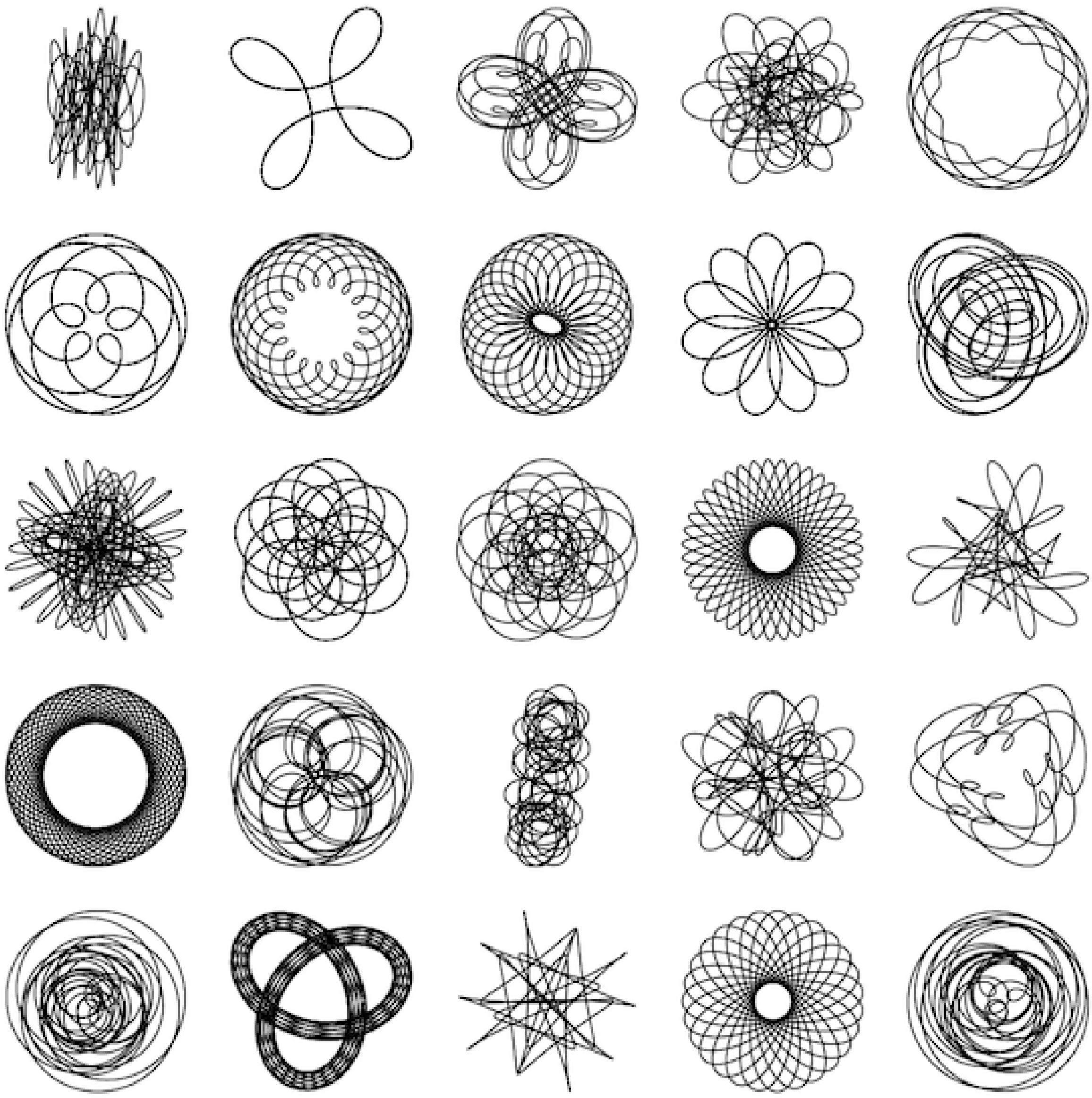
A-Level Pure Mathematics
Year 2

DIFFERENTIATION IV



Parametric Differentiation • Implicit Differentiation

1.1 What are Parametric Equations ?

A key idea in mathematics is the separation of “what is happening in the *x* direction” from “what is happening in the *y* direction”. This concept is often first encountered by students in mechanics, where the motion of a projectile is analysed by separating the horizontal component of a projectile's motion from its vertical component.
To elaborate on this example; for a projectile in flight, the horizontal component of the projectile's motion is determined by the single equation,

$$Distance = Speed \times Time$$

whereas the vertical component of a projectile's motion is determined by the following five equations, often referred to collectively as the *suvat* equations,

$$v = u + at \qquad s \ \ displacement$$

$$s = vt - \frac{1}{2}at^2 \qquad u \ \ initial\ velocity$$

$$s = ut + \frac{1}{2}at^2 \qquad v \ \ final\ velocity$$

$$s = \left(\frac{v+u}{2}\right)t \qquad a \ \ acceleration\ (constant)$$

$$v^2 = u^2 + 2as \qquad t \ \ time$$

The Applied Mathematics topic of Projectiles, has at its heart the fact that the motion of a projectile in the *x* direction is independent of the motion in the *y*.

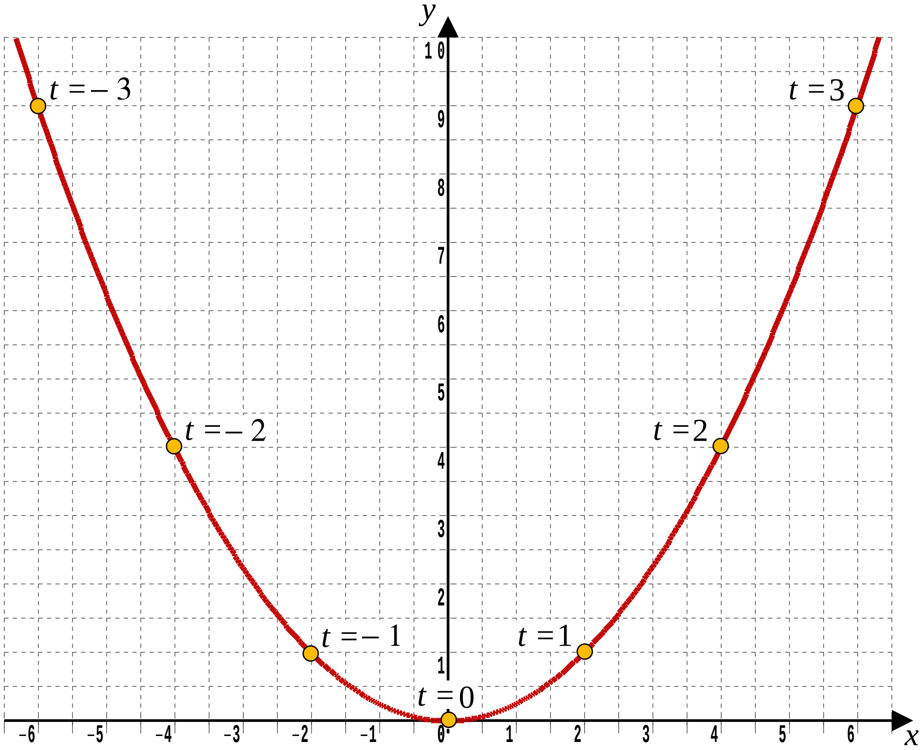
For a simple example to illustrate where, more generally, this separatist line of thought leads, consider the two equations; $x = 2t, \ y = t^2$
Such a pair of equations, one for *x* and a separate one for *y*, each in terms of time *t*, are an example of a pair of parametric equations. The *t* does not have to represent time, and in general it is termed a parameter. As *t* varies through all possible values, all possible points on the path are obtained, thus, in effect, providing another way of plotting a graph.

To plot $x = 2t, \ y = t^2$ work out the points on the path for particular values of t (e.g. when $t = -3, -2, -1, \ 0, \ 1, \ 2, \ 3)$ then join them up, dot-to-dot fashion.

