4.1 Tricky Trig Troubles

The more awkward examination questions on parametric differentiation invariably require a good grasp of the various trigonometric identities.
The following are frequently needed and not provided in the exam formulae booklet.

$cos^2\,\theta + sin^2\,\theta = 1$		$cos\,2\theta = cos^2\,\theta - sin^2\,\theta$
$1 + tan^2\,\theta = sec^2\,\theta$		$sin\,2\theta = 2\,sin\,\theta\,cos\,\theta$
$cot^2\,\theta + 1 = csc^2\,\theta$		$tan\,2\theta = \dfrac{2\,tan\,\theta}{1 - tan^2\,\theta}$

4.2 Example

For the parametric equations $x = sin^2\,\theta$

and $y = cos\,2\theta$

(i) Show that $\dfrac{dy}{dx} = -2$

(ii) Sketch the graph of the parametric equations.

Teaching Video : http://www.NumberWonder.co.uk/v9081/4.mp4

[3, 3 marks]

4.3 Exercise

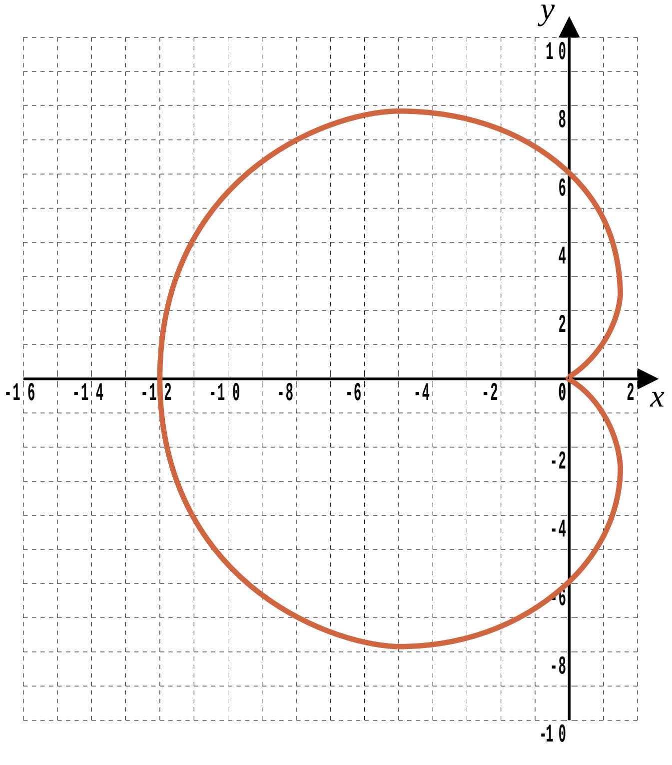
Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 60

Question 1

In the photograph, on the bottom of the empty cereal bowl, an interesting shape can be seen, made up from the light reflecting off the sides. This is an example of a caustic. A caustic is an envelope of light rays formed by reflection or refraction from a curved surface. More complicated caustics can be seen dancing about on the bottom of swimming pools, for example.

Photograph by Martin Hansen

The cereal bowl caustic is in the shape of a cardioid, from a Greek word that translates as “heart”. The graph below is of a perfect mathematical cardioid,



The cardioid in the graph was plotted from the parametric equations,

$$x = 6(1 - \cos\theta)\cos\theta$$

$$y = 6(1 - \cos\theta)\sin\theta$$

(i) What point corresponds to $\theta = \pi$?
Mark this point on the graph and label it A

[2 marks]

(ii) Without making any calculations, write down the equation of the tangent to the cardioid at point A and add this tangent to the graph.

[2 mark]

(iii) What point corresponds to $\theta = \frac{\pi}{2}$?
Mark this point on the graph and label it B

[2 marks]

(iv) Show that $\frac{dy}{dx} = \frac{\cos\theta - \cos 2\theta}{\sin 2\theta - \sin\theta}$

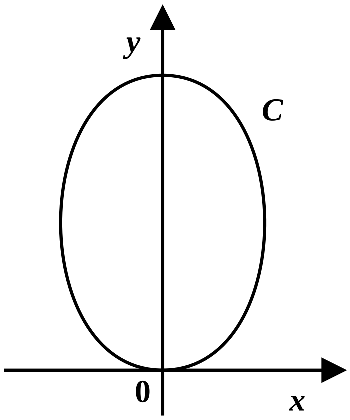
[6 marks]

(v) Find the equation of the tangent to the cardioid at point B
Draw this tangent on the graph.

