## 5.5 Answers to 5.4 Exercise



**Answer 1**

There are several ways of doing this question.
The following is most like the method used in previous questions where it's easy to get one equation for $\cos t$ and another for $\sin t$ these then being combined by use of the trigonometric identity $\cos^2 x + \sin^2 x = 1$

$$x = 10\cos t \quad \Rightarrow \quad \cos t = \frac{x}{10}$$

$$y = 4\sqrt{2}\sin t \quad \Rightarrow \quad \sin t = \frac{y}{4\sqrt{2}}$$

$$\cos^2 t + \sin^2 t = 1$$

$$\left(\frac{x}{10}\right)^2 + \left(\frac{y}{4\sqrt{2}}\right)^2 = 1$$

$$\frac{x^2}{100} + \frac{y^2}{32} = 1$$

Multiply through by $100 \times 32$

$$32x^2 + 100y^2 = 100 \times 32$$

Divide through by 4

$$8x^2 + 25y^2 = 800$$

$$8(66 - y^2) + 25y^2 = 800 \qquad \text{From the } C_2 \text{ equation}$$

$$528 - 8y^2 + 25y^2 = 800$$

$$17y^2 = 272$$

$$y^2 = 16$$

$$y = \pm 4$$

Using the $C_2$ equation as the easiest route to get $x$,

$$x^2 + y^2 = 66$$

$$x^2 = 66 - 16$$

$$x = \pm\sqrt{50}$$

$$= \pm 5\sqrt{2}$$

As $S$ is in the fourth quadrant, $S(5\sqrt{2}, -4)$

**[ 6 marks ]**

**Answer 2**

**( a )** $x = 4\, tan\, t \quad \Rightarrow \quad \frac{dx}{dt} = 4\, sec^2\, t \quad \Rightarrow \quad \frac{dt}{dx} = \frac{1}{4} cos^2\, t$

$$y = 5\sqrt{3}\, sin\, 2t \quad \Rightarrow \quad \frac{dy}{dt} = 10\sqrt{3}\, cos\, 2t$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= 10\sqrt{3}\, cos\, 2t \times \frac{1}{4} cos^2\, t$$

$$= \frac{5\sqrt{3}}{2} cos\, 2t\, cos^2\, t$$

As the coordinates of $P$ are known,

$$4\sqrt{3} = 4\, tan\, t$$

$$t = arctan\left(\sqrt{3}\right)$$

$$= \frac{\pi}{3}$$

$$\frac{dy}{dx} = \frac{5\sqrt{3}}{2} cos\left(\frac{2\pi}{3}\right) \left(cos\left(\frac{\pi}{3}\right)\right)^2$$

$$= \frac{5\sqrt{3}}{2}\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)^2$$

$$= -\frac{5\sqrt{3}}{16}$$

**[ 4 marks ]**

**( b )** $$\frac{dy}{dx} = 0$$

$$\frac{5\sqrt{3}}{2} cos\, 2t\, cos^2\, t = 0$$

Either $cos\, 2t = 0$ or $cos^2\, t = 0$

For $cos^2\, t = 0 \Rightarrow cos\, t = 0 \Rightarrow t = \frac{\pi}{2}$

but this is outside the domain specified and so is not a solution

That leaves $cos\, 2t = 0 \Rightarrow 2t = \frac{\pi}{2} \Rightarrow t = \frac{\pi}{4}$

This value of $t$ is now used to yield the point, $Q\left(4,\ 5\sqrt{3}\right)$

**[ 2 marks ]**

**Answer 3**

$$
\begin{aligned}
x &= 4\,cos\left(t + \frac{\pi}{6}\right)\\
&= 4\,cos\,t\,cos\left(\frac{\pi}{6}\right) - 4\,sin\,t\,sin\left(\frac{\pi}{6}\right)\\
&= 2\sqrt{3}\,cos\,t - 2\,sin\,t \qquad \text{but} \qquad y = 2\,sin\,t\\
x &= 2\sqrt{3}\,cos\,t - y\\
x + y &= 2\sqrt{3}\,cos\,t\\
cos\,t &= \frac{x+y}{2\sqrt{3}}
\end{aligned}
$$

$$cos^2 t + sin^2 t = 1$$

$$\left(\frac{x+y}{2\sqrt{3}}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$$

$$\frac{(x+y)^2}{12} + \frac{y^2}{4} = 1$$

Multiply through by 12

$$(x+y)^2 + 3y^2 = 12$$

That is $a = 3$ and $b = 12$

**[ 5 marks ]**

**Answer 4**

**( a )** $x = sec\, t \Rightarrow \dfrac{dx}{dt} = sec\, t\, tan\, t$

$y = tan\, t \Rightarrow \dfrac{dy}{dt} = sec^2\, t$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= sec^2\, t \times \frac{1}{sec\, t\, tan\, t}$$

$$= sec\, t \times cot\, t$$

$$= \frac{1}{cos\, t} \times \frac{cos\, t}{sin\, t}$$

$$= \frac{1}{sin\, t}$$

$$= csc\, t \qquad \square$$

**[ 4 marks ]**

**( b )** When $t = \dfrac{\pi}{4}$

$x = sec\left(\dfrac{\pi}{4}\right) \Rightarrow x = \dfrac{1}{cos\left(\frac{\pi}{4}\right)} \Rightarrow x = \sqrt{2}$