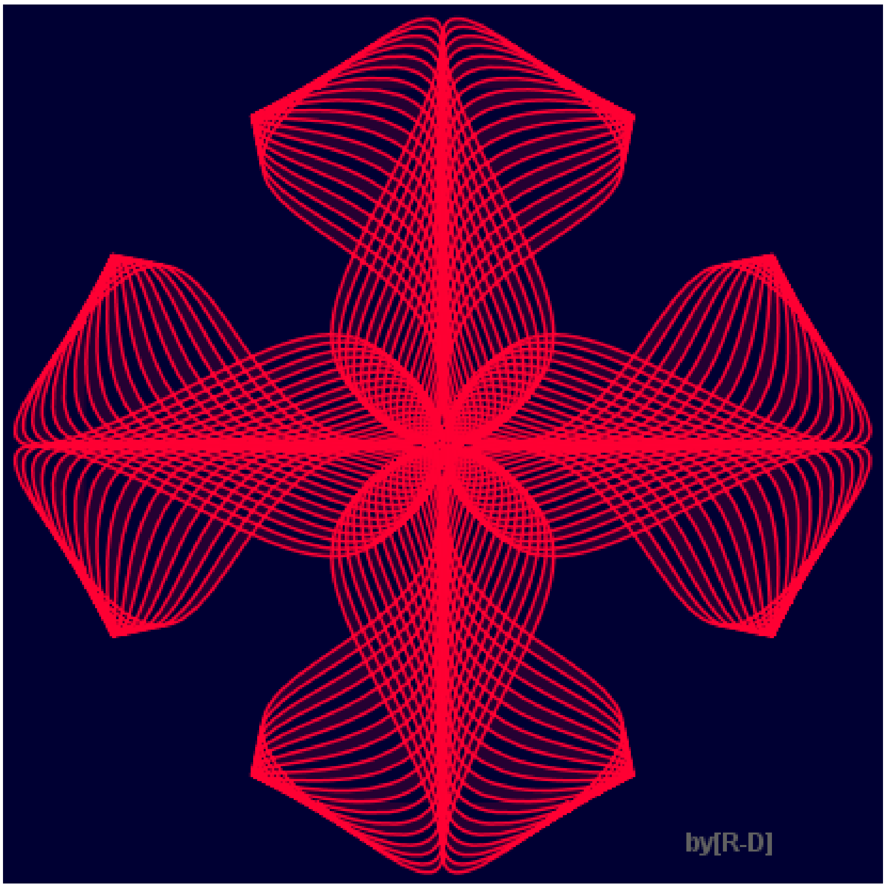
A table keeps the working organised;

t	-3	-2	-1	0	1	2	3
$x = 2t$	-6	-4	-2	0	2	4	6
$y = t^2$	9	4	1	0	1	4	9
(x, y)	$(-6, 9)$	$(-4, 4)$	$(-2, 1)$	$(0, 0)$	$(2, 1)$	$(4, 4)$	$(6, 9)$

And the resulting graph is,



Rather than view this as a static object, it can be considered a path along which a point moves. Isaac Newton saw the point as a particle. It arrives top left at a brisk pace, then slows down as it approaches the origin, and then speeds up again as it leaves, top right.



Some spectacular paths can be generated using parametric equations.

1.2 Exercise

Marks Available : 50

Question 1

(i) A curve is described by the parametric equations

$$x = 3 \sin 2\theta°$$

$$y = 6 \cos \theta° - 3\sqrt{2} \sin^2 \theta°$$

Complete the following tables and graph the resulting curve.
Work to 1 decimal place.

θ (in degrees)	0	15	30	45	60	75	90
$x = 3 \sin 2\theta°$							
$y = 6 \cos \theta° - 3\sqrt{2} \sin^2 \theta°$							

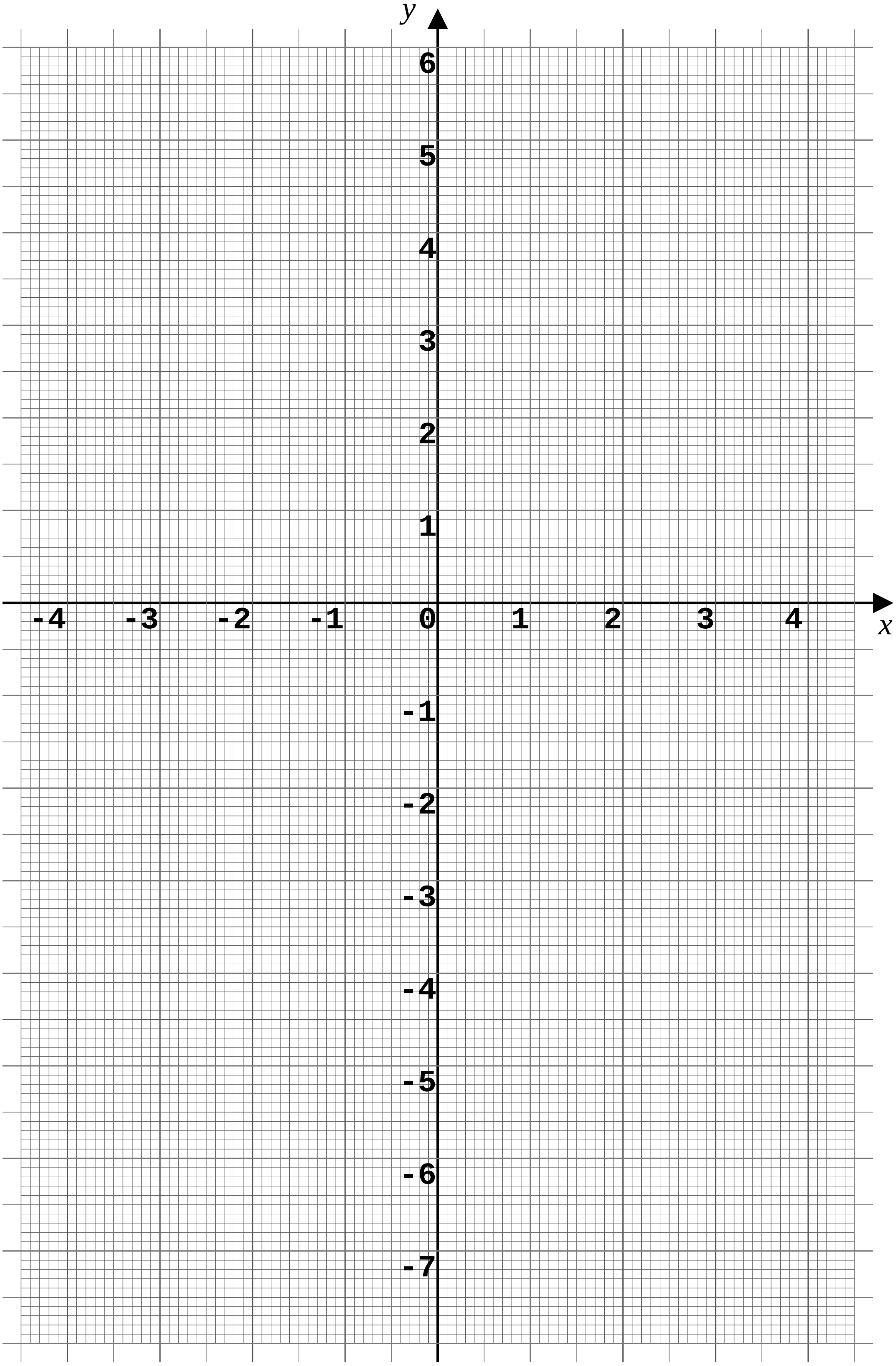
θ (in degrees)	105	120	135	150	165	180
$x = 3 \sin 2\theta°$						
$y = 6 \cos \theta° - 3\sqrt{2} \sin^2 \theta°$						

θ (in degrees)	195	210	225	240	255	270
$x = 3 \sin 2\theta°$						
$y = 6 \cos \theta° - 3\sqrt{2} \sin^2 \theta$						

θ (in degrees)	285	300	315	330	345	360
$x = 3 \sin 2\theta°$						
$y = 6 \cos \theta° - 3\sqrt{2} \sin^2 \theta°$						

[10 marks]

(ii) Plot your part (i) curve.



