4.4 Answers

4.4.1 Solution to 4.2 Example (Teaching Video)

(i) $x = sin^2\,\theta \;\Rightarrow\; \dfrac{dx}{d\theta} = 2\,sin\,\theta\,cos\,\theta \;\Rightarrow\; \dfrac{d\theta}{dx} = \dfrac{1}{sin\,2\theta}$

$y = cos\,2\theta \;\Rightarrow\; \dfrac{dy}{d\theta} = -\,2\,sin\,2\theta$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$

$$= -\,2\,sin\,2\theta \times \frac{1}{sin\,2\theta}$$

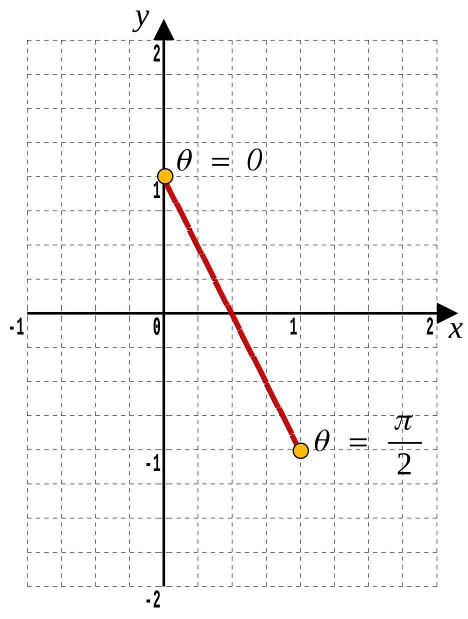
$$= -\,2$$

[3 marks]

(ii) The graph is a straight line with gradient – 2

- As x is the square of a function it can never be negative
- Plus, as x is the *sine* of a function, $0 \leqslant x \leqslant 1$
- As y is the *cosine* of a function, $-\,1 \leqslant y \leqslant 1$
- When $\theta = 0$ the point is $(\,0,\;1\,)$
- When $\theta = \pi$ the point is $(\,1,\,-\,1\,)$

Putting this altogether gives the graph as $y = -\,2\,x + 1, \quad 0 \leqslant x \leqslant 1$



[3 marks]

An alternative “clever” solution...

$$
\begin{aligned}
x &= sin^2\,\theta \\
y &= cos^2\,\theta - sin^2\,\theta \\
&= \left(1 - sin^2\,\theta\right) - sin^2\,\theta \\
&= 1 - 2\,sin^2\,\theta \\
&= 1 - 2x
\end{aligned}
$$

which is a straight line with gradient – 2

4.4.2 Solutions to 4.3 Exercise

Answer 1

(i) $x = 6(1 - \cos\pi)\cos\pi$ $\quad\quad$ $y = 6(1 - \cos\pi)\sin\pi$

$= 6(1 - (-1))(-1)$ $\quad\quad$ $= 6(1 - (-1))\,0$

$= -12$ $\quad\quad$ $= 0$

The point corresponding to $\theta = \pi$ is $A(-12, 0)$

See the graph on the next page

[2 marks]

(ii) $x = -12$

See the graph on the next page

[2 marks]

(iii) $x = 6\left(1 - \cos\left(\frac{\pi}{2}\right)\right)\cos\left(\frac{\pi}{2}\right)$ $\quad\quad$ $y = 6\left(1 - \cos\left(\frac{\pi}{2}\right)\right)\sin\left(\frac{\pi}{2}\right)$

$= 6(1 - 0)\,0$ $\quad\quad$ $= 6(1 - 0)\,1$

$= 0$ $\quad\quad$ $= 6$

The point corresponding to $\theta = \frac{\pi}{2}$ is $B(0, 6)$

See the graph on the next page

[2 marks]

(iv)

$$x = 6(1 - \cos\theta)\cos\theta$$
$$= 6\cos\theta - 6\cos^2\theta$$
$$\frac{dx}{d\theta} = -6\sin\theta + 6 \times 2\cos\theta\sin\theta$$
$$= 6\sin 2\theta - 6\sin\theta$$

$$y = 6(1 - \cos\theta)\sin\theta$$
$$= 6\sin\theta - 3 \times 2\sin\theta\cos\theta$$
$$= 6\sin\theta - 3\sin 2\theta$$
$$\frac{dy}{d\theta} = 6\cos\theta - 6\cos 2\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$
$$= \frac{6\cos\theta - 6\cos 2\theta}{6\sin 2\theta - 6\sin\theta}$$
$$= \frac{\cos\theta - \cos 2\theta}{\sin 2\theta - \sin\theta} \qquad \square$$