[4 marks]

Question 2

A-Level Examination Question from June 2012, Paper C4, Q6 (Edexcel)



The curve C has parametric equations

$$x = \sqrt{3} \sin 2t, \qquad y = 4\cos^2 t, \qquad 0 \leqslant t \leqslant \pi$$

(a) Show that $\dfrac{dy}{dx} = k\sqrt{3} \tan 2t$ where k is a constant to be determined

[5 marks]

(b) Find the equation of the tangent to C at the point where $t = \dfrac{\pi}{3}$

Give your answer in the form $y = ax + b$, where a and b are constants

[4 marks]

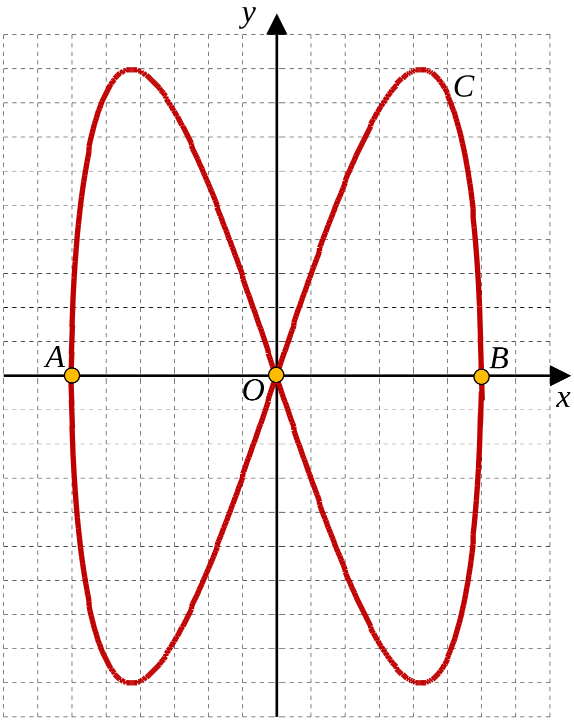
(c) Find a Cartesian equation of C

(Note : Any algebra that combines the parametric equations and then leads to an answer that has x and y in it, and no t, gets full marks)

[3 marks]

Question 3

A-Level Examination Question from January 2018, Paper C34, Q11 (Edexcel)



The curve C has parametric equations,

$$x = 3\cos t \qquad y = 9\sin 2t \qquad 0 \leqslant t \leqslant 2\pi$$

The curve C meets the x-axis at the origin and at the points A and B, as shown.

(a) Write down the coordinates of A and B

[2 marks]

(b) Find the values of t at which the curve passes through the origin.

[2 marks]

(c) Find an expression for $\dfrac{dy}{dx}$ in terms of t, and hence find the gradient of the curve when $t = \dfrac{\pi}{6}$

[4 marks]

(d) Show that the Cartesian equation for the curve C can be written in the form

$$y^2 = a x^2 (b - x^2)$$

where a and b are integers to be determined

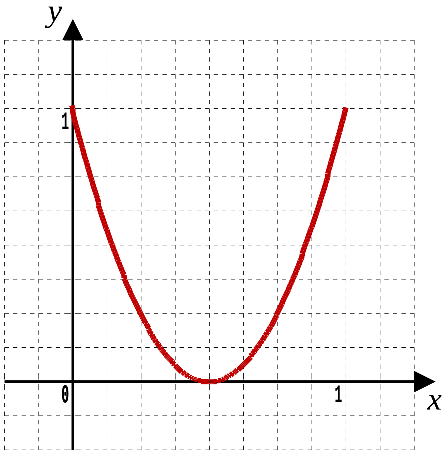
[4 marks]

Question 4

The graph is of the parametric equations,

$$x = \sin^2 \theta$$

$$y = \cos^2 2\theta$$



When $\theta = 0$, the corresponding point on the graph is (0, 1)

As θ increases the point moves along the path and is at (0.5, 0) when $\theta = \dfrac{\pi}{4}$

Further increase in θ continues the journey toward (1, 1) reached when $\theta = \dfrac{\pi}{2}$

The direction of travel now reverses and the point heads back the way it came.

(i) Determine the Cartesian equation of the curve.

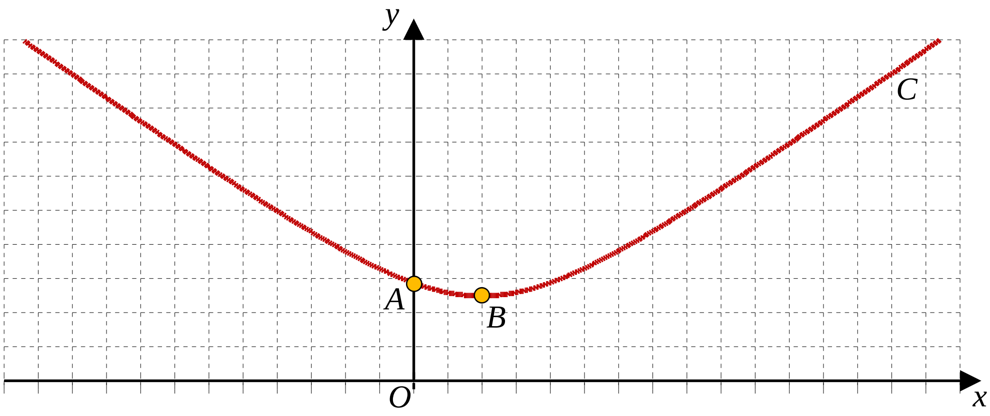
[4 marks]