[10 marks]

Question 2

A curve is described by the parametric equations

$$x = 4t^2$$

$$y = 16t\left(t^2 - 1\right)$$

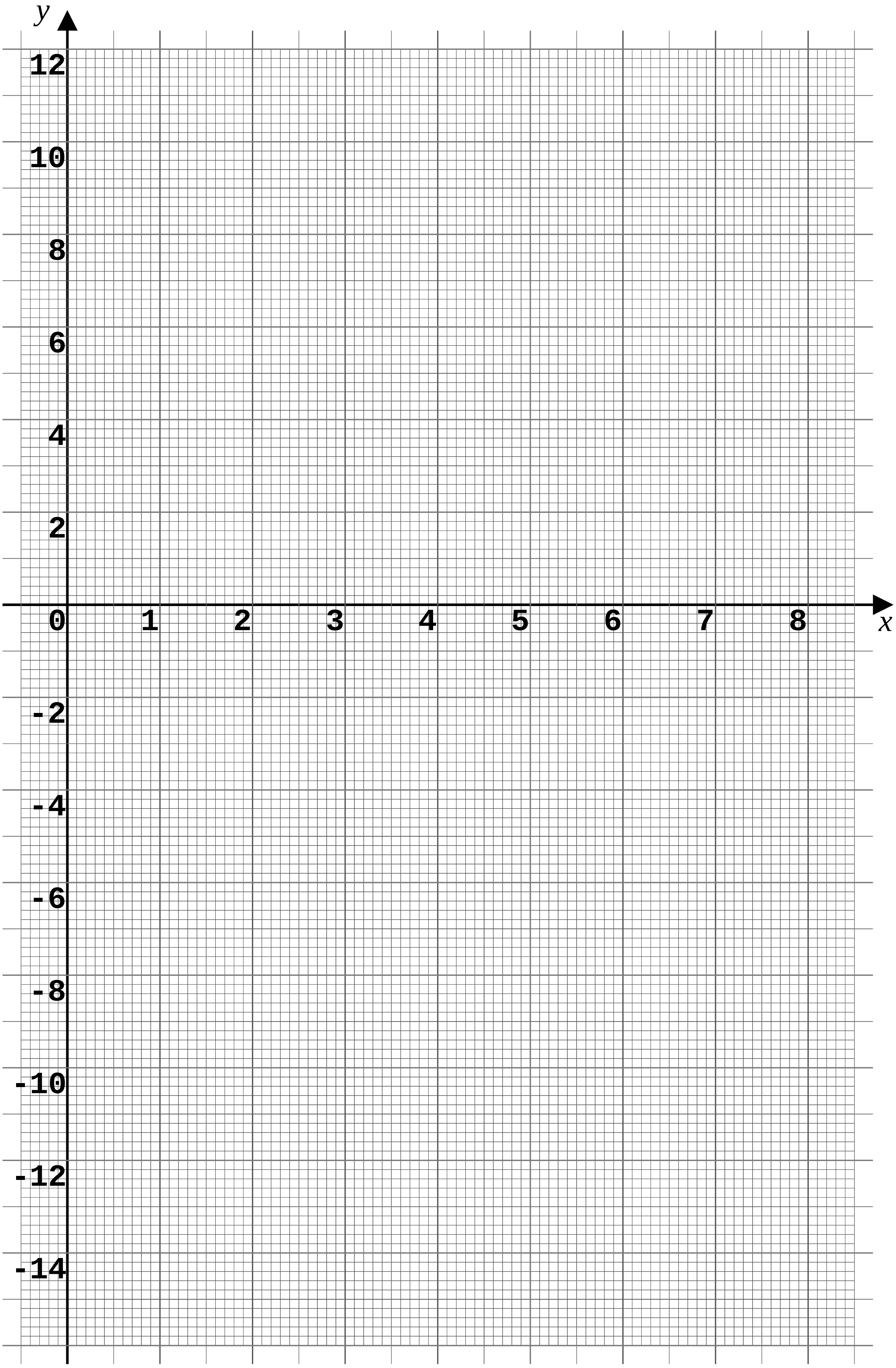
(i) Complete the following table.
Do not round off any table entries.

t	-1.25	-1	-0.75	-0.5	-0.25
$x = 4t^2$					
$y = 16t\left(t^2 - 1\right)$					

t	0	0.25	0.5	0.75	1	1.25
$x = 4t^2$						
$y = 16t\left(t^2 - 1\right)$						

[8 marks]

(ii) Plot your part (i) curve.
You may round off coordinates to 1 decimal place if you wish.



[7 marks]

Question 3

A curve is described by the parametric equations

$$x = 4\sin\theta$$

$$y = 6\cos\theta$$

(i) Complete the following table.
Work to 1 decimal place.
Make sure you are working in **RADIANS**

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
$x = 4\sin\theta$							
$y = 6\cos\theta$							

[6 marks]

(ii) Plot your part (i) curve using the graph paper on the following page.

[4 marks]

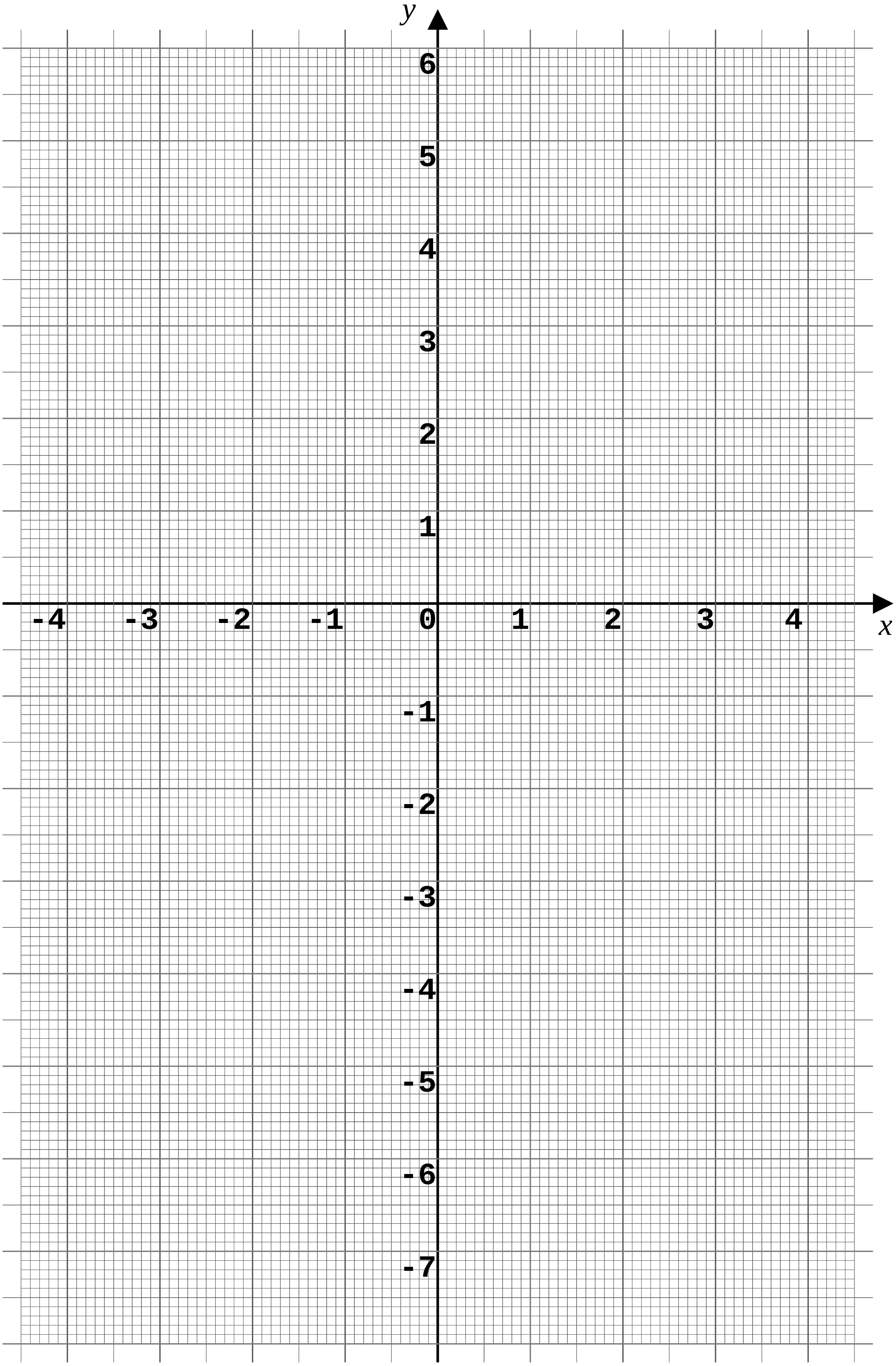
(iii) The table in part (i) was for $0 \leqslant \theta \leqslant \pi$