[6 marks]

(v) B corresponded to the point $(\ 0,\ \ 6\ )$ when $\theta\ =\ \dfrac{\pi}{2}$

Working out the gradient at this point,

$$\frac{dy}{dx} = \frac{\cos\left(\frac{\pi}{2}\right) - \cos(\pi)}{\sin(\pi) - \sin\left(\frac{\pi}{2}\right)}$$

$$= \frac{0 - (-1)}{0 - 1}$$

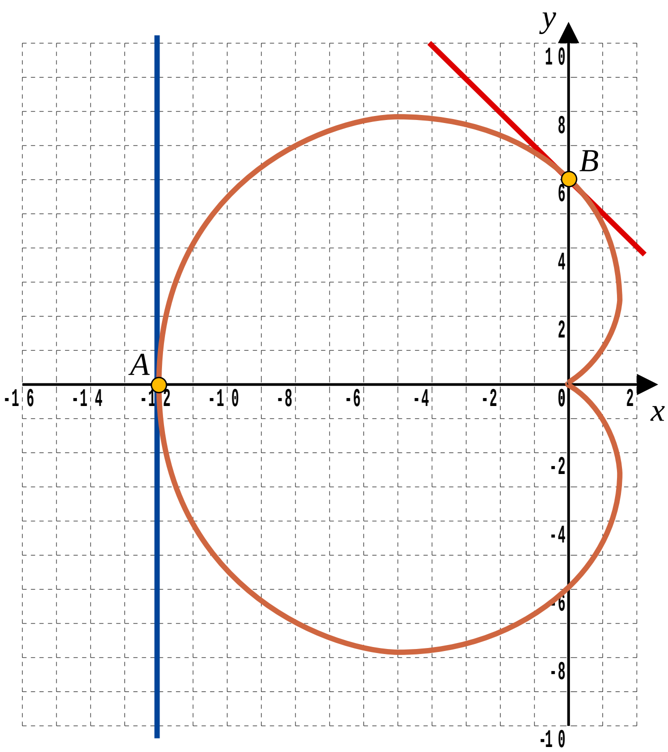
$$= -1$$

Thus the equation of the tangent to the curve at $B(\ 0, 6\ )$ is

$$y = -x + 6$$

See the graph below

[4 marks]



Answer 2

(a) $x = \sqrt{3}\, sin\, 2t \;\Rightarrow\; \dfrac{dx}{dt} = 2\sqrt{3}\, cos\, 2t$

$y = 4\, cos^2 t \;\Rightarrow\; \dfrac{dy}{dt} = -4 \times 2\, cos\, t\, sin\, t$

$= -4\, sin\, 2t$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= -4\, sin\, 2t \times \frac{1}{2\sqrt{3}\, cos\, 2t}$$

$$= -\frac{2}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}\, tan\, 2t$$

$$= -\frac{2}{3}\sqrt{3}\, tan\, 2t$$

That is, $k = -\dfrac{2}{3}$

[5 marks]

(b) When $t = \dfrac{\pi}{3}$ $\quad x = \sqrt{3}\, sin\left(\dfrac{2\pi}{3}\right) \Rightarrow x = \dfrac{3}{2}$

$y = 4\, cos^2\left(\dfrac{\pi}{3}\right) \Rightarrow y = 1$

$\dfrac{dy}{dx} = -\dfrac{2}{3}\sqrt{3}\, tan\left(\dfrac{2\pi}{3}\right) \Rightarrow \dfrac{dy}{dx} = 2$

$y = mx + c$

$1 = 2 \times \dfrac{3}{2} + c$

$c = -2$

$\therefore\; y = 2x - 2$

[4 marks]