(ii) Show that $\dfrac{dy}{dx} = -\dfrac{2 \sin 4\theta}{\sin 2\theta} \qquad \theta \neq \dfrac{n\pi}{2}, \ n \in \mathbb{Z}$

[4 marks]

Question 5

A-Level Examination Question from January 2017, Paper C34, Q13 (Edexcel)



The curve shown has parametric equations,

$$x = 1 + \sqrt{3}\tan\theta, \qquad y = 5\sec\theta, \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

The curve C crosses the y-axis at A and has a minimum turning point at B, as shown.

(a) Find the exact coordinates of A

[3 marks]

(b) Show that $\dfrac{dy}{dx} = \lambda \sin\theta$, giving the exact value of the constant λ

[4 marks]

(c) Find the coordinates of B

[2 marks]

(d) Show that the Cartesian equation for the curve C can be written in the form

$$y = k\sqrt{x^2 - 2x + 4}$$

where k is a simplified surd to be found

[3 marks]

4.4 Answers

4.4.1 Solution to 4.2 Example (Teaching Video)

(i) $x = \sin^2\theta \Rightarrow \frac{dx}{d\theta} = 2\sin\theta\cos\theta \Rightarrow \frac{d\theta}{dx} = \frac{1}{\sin 2\theta}$

$y = \cos 2\theta \Rightarrow \frac{dy}{d\theta} = -2\sin 2\theta$

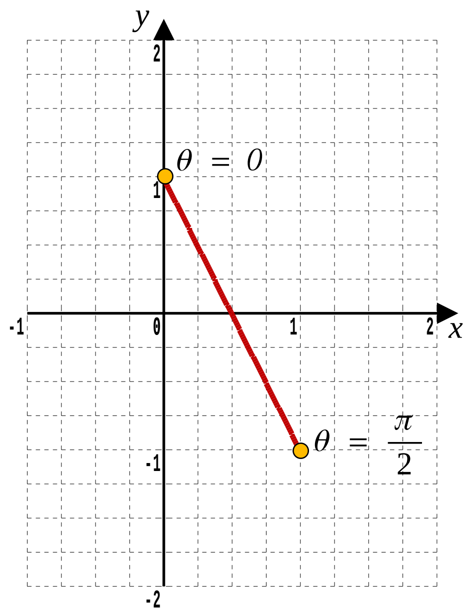
$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$
$$= -2\sin 2\theta \times \frac{1}{\sin 2\theta}$$
$$= -2$$

[3 marks]

(ii) The graph is a straight line with gradient – 2

- As x is the square of a function it can never be negative
- Plus, as x is the *sine* of a function, $0 \leqslant x \leqslant 1$
- As y is the *cosine* of a function, $-1 \leqslant y \leqslant 1$
- When $\theta = 0$ the point is $(0, 1)$
- When $\theta = \pi$ the point is $(1, -1)$

Putting this altogether gives the graph as $y = -2x + 1, \quad 0 \leqslant x \leqslant 1$



[3 marks]

An alternative “clever” solution...

$$\begin{aligned} x &= sin^2\,\theta \\ y &= cos^2\,\theta - sin^2\,\theta \\ &= \left(1 - sin^2\,\theta\right) - sin^2\,\theta \\ &= 1 - 2\,sin^2\,\theta \\ &= 1 - 2x \end{aligned}$$

which is a straight line with gradient – 2

4.4.2 Solutions to 4.3 Exercise

Answer 1

(i) $x = 6(1 - \cos\pi)\cos\pi$ $\quad$ $y = 6(1 - \cos\pi)\sin\pi$

$= 6(1 - (-1))(-1)$ $\quad$ $= 6(1 - (-1))\,0$

$= -12$ $\quad$ $= 0$

The point corresponding to $\theta = \pi$ is $A(-12, 0)$

See the graph on the next page

[2 marks]

(ii) $x = -12$

See the graph on the next page

[2 marks]

(iii) $x = 6\left(1 - \cos\left(\frac{\pi}{2}\right)\right)\cos\left(\frac{\pi}{2}\right)$ $\quad$ $y = 6\left(1 - \cos\left(\frac{\pi}{2}\right)\right)\sin\left(\frac{\pi}{2}\right)$

$= 6(1 - 0)\,0$ $\quad$ $= 6(1 - 0)\,1$

$= 0$ $\quad$ $= 6$

The point corresponding to $\theta = \frac{\pi}{2}$ is $B(0, 6)$

See the graph on the next page

[2 marks]