Draw the additional part of the graph for $\pi < \theta \leqslant 2\pi$

[3 marks]

(iv) Explain why the graph for $2\pi \leqslant \theta \leqslant 4\pi$ would not look any different to that for $0 \leqslant \theta \leqslant 2\pi$

[2 marks]


y
6
5
4
3
2
1
-4
-3
-2
-1
0
1
2
3
4
x
-1
-2
-3
-4
-5
-6
-7

1.3 Answers to 1.2 Exercise

Answer 1

(i)

θ (in degrees)	0	15	30	45	60	75	90
$x = 3\ sin\ 2\theta°$	0	1.5	2.6	3	2.6	1.5	0
$y = 6\ cos\ \theta° - 3\sqrt{2}\ sin^2\ \theta°$	6	5.5	4.1	2.1	– 0.2	– 2.4	– 4.2

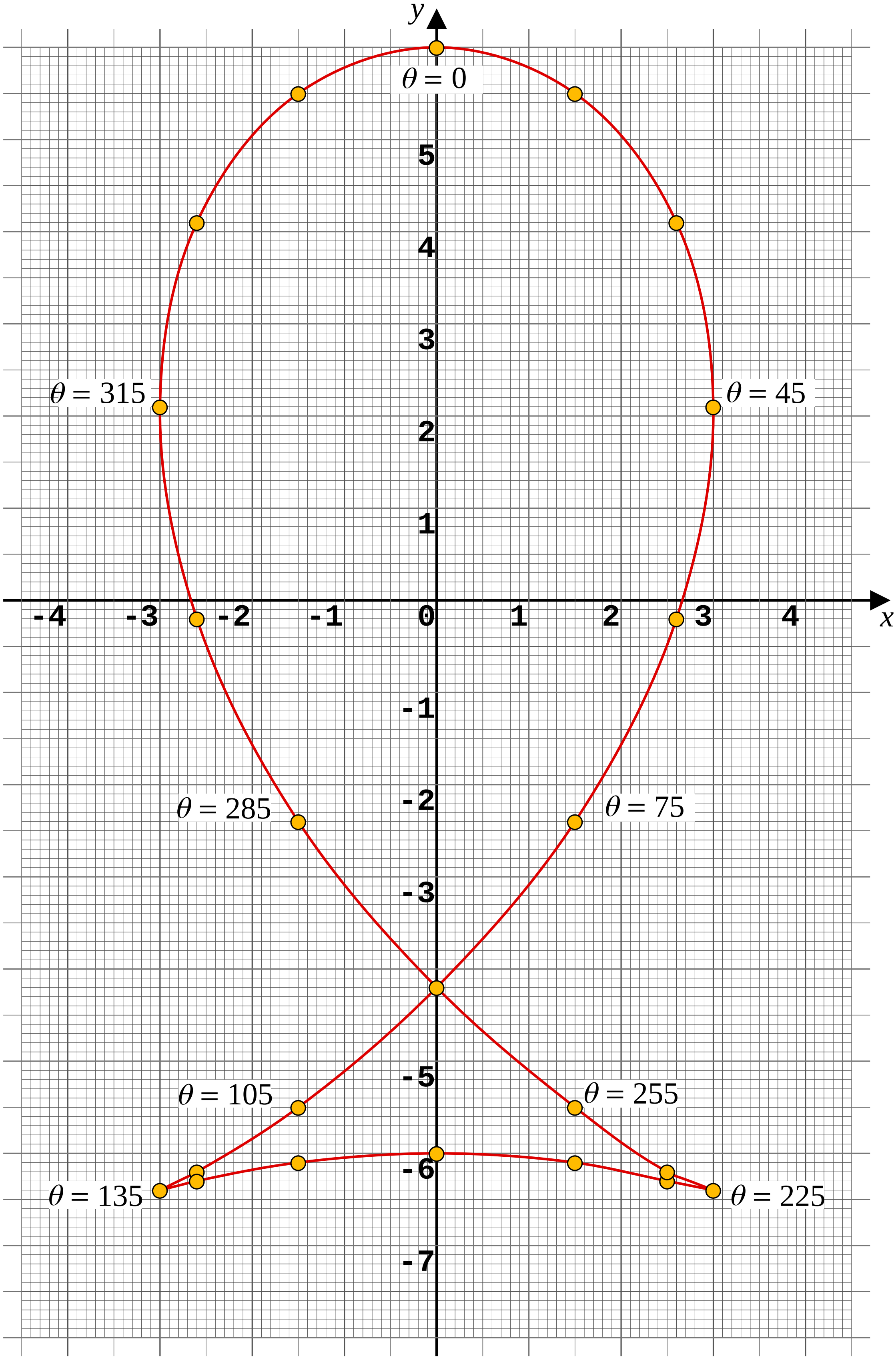
θ (in degrees)	105	120	135	150	165	180
$x = 3\ sin\ 2\theta°$	– 1.5	– 2.6	– 3	– 2.6	– 1.5	0
$y = 6\ cos\ \theta° - 3\sqrt{2}\ sin^2\ \theta°$	– 5.5	– 6.2	– 6.4	– 6.3	– 6.1	– 6

θ (in degrees)	195	210	225	240	255	270
$x = 3\ sin\ 2\theta°$	1.5	2.6	3	2.6	1.5	0
$y = 6\ cos\ \theta° - 3\sqrt{2}\ sin^2\ \theta$	– 6.1	– 6.3	– 6.4	– 6.2	– 5.5	– 4.2

θ (in degrees)	285	300	315	330	345	360
$x = 3\ sin\ 2\theta°$	– 1.5	– 2.6	– 3	– 2.6	– 1.5	0
$y = 6\ cos\ \theta° - 3\sqrt{2}\ sin^2\ \theta°$	– 2.4	– 0.2	2.1	4.1	5.5	6

[10 marks]

(ii)



y
x
-4
-3
-2
-1
0
1
2
3
4
5
4
3
2
1
-1
-2
-3
-5
-6
-7
$\theta = 0$
$\theta = 315$
$\theta = 45$
$\theta = 285$
$\theta = 75$
$\theta = 105$
$\theta = 255$
$\theta = 135$
$\theta = 225$

[7 marks]