(c) Clever Method

$$
\begin{aligned}
y &= 4\cos^2 t \\
&= 2\cos^2 t + 2\cos^2 t \\
&= 2\cos^2 t + 2\sin^2 t + 2\cos^2 t - 2\sin^2 t \\
&= 2 + 2\cos 2t \\
\therefore\ \cos 2t &= \frac{y-2}{2}
\end{aligned}
$$

$$
\begin{aligned}
x &= \sqrt{3}\sin 2t \\
\sin 2t &= \frac{x}{\sqrt{3}}
\end{aligned}
$$

$$
\begin{aligned}
\cos^2 2t + \sin^2 2t &= 1 \\
\frac{(y-2)^2}{4} + \frac{x^2}{3} &= 1 \\
3(y-2)^2 + 4x^2 &= 12 \qquad \text{which is the equation of an ellipse}
\end{aligned}
$$

More Direct Method

$$
\begin{aligned}
x &= \sqrt{3}\sin 2t \\
&= 2\sqrt{3}\sin t\cos t \\
x^2 &= 12\sin^2 t\cos^2 t \\
&= 12\left(1 - \cos^2 t\right)\cos^2 t \\
&= 12\left(1 - \frac{y}{4}\right)\left(\frac{y}{4}\right)
\end{aligned}
$$

[3 marks]

Answer 3

(a) The curve will cross the x-axis when $y = 0$

$$\therefore y = 9\,sin\,2t \qquad 0 \leqslant t \leqslant 2\pi$$

$$9\,sin\,2t = 0$$

$$sin\,2t = 0$$

$$2t = 0,\ \pi,\ 2\pi,\ 3\pi,\ 4\pi,\ \ldots$$

$$t = 0,\ \frac{\pi}{2},\ \pi,\ \frac{3\pi}{2},\ 2\pi,\ \ldots$$

These values of t can now be inserted into the x equation to get points, remembering that all these t values correspond to a $y = 0$

Thus, using $x = 3\,cos\,t$ yields the points,

$$(3,\ 0),\ (0,\ 0),\ (-3,\ 0),\ (0,\ 0),\ (3,\ 0),\ \ldots$$

Clearly, $A(-3,\ 0)$ and $B(3,\ 0)$

[2 marks]

(b) From (a), the curve passes through $(0,\ 0)$ when $t = \frac{\pi}{2},\ \frac{3\pi}{2}$

[2 marks]

(c) $x = 3\,cos\,t \quad \Rightarrow \quad \frac{dx}{dt} = -3\,sin\,t$

$y = 9\,sin\,2t \quad \Rightarrow \quad \frac{dy}{dt} = 18\,cos\,2t$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= 18\,cos\,2t \times \frac{1}{(-3\,sin\,t)}$$

$$= -\frac{6\,cos\,2t}{sin\,t}$$

When $t = \frac{\pi}{6}$, $\frac{dy}{dx} = -\frac{6 \times \frac{1}{2}}{\frac{1}{2}}$

$$= -6$$

[4 marks]

(d) $x = 3\cos t \Rightarrow \cos t = \dfrac{x}{3}$

$y = 9\sin 2t \Rightarrow y = 18\sin t \cot t$

$$y^2 = 18^2 \sin^2 t \cos^2 t$$
$$= 18^2 \left(1 - \cos^2 t \right) \cos^2 t$$
$$= 18^2 \left(1 - \frac{x^2}{9} \right)\left(\frac{x^2}{9} \right)$$
$$= 4x^2 \left(9 - x^2 \right)$$

That is, $a = 4$ and $b = 9$

[4 marks]

Answer 4

(i) $x = \sin^2 \theta$

$$y = \cos^2 2\theta$$
$$= (\cos 2\theta)^2$$
$$= \left(\cos^2 \theta - \sin^2 \theta \right)^2$$
$$= \left(1 - \sin^2 \theta - \sin^2 \theta \right)^2$$
$$= \left(1 - 2\sin^2 \theta \right)^2$$
$$= (1 - 2x)^2$$
$$= 1 - 4x + 4x^2$$

[4 marks]

(ii) $x = \sin^2 \theta \quad \Rightarrow \dfrac{dx}{d\theta} = 2\sin\theta\cos\theta$

$$= \sin 2\theta$$

$y = \cos^2 2\theta \Rightarrow \dfrac{dy}{d\theta} = 2\cos 2\theta \left(-\sin 2\theta \right) \times 2$

$$= -2\sin 4\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$

$$\therefore \frac{dy}{dx} = -\frac{2\sin 4\theta}{\sin 2\theta} \qquad \theta \neq \frac{n\pi}{2} \quad n \in \mathbb{Z} \qquad \square$$

[4 marks]