(iv)

$$x = 6(1 - \cos\theta)\cos\theta$$
$$= 6\cos\theta - 6\cos^2\theta$$
$$\frac{dx}{d\theta} = -6\sin\theta + 6 \times 2\cos\theta\sin\theta$$
$$= 6\sin 2\theta - 6\sin\theta$$

$$y = 6(1 - \cos\theta)\sin\theta$$
$$= 6\sin\theta - 3 \times 2\sin\theta\cos\theta$$
$$= 6\sin\theta - 3\sin 2\theta$$
$$\frac{dy}{d\theta} = 6\cos\theta - 6\cos 2\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$
$$= \frac{6\cos\theta - 6\cos 2\theta}{6\sin 2\theta - 6\sin\theta}$$
$$= \frac{\cos\theta - \cos 2\theta}{\sin 2\theta - \sin\theta} \quad \square$$

[6 marks]

(v) B corresponded to the point $(\,0,\ 6\,)$ when $\theta = \dfrac{\pi}{2}$

Working out the gradient at this point,

$$\frac{dy}{dx} = \frac{\cos\left(\frac{\pi}{2}\right) - \cos(\pi)}{\sin(\pi) - \sin\left(\frac{\pi}{2}\right)}$$

$$= \frac{0 - (-1)}{0 - 1}$$

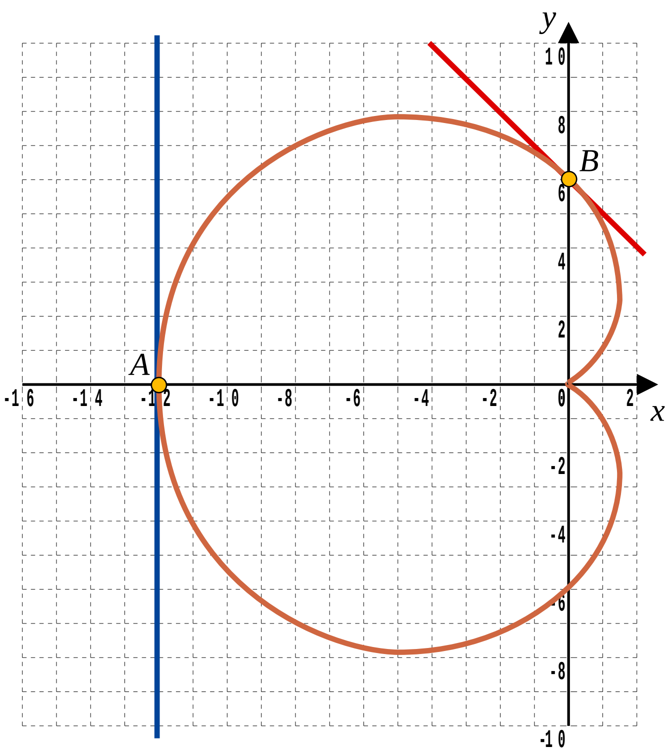
$$= -1$$

Thus the equation of the tangent to the curve at $B(\,0,\,6\,)$ is

$$y = -x + 6$$

See the graph below

[4 marks]



Answer 2

(a) $x = \sqrt{3}\,sin\,2t \;\Rightarrow\; \dfrac{dx}{dt} = 2\sqrt{3}\,cos\,2t$

$y = 4\,cos^2\,t \;\Rightarrow\; \dfrac{dy}{dt} = -4 \times 2\,cos\,t\,sin\,t$

$= -4\,sin\,2t$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= -4\,sin\,2t \times \frac{1}{2\sqrt{3}\,cos\,2t}$$

$$= -\frac{2}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}\,tan\,2t$$

$$= -\frac{2}{3}\sqrt{3}\,tan\,2t$$

That is, $k = -\dfrac{2}{3}$

[5 marks]

(b) When $t = \dfrac{\pi}{3}$ $\quad x = \sqrt{3}\,sin\left(\dfrac{2\pi}{3}\right) \Rightarrow x = \dfrac{3}{2}$

$y = 4\,cos^2\left(\dfrac{\pi}{3}\right) \Rightarrow y = 1$

$$\frac{dy}{dx} = -\frac{2}{3}\sqrt{3}\,tan\left(\frac{2\pi}{3}\right) \Rightarrow \frac{dy}{dx} = 2$$

$y = mx + c$

$1 = 2 \times \dfrac{3}{2} + c$

$c = -2$

$\therefore\; y = 2x - 2$

[4 marks]

(c) Clever Method

$$\begin{aligned} y &= 4\,cos^2 t \\ &= 2\,cos^2 t + 2\,cos^2 t \\ &= 2\,cos^2 t + 2\,sin^2 t + 2\,cos^2 t - 2\,sin^2 t \\ &= 2 + 2\,cos\,2t \\ \therefore\ cos\,2t &= \frac{y-2}{2} \end{aligned}$$

$$\begin{aligned} x &= \sqrt{3}\,sin\,2t \\ sin\,2t &= \frac{x}{\sqrt{3}} \end{aligned}$$

$$\begin{aligned} cos^2 2t + sin^2 2t &= 1 \\ \frac{(y-2)^2}{4} + \frac{x^2}{3} &= 1 \\ 3\,(y-2)^2 + 4x^2 &= 12 \end{aligned}$$

which is the equation of an ellipse

More Direct Method

$$\begin{aligned} x &= \sqrt{3}\,sin\,2t \\ &= 2\sqrt{3}\,sin\,t\,cos\,t \\ x^2 &= 12\,sin^2 t\,cos^2 t \\ &= 12\left(1 - cos^2 t\right)cos^2 t \\ &= 12\left(1 - \frac{y}{4}\right)\left(\frac{y}{4}\right) \end{aligned}$$