Answer 2

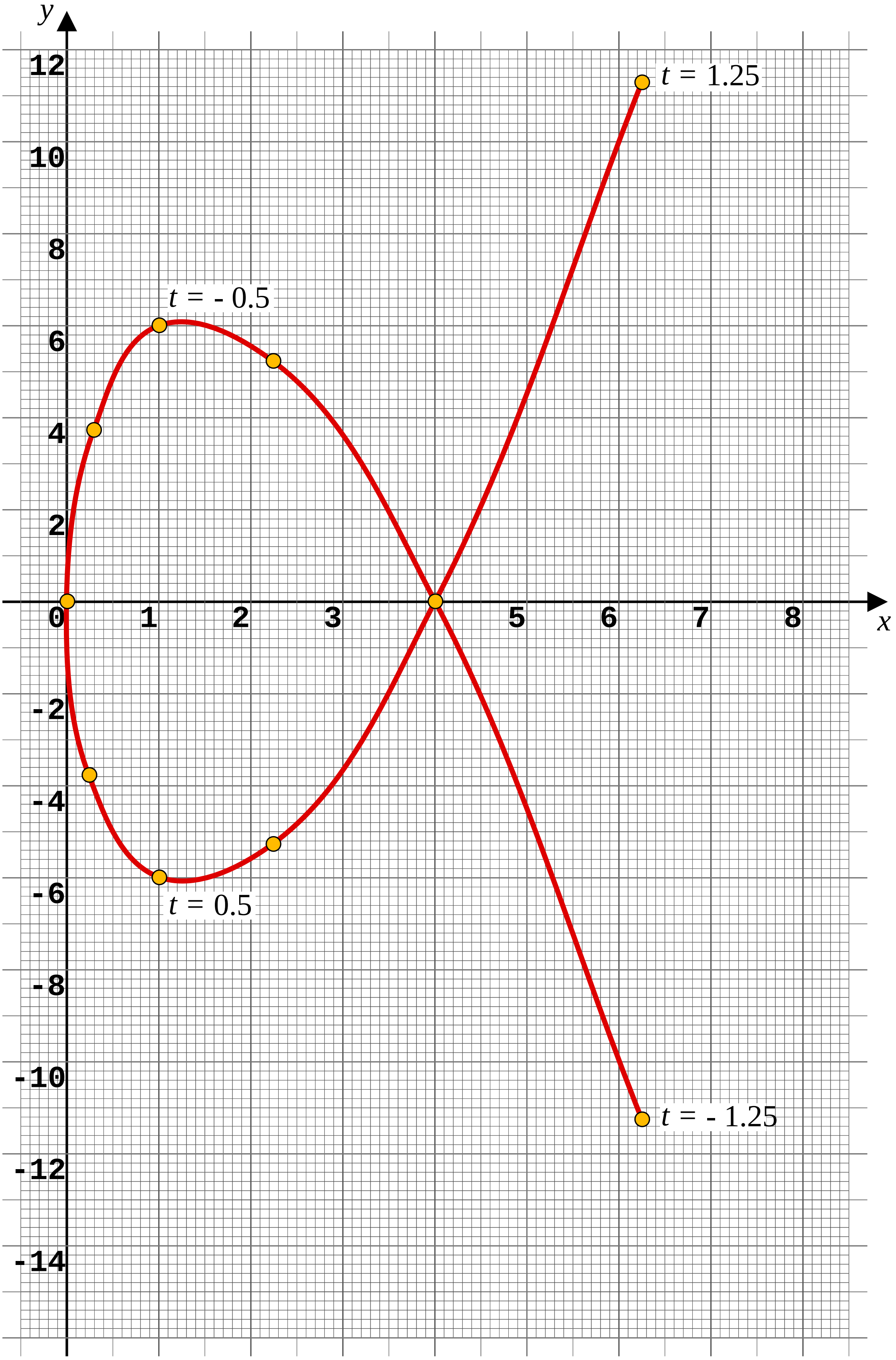
(i)

t	-1.25	-1	-0.75	-0.5	-0.25
$x = 4t^2$	6.25	4	2.25	1	0.25
$y = 16t(t^2 - 1)$	-11.25	0	5.25	6	3.75

t	0	0.25	0.5	0.75	1	1.25
$x = 4t^2$	0	0.25	1	2.25	4	6.25
$y = 16t(t^2 - 1)$	0	-3.75	-6	-5.25	0	11.25

[8 marks]

(ii)



[7 marks]

Answer 3

(i)

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
$x = 4\,sin\;\theta$	0	2	3.5	4	3.5	2	0
$y = 6\cos\theta$	6	5.2	3	0	– 3	– 5.2	– 6

[6 marks]

(ii) See the graph on the following page, red curve.

[4 marks]

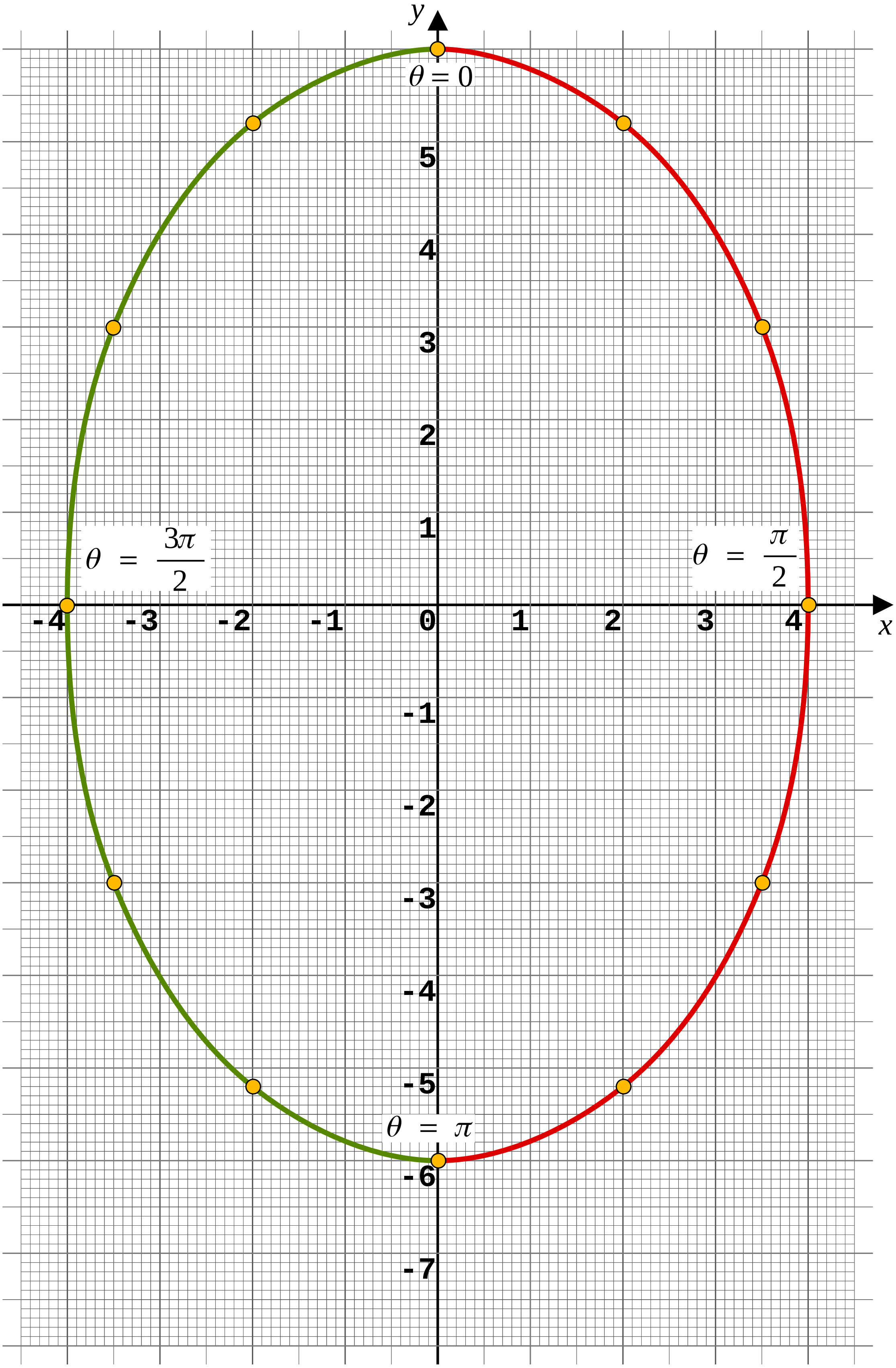
(iii) See the graph on the following page, green curve.

[3 marks]

(iv) The same ellipse will be traced out because,

$4\,sin\,(\theta + 2\pi) = 4\,sin\,\theta$ and $6\,cos\,(\theta + 2\pi) = 6\,cos\,\theta$

[2 marks]



y
x
$\theta = 0$
$\theta = \frac{\pi}{2}$
$\theta = \pi$
$\theta = \frac{3\pi}{2}$
5
4
3
2
1
0
-1
-2
-3
-4
-5
-6
-7
-4
-3
-2
-1
1
2
3
4

2.1 Parametric to Cartesian Manipulations

It may be possible to convert a pair of parametric equations into Cartesian form.

Example #1

Consider the following parametric equations which describe an ellipse†

$$x = 4 \sin \theta$$

$$y = 6 \cos \theta$$

In this case it is possible to eliminate the parameter θ, and obtain a single equation containing only numbers and the variables x and y.