$y = tan\left(\dfrac{\pi}{4}\right) \Rightarrow y = 1$

$\dfrac{dy}{dx} = csc\left(\dfrac{\pi}{4}\right) \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{sin\left(\frac{\pi}{4}\right)} \Rightarrow \dfrac{dy}{dx} = \sqrt{2}$

$$y = mx + c$$

$$1 = \sqrt{2} \times \sqrt{2} + c$$

$$c = -1$$

The equation of the tangent to $C$ at $\left(\sqrt{2},\ 1\right)$ is thus

$$y = \sqrt{2}\, x - 1$$

**[ 4 marks ]**

**Answer 5**

**( a )** $x = sin^2 t \Rightarrow \dfrac{dx}{dt} = 2\, sin\, t\, cos\, t$

$$y = 2\, tan\, t \Rightarrow \frac{dy}{dt} = 2\, sec^2 t$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= 2\, sec^2 t \times \frac{1}{2\, sin\, t\, cos\, t}$$

$$= sec^2 t\, csc\, t\, sec\, t$$

$$= sec^3 t\, csc\, t$$

$$= \frac{1}{cos^3 t\, sin\, t}$$

**[ 4 marks ]**

**( b )** When $t = \dfrac{\pi}{3}$

$$x = sin^2\left(\frac{\pi}{3}\right) \Rightarrow x = \frac{3}{4}$$

$$y = 2\, tan\left(\frac{\pi}{3}\right) \Rightarrow y = 2\sqrt{3}$$

$$\frac{dy}{dx} = \frac{1}{cos^3\left(\frac{\pi}{3}\right) sin\left(\frac{\pi}{3}\right)} \Rightarrow \frac{dy}{dx} = 8 \times \frac{2}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{16\sqrt{3}}{3}$$

$$y = mx + c$$

$$2\sqrt{3} = \frac{16\sqrt{3}}{3} \times \frac{3}{4} + c$$

$$2\sqrt{3} = 4\sqrt{3} + c$$

$$c = -2\sqrt{3}$$

The equation of the tangent to $C$ at $\left(\dfrac{3}{4},\ 2\sqrt{3}\right)$ is thus

$$y = \frac{16\sqrt{3}}{3} x - 2\sqrt{3}$$

This will cut the $x$-axis when $y = 0$

$$\frac{16\sqrt{3}}{3} x = 2\sqrt{3}$$

$$x = \frac{3}{8}$$

That is, at $\left(\dfrac{3}{8},\ 0\right)$

**[ 6 marks ]**

**Answer 6**

**( a )**

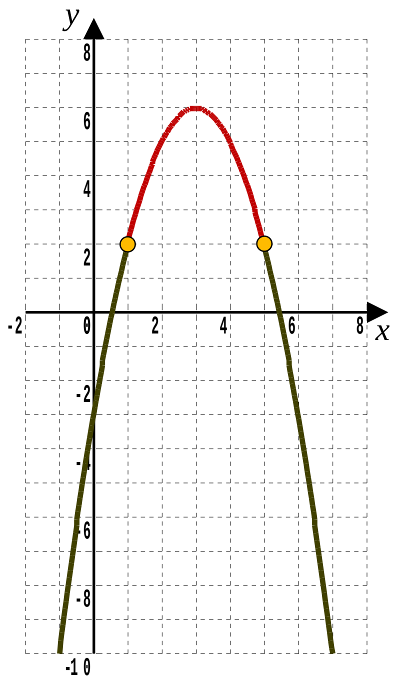
$$\begin{aligned}
\text{RHS} &= 6 - (x - 3)^2 \\
&= 6 - (3 + 2\sin t - 3)^2 \\
&= 6 - \left(4\sin^2 t\right) \\
&= 4 + (2) - 4\sin^2 t \qquad \text{As the parametric for } y \text{ has an isolated 4} \\
&= 4 + \left(2\cos^2 t + 2\sin^2 t\right) - 4\sin^2 t \\
&= 4 + 2\cos^2 t - 2\sin^2 t \\
&= 4 + 2\left(\cos^2 t - \sin^2 t\right) \\
&= 4 + 2\cos 2t \\
&= y \\
&= \text{LHS} \qquad \square
\end{aligned}$$

**[ 2 marks ]**

**( b )** **( i )** It's an upside down parabola due to the $-x^2$ term

It has a maximum at $(3, 6)$ from the given completed square form

**( ii )** As $-1 \leqslant \sin t \leqslant 1 \Rightarrow 1 \leqslant x \leqslant 5$




It's the red part of the graph that is wanted.
The two green parts cannot be reached by
the parametric equations.

**[ 3 marks ]**

**( c )** $x + y = k \Rightarrow y = -x + k$

This is a line loping downwards at 45°

The lower bound on $k$ will be when the line goes through the endpoint when $x = 5$.

$$\begin{aligned} y &= 6 - (5 - 3)^2 \\ &= 2 \\ y &= -x + k \\ 2 &= -5 + k \\ k &= 7 \end{aligned}$$

For the upper bound, we need then the curve has a gradient of $-1$ corresponding to the line being a tangent to the curve.

$$\begin{aligned} y &= 6 - (x - 3)^2 \\ &= 6 - \left(x^2 - 6x + 9\right) \\ &= -x^2 + 6x - 3 \\ \frac{dy}{dx} &= -2x + 6 \\ \text{Thus, for } -2x + 6 &= -1 \\ -2x &= -7 \\ x &= \frac{7}{2} \\ y &= 6 - \left(\frac{7}{2} - 3\right)^2 \\ &= 6 - \frac{1}{4} \\ &= \frac{23}{4} \\ y &= -x + k \\ \frac{23}{4} &= -\frac{7}{2} + k \\ k &= \frac{37}{4} \end{aligned}$$

$$\text{Thus } 7 \leqslant k < \frac{37}{4}$$

**[ 5 marks ]**