[3 marks]

Answer 3

(a) The curve will cross the x-axis when $y = 0$

$$\therefore y = 9\,sin\,2t \qquad 0 \leqslant t \leqslant 2\pi$$

$$9\,sin\,2t = 0$$

$$sin\,2t = 0$$

$$2t = 0,\ \pi,\ 2\pi,\ 3\pi,\ 4\pi,\ \ldots$$

$$t = 0,\ \frac{\pi}{2},\ \pi,\ \frac{3\pi}{2},\ 2\pi,\ \ldots$$

These values of t can now be inserted into the x equation to get points, remembering that all these t values correspond to a $y = 0$

Thus, using $x = 3\,cos\,t$ yields the points,

$$(3,\ 0),\ (0,\ 0),\ (-3,\ 0),\ (0,\ 0),\ (3,\ 0),\ \ldots$$

Clearly, $A(-3,\ 0)$ and $B(3,\ 0)$

[2 marks]

(b) From (a), the curve passes through $(0,\ 0)$ when $t = \frac{\pi}{2},\ \frac{3\pi}{2}$

[2 marks]

(c) $x = 3\,cos\,t \quad \Rightarrow \quad \frac{dx}{dt} = -3\,sin\,t$

$y = 9\,sin\,2t \quad \Rightarrow \quad \frac{dy}{dt} = 18\,cos\,2t$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= 18\,cos\,2t \times \frac{1}{(-3\,sin\,t)}$$

$$= -\frac{6\,cos\,2t}{sin\,t}$$

When $t = \frac{\pi}{6}$, $\frac{dy}{dx} = -\frac{6 \times \frac{1}{2}}{\frac{1}{2}}$

$$= -6$$

[4 marks]

(d) $x = 3\cos t \Rightarrow \cos t = \dfrac{x}{3}$

$y = 9\sin 2t \Rightarrow y = 18\sin t \cot t$

$$\begin{aligned} y^2 &= 18^2 \sin^2 t \cos^2 t \\ &= 18^2 \left(1 - \cos^2 t \right) \cos^2 t \\ &= 18^2 \left(1 - \frac{x^2}{9} \right)\left(\frac{x^2}{9} \right) \\ &= 4x^2 \left(9 - x^2 \right) \end{aligned}$$

That is, $a = 4$ and $b = 9$

[4 marks]

Answer 4

(i) $x = \sin^2 \theta$

$$\begin{aligned} y &= \cos^2 2\theta \\ &= (\cos 2\theta)^2 \\ &= \left(\cos^2 \theta - \sin^2 \theta \right)^2 \\ &= \left(1 - \sin^2 \theta - \sin^2 \theta \right)^2 \\ &= \left(1 - 2\sin^2 \theta \right)^2 \\ &= (1 - 2x)^2 \\ &= 1 - 4x + 4x^2 \end{aligned}$$

[4 marks]

(ii) $x = \sin^2 \theta \quad \Rightarrow \dfrac{dx}{d\theta} = 2\sin\theta\cos\theta$

$= \sin 2\theta$

$y = \cos^2 2\theta \Rightarrow \dfrac{dy}{d\theta} = 2\cos 2\theta \left(-\sin 2\theta \right) \times 2$

$= -2\sin 4\theta$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$

$$\therefore \frac{dy}{dx} = -\frac{2\sin 4\theta}{\sin 2\theta} \qquad \theta \neq \frac{n\pi}{2} \;\; n \in \mathbb{Z} \qquad \square$$

[4 marks]

Answer 5

(a) The curve will cross the y- axis when $x = 0$,

i.e. When $1 + \sqrt{3}\ tan\ \theta = 0$

$$tan\ \theta = -\frac{1}{\sqrt{3}}$$

$$\theta = -\frac{\pi}{6} \quad : \text{ The only solution } -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$y = 5\,sec\ \theta$$

$$= \frac{5}{cos\left(\frac{\pi}{6}\right)}$$

$$= \frac{10\sqrt{3}}{3}$$

The coordinates of A are thus $\left(0,\ \frac{10\sqrt{3}}{3}\right)$

[3 marks]

(b) $x = 1 + \sqrt{3}\ tan\ \theta \Rightarrow \frac{dx}{d\theta} = \sqrt{3}\ sec^2\ \theta$