Teaching Video : http://www.NumberWonder.co.uk/v9081/2a.mp4

[4 marks]

Example #2

Express in the form $y^2 = f(x)$ the following parametric equations‡

$$x = 4t^2$$

$$y = 16t(t^2 - 1)$$

Teaching Video : http://www.NumberWonder.co.uk/v9081/2b.mp4

[4 marks]

† These parametric equations were graphed in Lesson 1, Exercise 1.2, Question 3

‡ These parametric equations were graphed in Lesson 1, Exercise 1.2, Question 2

2.2 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 40

Question 1

Find an equation of the form $a x^2 + b y^2 = c$, where a, b and c are integer constants to be found, for the following pair of parametric equations;

$$x = 15 \sin \theta$$

$$y = 20 \cos \theta$$

[4 marks]

Question 2

Find the Cartesian equations of each of these curves in the form $y = f(x)$

(i) $x = 8t$, $y = \dfrac{8}{t}$

(ii) $x = \dfrac{20}{t}$, $y = t^2$

[3, 3 marks]

(iii) $x = 14t$, $y = 7t^2$

(iv) $x = t - 2$, $y = t^2 + 3$

[3, 3 marks]

Question 3

Use trigonometric identities to find the Cartesian equations of each of these curves,

(i) $x = 2\,cos\,\theta$

$y = 3\,sin\,\theta$

(ii) $x = 3\,sec\,\theta$

$y = 5\,tan\,\theta$

[3, 3 marks]

Question 4

Find the Cartesian equations of each of these curves in the form $y = f\,(x)$

(i) $x = 2 + 3\,t$

$y = \dfrac{1}{t}$

(ii) $x = 3 + 2\,t$

$y = 4\,t^2 - 9$

[3, 3 marks]

Question 5

Find the Cartesian equations of the curve with parametric equations ;

$$x = 2 \cos \theta$$

$$y = 5 \sin 2\theta$$

Give your answer in the form $y^2 = f(x)$

Hint : Make use of the trigonometric identity $\sin 2\theta = 2 \sin \theta \cos \theta$

[6 marks]

Question 6

A curve C has parametric equations

$$x = 2 \sin \theta$$

$$y = 1 - \cos 2\theta$$

Find a Cartesian equation for C in the form $y = f(x)$

[6 marks]

2.3 Homework

A-Level Pure Mathematics : Year 2

Differentiation IV

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 20

Question 1

Find a Cartesian equations of each of these curves in the form $y = f(x)$,

(i) $x = \dfrac{12}{t}$, $y = t^2 + t$

(ii) $x = 4\sqrt{t}$, $y = 7t^2$

[3, 3 marks]

(iii) $x = e^{2t}$, $y = e^{6t} - 1$

(iv) $x = \sqrt{t + 1}$, $y = t^2$

[3, 3 marks]

Question 2

Find an equation of the form $a\,x^2 + b\,y^2 = c$, where a, b and c are integer constants to be found, for the following pair of parametric equations

$$x = 35\,cos\,\theta°$$

$$y = 20\,sin\,\theta°$$

[4 marks]

Question 3

Show that the parametric equations

$$x = \frac{1}{t - 1}$$

$$y = \frac{1}{t + 1}$$

can be written in the form $y = \dfrac{x}{a + bx}$ where a and b are integers to be determined.

[4 marks]

2.4 Answers

2.4.1 Solutions to 2.1 Examples (Teaching Videos)

Example #1

$$x = 4\sin\theta \Rightarrow \sin\theta = \frac{x}{4}$$

$$y = 6\cos\theta \Rightarrow \cos\theta = \frac{y}{6}$$

Trigonometric Identity : $\cos^2\theta + \sin^2\theta = 1$

$$\therefore \left(\frac{y}{6}\right)^2 + \left(\frac{x}{4}\right)^2 = 1$$

Multiply through by $6^2\,4^2$

$$4^2 y^2 + 6^2 x^2 = 6^2\,4^2$$

$$16y^2 + 36x^2 = 576$$

[4 marks]

Example #2

$$x = 4t^2 \qquad \Rightarrow t^2 = \frac{x}{4}$$

$$y = 16t(t^2 - 1) \Rightarrow y^2 = 256t^2(t^2 - 1)^2$$

$$= 256 \times \frac{x}{4}\left(\frac{x}{4} - 1\right)^2$$

$$= 64x\left(\frac{x^2}{16} - \frac{x}{2} + 1\right)$$

$$y^2 = 4x^3 - 32x^2 + 64x$$

[4 marks]

2.4.2 Solutions to 2.2 Exercise

Answer 1

$$x = 15\sin\theta \Rightarrow \sin\theta = \frac{x}{15}$$

$$y = 20\cos\theta \Rightarrow \cos\theta = \frac{y}{20}$$

Trigonometric Identity : $\cos^2\theta + \sin^2\theta = 1$

$$\therefore \left(\frac{y}{20}\right)^2 + \left(\frac{x}{15}\right)^2 = 1$$

Multiply through by $20^2\,15^2$

$$15^2 y^2 + 20^2 x^2 = 300^2$$

$$225y^2 + 400x^2 = 90000$$

$$9y^2 + 16x^2 = 3600$$

[4 marks]

Answer 2

(i) $x = 8t \quad \Rightarrow t = \frac{x}{8}$

$y = \frac{8}{t} \quad \Rightarrow 8 = yt \quad \Rightarrow 8 = y \times \frac{x}{8} \Rightarrow y = \frac{64}{x}$

[3 marks]

(ii) $x = \frac{20}{t} \quad \Rightarrow t^2 = \frac{400}{x^2}$

$y = t^2 \quad \Rightarrow y = \frac{400}{x^2}$

[3 marks]

(iii) $x = 14t \quad \Rightarrow t = \frac{x}{14}$

$y = 7t^2 \quad \Rightarrow y = \frac{7x^2}{14^2} \Rightarrow y = \frac{x^2}{28}$

[3 marks]

(iv) $x = t - 2 \Rightarrow t^2 = (x + 2)^2$

$y = t^2 + 3 \quad \Rightarrow y = x^2 + 4x + 7$

[3 marks]

Answer 3

(i)

$$x = 2\cos\theta \Rightarrow \cos\theta = \frac{x}{2}$$

$$y = 3\sin\theta \Rightarrow \sin\theta = \frac{y}{3}$$

Trigonometric Identity : $\cos^2\theta + \sin^2\theta = 1$

$$\therefore \left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$$

Multiply through by $2^2\,3^2$

$$3^2 x^2 + 2^2 y^2 = 6^2$$

$$9x^2 + 4y^2 = 36$$

[3 marks]

(ii)

$$x = 3\sec\theta \Rightarrow \sec\theta = \frac{x}{3}$$

$$y = 5\tan\theta \Rightarrow \tan\theta = \frac{y}{5}$$

Trigonometric Identity : $\cos^2\theta + \sin^2\theta = 1$

$$\frac{\cos^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$\therefore 1 + \left(\frac{y}{5}\right)^2 = \left(\frac{x}{3}\right)^2$$

Multiply through by $5^2\,3^2$

$$5^2\,3^2 + 3^2 y^2 = 5^2 x^2$$

$$225 + 9y^2 = 25x^2$$

$$25x^2 - 9y^2 = 225$$

[3 marks]

Answer 4

(i) $x = 2 + 3t \Rightarrow t = \dfrac{x - 2}{3}$

$y = \dfrac{1}{t} \qquad \Rightarrow y = \dfrac{3}{x - 2}$

[3 marks]

(ii) $x = 3 + 2t \Rightarrow t = \dfrac{x - 3}{2} \Rightarrow t^2 = \dfrac{x^2 - 6x + 9}{4}$

$y = 4t^2 - 9 \Rightarrow y = x^2 - 6x$

[3 marks]

Answer 5

$x = 2\cos\theta \Rightarrow x^2 = 4\cos^2\theta \Rightarrow \cos^2\theta = \dfrac{x^2}{4}$

$$
\begin{aligned}
y = 5\sin 2\theta \Rightarrow \quad y &= 10\sin\theta\cos\theta \\
y^2 &= 100\sin^2\theta\cos^2\theta \\
&= 100\left(1 - \cos^2\theta\right)\cos^2\theta \\
&= 100\left(1 - \frac{x^2}{4}\right)\left(\frac{x^2}{4}\right) \\
&= 25x^2\left(1 - \frac{x^2}{4}\right)
\end{aligned}
$$