Answer 5

(a) The curve will cross the y- axis when $x = 0$,

i.e. When $1 + \sqrt{3}\ \tan\theta = 0$

$$\tan\theta = -\frac{1}{\sqrt{3}}$$

$$\theta = -\frac{\pi}{6} \quad : \text{ The only solution } -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$y = 5\sec\theta$$

$$= \frac{5}{\cos\left(\frac{\pi}{6}\right)}$$

$$= \frac{10\sqrt{3}}{3}$$

The coordinates of A are thus $\left(0,\ \frac{10\sqrt{3}}{3}\right)$

[3 marks]

(b) $x = 1 + \sqrt{3}\ \tan\theta \Rightarrow \frac{dx}{d\theta} = \sqrt{3}\ \sec^2\theta$

$y = 5\sec\theta \qquad \Rightarrow \frac{dy}{d\theta} = 5\sec\theta\tan\theta$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$

$$= 5\sec\theta\tan\theta \times \frac{1}{\sqrt{3}\ \sec^2\theta}$$

$$= 5\tan\theta \times \frac{1}{\sqrt{3}\ \sec\theta}$$

$$= 5\frac{\sin\theta}{\cos\theta} \times \frac{\cos\theta}{\sqrt{3}}$$

$$= \frac{5}{\sqrt{3}}\sin\theta$$

$$= \frac{5\sqrt{3}}{3}\sin\theta \qquad \therefore \lambda = \frac{5\sqrt{3}}{3}$$

[4 marks]

(c) The minimum point is where $\frac{dy}{dx} = 0 \Rightarrow \frac{5\sqrt{3}}{3}\sin\theta = 0$

$$\sin\theta = 0$$

$$\theta = 0$$

Thus, B will be the point $(1,\ 5)$

(Note that the graph has different scales on the x and y axis)

[2 marks]

(d) “Many Little Steps” Method

$$y = 5\,sec\,\theta \Rightarrow y = \frac{5}{cos\,\theta}$$

$$x = 1 + \sqrt{3}\,tan\,\theta$$

$$x - 1 = \sqrt{3} \times \frac{sin\,\theta}{cos\,\theta}$$

$$(x - 1)^2 = 3 \times \frac{sin^2\,\theta}{cos^2\,\theta}$$

$$\frac{x^2 - 2x + 1}{3} = \frac{1 - cos^2\,\theta}{cos^2\,\theta}$$

$$\frac{x^2 - 2x + 1}{3} = \frac{1}{cos^2\,\theta} - \frac{cos^2\,\theta}{cos^2\,\theta}$$

$$\frac{x^2 - 2x + 1}{3} = \frac{25}{25} \times \frac{1}{cos^2\,\theta} - 1$$

$$\frac{x^2 - 2x + 1}{3} + 1 = \frac{1}{25} \times \frac{25}{cos^2\,\theta}$$

$$\frac{x^2 - 2x + 1}{3} + \frac{3}{3} = \frac{1}{25} \times y^2$$

$$25\left(\frac{x^2 - 2x + 4}{3}\right) = y^2$$

$$y = \pm\frac{5}{\sqrt{3}}\sqrt{x^2 - 2x + 4}$$

$$= \pm\frac{5\sqrt{3}}{3}\sqrt{x^2 - 2x + 4}$$

That is, $k = \pm\frac{5\sqrt{3}}{3}$

“Trigonometric Formula” Method

$$x = 1 + \sqrt{3}\ tan\,\theta$$

$$\therefore\ tan\,\theta = \frac{x - 1}{\sqrt{3}}$$

$$\begin{aligned}
y &= 5\,sec\,\theta \\
y^2 &= 25\,sec^2\,\theta \\
&= 25\left(1 + tan^2\,\theta \right) \\
&= 25\left(1 + \left(\frac{x - 1}{\sqrt{3}} \right)^2 \right) \\
&= 25\left(\frac{3}{3} + \frac{x^2 - 2x + 1}{3} \right) \\
&= \frac{25}{3}\left(x^2 - 2x + 4 \right) \\
y &= \pm\frac{5}{\sqrt{3}}\sqrt{x^2 - 2x + 4} \\
&= \pm\frac{5\sqrt{3}}{3}\sqrt{x^2 - 2x + 4}
\end{aligned}$$

That is, $k = \pm\dfrac{5\sqrt{3}}{3}$

[3 marks]