$y = 5\,sec\ \theta \Rightarrow \frac{dy}{d\theta} = 5\,sec\ \theta\ tan\ \theta$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$

$$= 5\,sec\ \theta\ tan\ \theta \times \frac{1}{\sqrt{3}\ sec^2\ \theta}$$

$$= 5\,tan\ \theta \times \frac{1}{\sqrt{3}\ sec\ \theta}$$

$$= 5\,\frac{sin\ \theta}{cos\ \theta} \times \frac{cos\ \theta}{\sqrt{3}}$$

$$= \frac{5}{\sqrt{3}}\,sin\ \theta$$

$$= \frac{5\sqrt{3}}{3}\,sin\ \theta \qquad \therefore \lambda = \frac{5\sqrt{3}}{3}$$

[4 marks]

(c) The minimum point is where $\frac{dy}{dx} = 0 \Rightarrow \frac{5\sqrt{3}}{3}\,sin\ \theta = 0$

$$sin\ \theta = 0$$

$$\theta = 0$$

Thus, B will be the point $(1,\ 5)$

(Note that the graph has different scales on the x and y axis)

[2 marks]

(d) “Many Little Steps” Method

$$y = 5\,sec\,\theta \Rightarrow y = \frac{5}{cos\,\theta}$$

$$x = 1 + \sqrt{3}\,tan\,\theta$$

$$x - 1 = \sqrt{3} \times \frac{sin\,\theta}{cos\,\theta}$$

$$(x - 1)^2 = 3 \times \frac{sin^2\,\theta}{cos^2\,\theta}$$

$$\frac{x^2 - 2x + 1}{3} = \frac{1 - cos^2\,\theta}{cos^2\,\theta}$$

$$\frac{x^2 - 2x + 1}{3} = \frac{1}{cos^2\,\theta} - \frac{cos^2\,\theta}{cos^2\,\theta}$$

$$\frac{x^2 - 2x + 1}{3} = \frac{25}{25} \times \frac{1}{cos^2\,\theta} - 1$$

$$\frac{x^2 - 2x + 1}{3} + 1 = \frac{1}{25} \times \frac{25}{cos^2\,\theta}$$

$$\frac{x^2 - 2x + 1}{3} + \frac{3}{3} = \frac{1}{25} \times y^2$$

$$25\left(\frac{x^2 - 2x + 4}{3}\right) = y^2$$

$$y = \pm \frac{5}{\sqrt{3}} \sqrt{x^2 - 2x + 4}$$

$$= \pm \frac{5\sqrt{3}}{3} \sqrt{x^2 - 2x + 4}$$

That is, $k = \pm \dfrac{5\sqrt{3}}{3}$

"Trigonometric Formula" Method

$$x = 1 + \sqrt{3}\ tan\ \theta$$

$$\therefore\ tan\ \theta = \frac{x - 1}{\sqrt{3}}$$

$$y = 5\,sec\ \theta$$

$$y^2 = 25\,sec^2\ \theta$$

$$= 25\left(1 + tan^2\ \theta\right)$$

$$= 25\left(1 + \left(\frac{x - 1}{\sqrt{3}}\right)^2\right)$$

$$= 25\left(\frac{3}{3} + \frac{x^2 - 2x + 1}{3}\right)$$

$$= \frac{25}{3}\left(x^2 - 2x + 4\right)$$

$$y = \pm\frac{5}{\sqrt{3}}\sqrt{x^2 - 2x + 4}$$

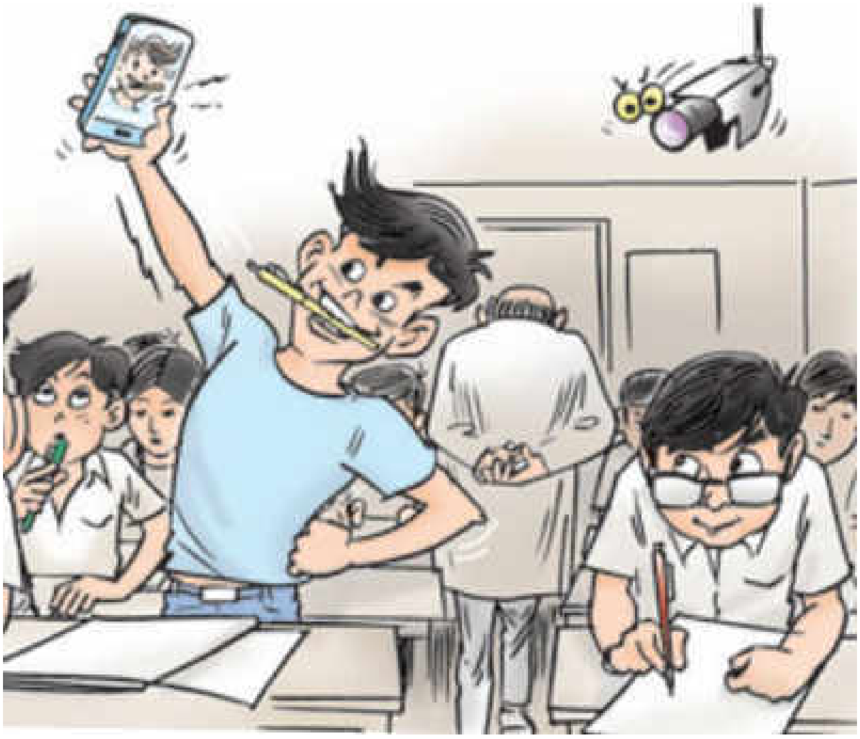
$$= \pm\frac{5\sqrt{3}}{3}\sqrt{x^2 - 2x + 4}$$