[6 marks]

Answer 6

$x = 2\sin\theta \Rightarrow \sin^2\theta = \dfrac{x^2}{4}$

$$
\begin{aligned}
y = 1 - \cos 2\theta \Rightarrow y &= 1 - \left(\cos^2\theta - \sin^2\theta\right) \\
&= 1 - \left(\left(1 - \sin^2\theta\right) - \sin^2\theta\right) \\
&= 1 - 1 + \sin^2\theta + \sin^2\theta \\
&= 2\sin^2\theta \\
&= \frac{x^2}{2}
\end{aligned}
$$

[6 marks]

2.4.3 Solutions to 2.3 Homework

Answer 1

(i) $x = \dfrac{12}{t} \Rightarrow t = \dfrac{12}{x}$

$$y = t^2 + t \Rightarrow y = \left(\frac{12}{x}\right)^2 + \frac{12}{x} \Rightarrow y = \frac{144}{x^2} + \frac{12}{x}$$

[3 marks]

(ii) $x = 4\sqrt{t} \Rightarrow t^2 = \dfrac{x^4}{256}$

$$y = 7t^2 \Rightarrow y = \frac{7x^4}{256}$$

[3 marks]

(iii) $x = e^{2t} \Rightarrow x^3 = e^{6t}$

$$y = e^{6t} - 1 \Rightarrow y = x^3 - 1$$

[3 marks]

(iv) $x = \sqrt{t+1} \Rightarrow x^2 = t + 1 \Rightarrow t = x^2 - 1$

$$y = t^2 \Rightarrow y = \left(x^2 - 1\right)^2$$

[3 marks]

Answer 2

$x = 35\cos\theta° \Rightarrow \cos\theta = \dfrac{x}{35}$

$y = 20\sin\theta° \Rightarrow \sin\theta = \dfrac{y}{20}$

Trigonometric Identity : $\cos^2\theta + \sin^2\theta = 1$

$$\therefore \left(\frac{x}{35}\right)^2 + \left(\frac{y}{20}\right)^2 = 1$$

Multiply through by $35^2\,20^2$

$$20^2 x^2 + 35^2 y^2 = 700^2$$

$$400x^2 + 1225y^2 = 490000$$

Divide through by 25

$$16x^2 + 49y^2 = 19600$$

[4 marks]

Answer 3

$$x = \frac{1}{t - 1} \Rightarrow t - 1 = \frac{1}{x} \Rightarrow t = \frac{1}{x} + 1$$

$$y = \frac{1}{t + 1} \Rightarrow t + 1 = \frac{1}{y} \Rightarrow t = \frac{1}{y} - 1$$

$$\text{Thus, } \frac{1}{y} - 1 = \frac{1}{x} + 1$$

$$\frac{1}{y} = \frac{1}{x} + 2$$

$$\frac{1}{y} = \frac{1 + 2x}{x}$$

$$y = \frac{x}{1 + 2x}$$

That is, $a = 1$ and $b = 2$

[4 marks]

Lesson 3

A-Level Pure Mathematics : Year 2
Differentiation IV

3.1 Differentiation Of Parametric Equations

To differentiate parametric equations two ideas are needed,

The Two Parametric Differentiation Key Facts

- $\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$ The Chain Rule
- $\frac{d\theta}{dx} = \frac{1}{\left(\frac{dx}{d\theta}\right)}$ The Inversion Rule

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A proper mathematically rigorous proof of either of these is difficult and tricky but the two results are easily remembered because it is as if the entities involved are behaving like ordinary fractions. For A-Level you should think of them this way, (as ordinary fractions) but keep in mind that they are "not really" and that there is an unresolved issue here for tackling on a University Maths course !

Example

Given the parametric equations $x = sin\,\theta$

$$y = cos\,\theta$$

(i) Sketch the part circle traced out as θ increases from 0 to π radians.

(ii) Find $\frac{dy}{dx}$ in terms of θ

(iii) State the value of the gradient when $\theta = \frac{3\pi}{4}$

Teaching Video : http://www.NumberWonder.co.uk/v9081/3.mp4

[2, 3, 1 marks]

3.2 Table of standard derivatives

$f(x)$	$f'(x)$	In Formula Book ?
x^n	$n x^{n-1}$	No
e^x	e^x	No
$\ln x$	$\frac{1}{x}$	No
$\sin x$	$\cos x$	No
$\cos x$	$-\sin x$	No
$\tan x$	$\sec^2 x$	Yes
$\csc x$	$-\csc x \cot x$	Yes
$\sec x$	$\sec x \tan x$	Yes
$\cot x$	$-\csc^2 x$	Yes
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$	Yes
$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$	Yes
$\arctan x$	$\frac{1}{1+x^2}$	Yes

When trigonometry and calculus mix, θ must be in **RADIANS**

3.3 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable

Marks Available : 50

Question 1

Given that $x = 5t^4$

$y = 3t^5$

Show that

$$\frac{dy}{dx} = \frac{3}{4}t$$

[3 marks]

Question 2

Given that $x = t^2 + t$

$y = 2t - t^2$

Find $\frac{dy}{dx}$ in terms of t

[3 marks]

Question 3

Given that $x = (4t + 2)^3$

$y = (3t + 4)^2$

Find $\frac{dy}{dx}$ in terms of t

[3 marks]

Question 4

Given that $x = 2\,cot\,\theta$

$y = 3\,tan\,\theta$

Show that

$$\frac{dy}{dx} = -\frac{3}{2}\,tan^2\,\theta$$

[3 marks]

Question 5

Find $\frac{dy}{dx}$ in terms of θ for the parametric equations $x = 3\,sin\,\theta$

and $y = 5\,cos\,\theta$

[3 marks]

Question 6

For the parametric equations $x = tan\,\theta$

and $y = sec\,\theta$

(i) Find $\frac{dy}{dx}$ in terms of θ

[3 marks]

(ii) What is the value of the gradient when θ is $\frac{\pi}{3}$ radians ?

[1 mark]

Question 7

The rule for the differentiation of products tells us that;

$$\text{If } f = u\,v \quad \text{then} \quad f' = u\,v' + u'\,v$$

Consider the parametric equations $x = -\,cos\,\theta$

and $y = \theta\,sin\,\theta$

Show that $\frac{dy}{dx} = \theta\,cot\,\theta + 1$

[4 marks]

Question 8

For the parametric equations $x = cot\,\theta$

and $y = csc\,\theta$

(i) Find $\dfrac{dy}{dx}$ in terms of θ

[3 marks]

(ii) What is the value of the gradient when θ is $\dfrac{\pi}{3}$ radians ?

[1 mark]

Question 9

For the parametric equations $x = -\,sin\,\theta$

and $y = \theta\,cos\,\theta$

Find $\dfrac{dy}{dx}$ in terms of θ

[4 marks]

Question 10

Consider the parametric equations $x = 5t^2$

and $y = 4t^3$

(i) Determine the value of x and the value of y when $t = 2$.
This is a point on the curve.

[1 mark]

(ii) Find $\frac{dy}{dx}$ in terms of t

[3 marks]

(iii) Use your part (ii) answer to show that when $t = 2$, $\frac{dy}{dx} = \frac{12}{5}$

[2 marks]

(iv) Use your part (i) point, and your part (iii) gradient, to find the equation of the tangent to the curve when $t = 2$.
Give the answer in the form $ax + by + c = 0$
where a, b and c are INTEGERS to be found.

[3 marks]

Question 11

A curve has the parametric equations $x = \cos t$

and $y = \sin 2t$

(i) Determine the value of x and the value of y when $t = \dfrac{\pi}{6}$

This is a point on the curve.

[1 mark]

(ii) Find $\dfrac{dy}{dx}$ in terms of t

[4 marks]

(iii) Use your part (ii) answer to show that when $t = \dfrac{\pi}{6}$, $\dfrac{dy}{dx} = -2$

[2 marks]

(iv) Use your part (i) point, and your part (iii) gradient, to find the equation of the tangent to the curve when $t = \dfrac{\pi}{6}$

Give an exact answer in the form $y = mx + c$ where m and c are constants.

