \frac{dy}{dt} = 2(3t + 4) \times 3$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$\frac{dy}{dx} = \frac{3t + 4}{2(4t + 2)^2}$$

[3 marks]

Answer 4

$$x = 2\cot\theta \Rightarrow \frac{dx}{d\theta} = -2\csc^2\theta \Rightarrow \frac{d\theta}{dx} = -\frac{\sin^2\theta}{2}$$

$$y = 3\tan\theta \Rightarrow \frac{dy}{d\theta} = 3\sec^2\theta = \frac{3}{\cos^2\theta}$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$

$$= \frac{3}{\cos^2\theta} \times \left(-\frac{\sin^2\theta}{2}\right)$$

$$= -\frac{3}{2}\tan^2\theta$$

[3 marks]

Answer 5

$$x = 3\sin\theta \Rightarrow \frac{dx}{d\theta} = 3\cos\theta \Rightarrow \frac{d\theta}{dx} = \frac{1}{3\cos\theta}$$

$$y = 5\cos\theta \Rightarrow \frac{dy}{d\theta} = -5\sin\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$

$$= -5\sin\theta \times \frac{1}{3\cos\theta}$$

$$= -\frac{5}{3}\tan\theta$$

[3 marks]

Answer 6

(i) $x = \tan\theta \Rightarrow \frac{dx}{d\theta} = \sec^2\theta \Rightarrow \frac{d\theta}{dx} = \cos^2\theta$

$$y = \sec\theta \Rightarrow \frac{dy}{d\theta} = \sec\theta\tan\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$

$$= \sec\theta\tan\theta\cos^2\theta$$

$$= \frac{1}{\cos\theta} \times \frac{\sin\theta}{\cos\theta} \times \frac{\cos^2\theta}{1}$$

$$= \sin\theta$$

[3 marks]

(ii) $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

[1 mark]

Answer 7

$$x = -\cos\theta \Rightarrow \frac{dx}{d\theta} = \sin\theta \Rightarrow \frac{d\theta}{dx} = \frac{1}{\sin\theta}$$

$$y = \theta\sin\theta \Rightarrow \frac{dy}{d\theta} = \theta\cos\theta + \sin\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$

$$= \frac{\theta\cos\theta + \sin\theta}{\sin\theta}$$

$$= \theta\cot\theta + 1 \qquad \square$$

[4 marks]

Answer 8

(i) $x = \cot\theta \Rightarrow \frac{dx}{d\theta} = -\csc^2\theta \Rightarrow \frac{d\theta}{dx} = -\sin^2\theta$

$y = \csc\theta \Rightarrow \frac{dy}{d\theta} = -\csc\theta\cot\theta$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$

$$= -\csc\theta\cot\theta \times \left(-\sin^2\theta\right)$$

$$= \frac{1}{\sin\theta} \times \frac{\cos\theta}{\sin\theta} \times \sin^2\theta$$

$$= \cos\theta$$

[3 marks]

(ii) $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$

[1 mark]

Answer 9

$x = -\sin\theta \Rightarrow \frac{dx}{d\theta} = -\cos\theta \Rightarrow \frac{d\theta}{dx} = -\frac{1}{\cos\theta}$

$y = \theta\cos\theta \Rightarrow \frac{dy}{d\theta} = \theta(-\sin\theta) + \cos\theta$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$

$$= \frac{-\theta\sin\theta + \cos\theta}{(-\cos\theta)}$$

$$= \theta\tan\theta - 1$$

[4 marks]

Answer 10

(i) (20, 32)

[1 mark]

(ii) $x = 5t^2 \Rightarrow \frac{dx}{dt} = 10t \Rightarrow \frac{dt}{dx} = \frac{1}{10t}$

$y = 4t^3 \Rightarrow \frac{dy}{dt} = 12t^2$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= 12t^2 \times \frac{1}{10t}$$

$$= \frac{6t}{5}$$

[3 marks]

(iii) When $t = 2$, $\frac{dy}{dx} = \frac{12}{5}$

[2 marks]

(iv)

$$y = mx + c$$

$$32 = \frac{12}{5} \times 20 + c$$

$$32 = 48 + c$$

$$c = -16$$

$$\therefore \; y = \frac{12}{5}x - 16$$

Multiply through by 5

$$5y = 12x - 80$$

$$12x - 5y - 80 = 0$$

[3 marks]

Answer 11

(i) $\left(\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2} \right)$

[1 mark]

(ii) $x = \cos t \Rightarrow \frac{dx}{dt} = -\sin t \Rightarrow \frac{dt}{dx} = -\frac{1}{\sin t}$

$y = \sin 2t \Rightarrow \frac{dy}{dt} = 2\cos 2t$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= -\frac{2\cos 2t}{\sin t}$$

[4 marks]

(iii) When $t = \frac{\pi}{6}$, $\frac{dy}{dx} = -\frac{2\cos 2\left(\frac{\pi}{6}\right)}{\sin\left(\frac{\pi}{6}\right)}$

$$= -\frac{2 \times \left(\frac{1}{2}\right)}{\left(\frac{1}{2}\right)}$$

$$= -2$$

[2 marks]

(iv) $y = mx + c$

$$\frac{\sqrt{3}}{2} = -2 \times \left(\frac{\sqrt{3}}{2}\right) + c$$

$$c = 3\left(\frac{\sqrt{3}}{2}\right)$$

$$\therefore \ y = -2x + 3\left(\frac{\sqrt{3}}{2}\right)$$

[3 marks]