That is, $k = \pm\dfrac{5\sqrt{3}}{3}$

[3 marks]

Lesson 5

5.1 From The Examination Hall



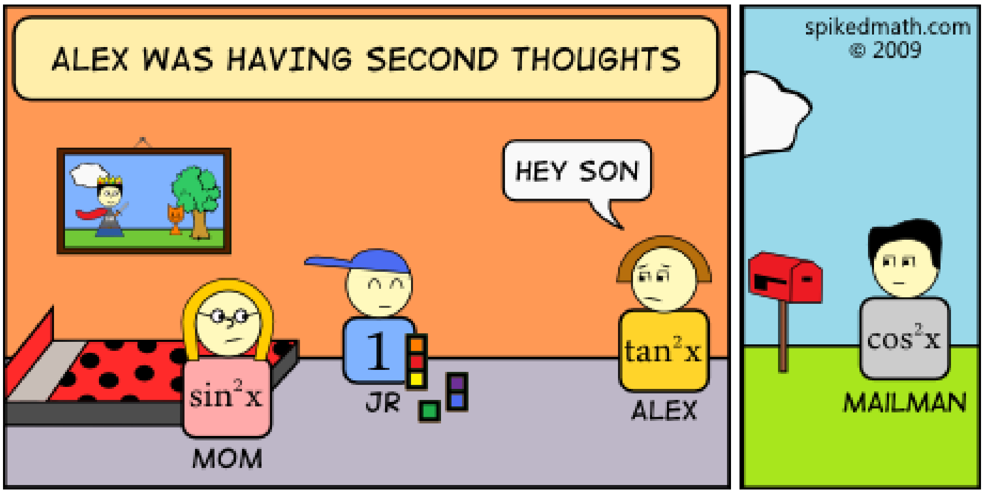
Exam Regulation #284 : No taking of selfies in the Examination Hall

It is highly likely that in your examination there will be a question involving parametric equations and so this lesson is a further practice session using old examination questions.

In the examination you may look up the derivatives of many functions in the examination formulae book but don't forget that many of the most useful trig identities are not given so you either need to remember how to derive them or commit them to memory.

5.2 The NOT Provided Trigonometric Identities

$\cos^2\theta + \sin^2\theta = 1$		$\cos 2\theta = \cos^2\theta - \sin^2\theta$
$1 + \tan^2\theta = \sec^2\theta$		$\sin 2\theta = 2\sin\theta\cos\theta$
$\cot^2\theta + 1 = \csc^2\theta$		$\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$



5.3 Table of Standard Derivatives

$f(x)$	$f'(x)$	In Formula Book ?
x^n	$n x^{n-1}$	No
e^x	e^x	No
$\ln x$	$\frac{1}{x}$	No
$\sin x$	$\cos x$	No
$\cos x$	$-\sin x$	No
$\tan x$	$\sec^2 x$	Yes
$\csc x$	$-\csc x \cot x$	Yes
$\sec x$	$\sec x \tan x$	Yes
$\cot x$	$-\csc^2 x$	Yes
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$	Yes
$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$	Yes
$\arctan x$	$\frac{1}{1+x^2}$	Yes

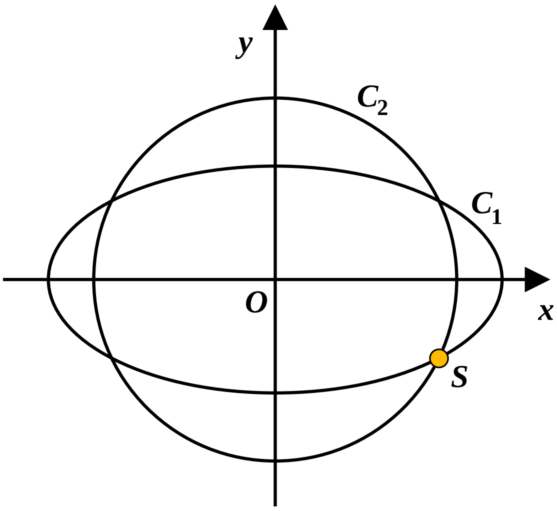
When trigonometry and calculus mix, θ must be in **RADIANS**