[3 marks]

3.4 Answers

3.4.1 Solution to the Example (Teaching Video)

(i) The graph is a circular path that starts at 12 O'clock when $\theta = 0$. It heads off clockwise, is at 3 O'clock when $\theta = \frac{\pi}{2}$ and stops at 6 O'clock when $\theta = \pi$.

[2 marks]

(ii) $x = \sin\theta \Rightarrow \frac{dx}{d\theta} = \cos\theta \Rightarrow \frac{d\theta}{dx} = \frac{1}{\cos\theta}$

$y = \cos\theta \Rightarrow \frac{dy}{d\theta} = -\sin\theta$

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{d\theta} \times \frac{d\theta}{dx} \\ &= -\sin\theta \times \frac{1}{\cos\theta} \\ &= -\tan\theta \end{aligned}$$

[3 marks]

(iii) 1, as is also obvious from the part (i) sketch

[1 mark]

3.4.2 Solutions to 3.3 Exercise

Answer 1

$x = 5t^4 \Rightarrow \frac{dx}{dt} = 20t^3 \Rightarrow \frac{dt}{dx} = \frac{1}{20t^3}$

$y = 3t^5 \Rightarrow \frac{dy}{dt} = 15t^4$

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{dt} \times \frac{dt}{dx} \\ &= 15t^4 \times \frac{1}{20t^3} \\ &= \frac{3}{4}t \qquad \square \end{aligned}$$

[3 marks]

Answer 2

$$x = t^2 + t \Rightarrow \frac{dx}{dt} = 2t + 1 \Rightarrow \frac{dt}{dx} = \frac{1}{2t + 1}$$

$$y = 2t - t^2 \Rightarrow \frac{dy}{dt} = 2 - 2t$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$\frac{dy}{dx} = \frac{2 - 2t}{2t + 1}$$

[3 marks]

Answer 3

$$x = (4t + 2)^3 \Rightarrow \frac{dx}{dt} = 3(4t + 2)^2 \times 4 \Rightarrow \frac{dt}{dx} = \frac{1}{3 \times 4(4t + 2)^2}$$

$$y = (3t + 4)^2 \Rightarrow \frac{dy}{dt} = 2(3t + 4) \times 3$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$\frac{dy}{dx} = \frac{3t + 4}{2(4t + 2)^2}$$

[3 marks]

Answer 4

$$x = 2\cot\theta \Rightarrow \frac{dx}{d\theta} = -2\csc^2\theta \Rightarrow \frac{d\theta}{dx} = -\frac{\sin^2\theta}{2}$$

$$y = 3\tan\theta \Rightarrow \frac{dy}{d\theta} = 3\sec^2\theta = \frac{3}{\cos^2\theta}$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$

$$= \frac{3}{\cos^2\theta} \times \left(-\frac{\sin^2\theta}{2}\right)$$

$$= -\frac{3}{2}\tan^2\theta$$

[3 marks]

Answer 5

$$x = 3\sin\theta \Rightarrow \frac{dx}{d\theta} = 3\cos\theta \Rightarrow \frac{d\theta}{dx} = \frac{1}{3\cos\theta}$$

$$y = 5\cos\theta \Rightarrow \frac{dy}{d\theta} = -5\sin\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$

$$= -5\sin\theta \times \frac{1}{3\cos\theta}$$

$$= -\frac{5}{3}\tan\theta$$

[3 marks]

Answer 6

(i) $$x = \tan\theta \Rightarrow \frac{dx}{d\theta} = \sec^2\theta \Rightarrow \frac{d\theta}{dx} = \cos^2\theta$$

$$y = \sec\theta \Rightarrow \frac{dy}{d\theta} = \sec\theta\tan\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$

$$= \sec\theta\tan\theta\cos^2\theta$$

$$= \frac{1}{\cos\theta} \times \frac{\sin\theta}{\cos\theta} \times \frac{\cos^2\theta}{1}$$

$$= \sin\theta$$

[3 marks]

(ii) $$\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

[1 mark]

Answer 7

$$x = -\cos\theta \Rightarrow \frac{dx}{d\theta} = \sin\theta \Rightarrow \frac{d\theta}{dx} = \frac{1}{\sin\theta}$$

$$y = \theta\sin\theta \Rightarrow \frac{dy}{d\theta} = \theta\cos\theta + \sin\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$

$$= \frac{\theta\cos\theta + \sin\theta}{\sin\theta}$$

$$= \theta\cot\theta + 1 \qquad \square$$

[4 marks]

Answer 8

(i) $x = cot\,\theta \Rightarrow \dfrac{dx}{d\theta} = -csc^2\,\theta \Rightarrow \dfrac{d\theta}{dx} = -sin^2\,\theta$

$y = csc\,\theta \Rightarrow \dfrac{dy}{d\theta} = -csc\,\theta\,cot\,\theta$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$

$$= -csc\,\theta\,cot\,\theta \times \left(-sin^2\,\theta\right)$$

$$= \frac{1}{sin\,\theta} \times \frac{cos\,\theta}{sin\,\theta} \times sin^2\,\theta$$

$$= cos\,\theta$$

[3 marks]

(ii) $cos\left(\dfrac{\pi}{3}\right) = \dfrac{1}{2}$

[1 mark]

Answer 9

$x = -sin\,\theta \Rightarrow \dfrac{dx}{d\theta} = -cos\,\theta \Rightarrow \dfrac{d\theta}{dx} = -\dfrac{1}{cos\,\theta}$

$y = \theta\,cos\,\theta \Rightarrow \dfrac{dy}{d\theta} = \theta\,(-sin\,\theta\,) + cos\,\theta$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$

$$= \frac{-\theta\,sin\,\theta + cos\,\theta}{(-cos\,\theta\,)}$$

$$= \theta\,tan\,\theta - 1$$

[4 marks]

Answer 10

(i) (20, 32)

[1 mark]

(ii) $x = 5t^2 \Rightarrow \dfrac{dx}{dt} = 10t \Rightarrow \dfrac{dt}{dx} = \dfrac{1}{10t}$

$y = 4t^3 \Rightarrow \dfrac{dy}{dt} = 12t^2$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= 12t^2 \times \frac{1}{10t}$$

$$= \frac{6t}{5}$$

[3 marks]

(iii) When $t = 2$, $\dfrac{dy}{dx} = \dfrac{12}{5}$

[2 marks]

(iv)

$$y = mx + c$$

$$32 = \frac{12}{5} \times 20 + c$$

$$32 = 48 + c$$

$$c = -16$$

$$\therefore\ y = \frac{12}{5}x - 16$$

Multiply through by 5

$$5y = 12x - 80$$

$$12x - 5y - 80 = 0$$

[3 marks]

Answer 11

(i) $\left(\frac{\sqrt{3}}{2}, \ \frac{\sqrt{3}}{2} \right)$

[1 mark]

(ii) $x = cos\, t \Rightarrow \frac{dx}{dt} = -sin\, t \Rightarrow \frac{dt}{dx} = -\frac{1}{sin\, t}$

$y = sin\, 2t \Rightarrow \frac{dy}{dt} = 2\, cos\, 2t$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= -\frac{2\, cos\, 2t}{sin\, t}$$

[4 marks]

(iii) When $t = \frac{\pi}{6}$, $\frac{dy}{dx} = -\frac{2\, cos\, 2\left(\frac{\pi}{6}\right)}{sin\, \left(\frac{\pi}{6}\right)}$

$$= -\frac{2 \times \left(\frac{1}{2}\right)}{\left(\frac{1}{2}\right)}$$

$$= -2$$

[2 marks]

(iv)

$$y = mx + c$$

$$\frac{\sqrt{3}}{2} = -2 \times \left(\frac{\sqrt{3}}{2}\right) + c$$

$$c = 3\left(\frac{\sqrt{3}}{2}\right)$$

$$\therefore \ y = -2x + 3\left(\frac{\sqrt{3}}{2}\right)$$

[3 marks]