5.4 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 45

Question 1

A-Level Examination Question from June 2019, Paper 2, Q4 (Edexcel)



The curve C_1 with parametric equations

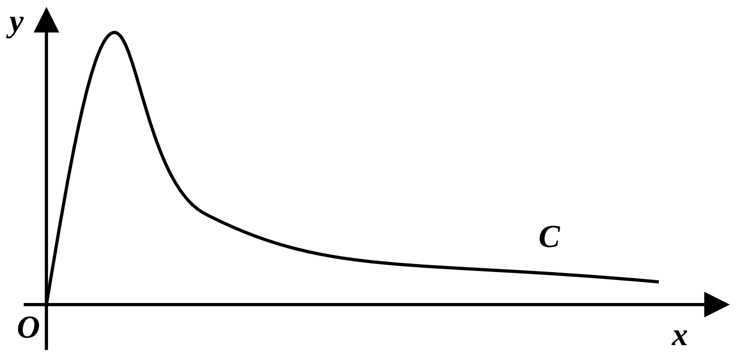
$$x = 10\cos t, \qquad y = 4\sqrt{2}\sin t, \qquad 0 \leqslant t < 2\pi$$

meets the circle C_2 with equation $x^2 + y^2 = 66$ at four distinct points as shown.
Given that one of these points, S, lies in the 4th quadrant, find the Cartesian coordinates of S

[6 marks]

Question 2

A-Level Examination Question from June 2016, Paper C4, Q5 (Edexcel)



The sketch is of the curve C with parametric equations

$$x = 4\tan t, \qquad y = 5\sqrt{3}\sin 2t, \qquad 0 \leqslant t < \frac{\pi}{2}$$

The point P lies on C and has coordinates $\left(4\sqrt{3}, \dfrac{15}{2}\right)$

(a) Find the exact value of $\dfrac{dy}{dx}$ at the point P.

Give your answer as a simplified surd.

[4 marks]

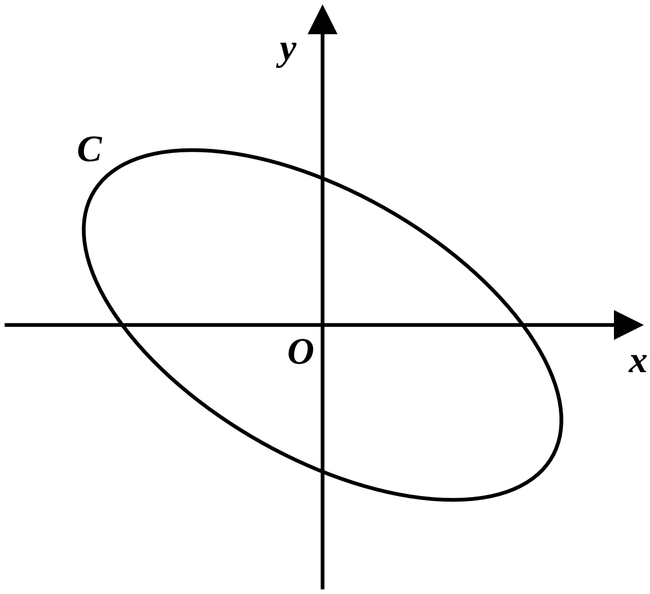
The point Q lies on the curve C, where $\dfrac{dy}{dx} = 0$

(b) Find the exact coordinates of the point Q

[2 marks]

Question 3

A-Level Specimen Paper 1 for the June 2021 (cancelled) Examination, Q14 (Edexcel)



The sketch is of the curve C with parametric equations

$$x = 4\cos\left(t + \frac{\pi}{6}\right), \qquad y = 2\sin t, \qquad 0 < t \leqslant 2\pi$$

Show that a Cartesian equation of C can be written in the form

$$(x + y)^2 + ay^2 = b$$

where a and b are integers to be found.

[5 marks]

Question 4

The curve C has parametric equations

$$x = \sec t, \qquad y = \tan t, \qquad 0 \leqslant t \leqslant \frac{\pi}{2}$$

(a) Prove that $\dfrac{dy}{dx} = \csc t$

[4 marks]

(b) Find the equation in the form $y = mx + c$ of the tangent to C at the point where $t = \dfrac{\pi}{4}$

[4 marks]

Question 5

A-Level Examination Question from June 2010, Paper C4, Q4 (Edexcel)

A curve C has parametric equations

$$x = sin^2 t, \qquad y = 2\, tan\, t, \qquad 0 \leqslant t \leqslant \frac{\pi}{2}$$

(a) Find $\frac{dy}{dx}$ in terms of t

[4 marks]

The tangent to C at the point where $t = \frac{\pi}{3}$ cuts the x-axis at the point P.

(b) Find the x-coordinate of P

[6 marks]

Question 6

A-Level Examination Question from June 2018, Paper 1, Q14 (Edexcel)

A curve C has parametric Equations

$$x = 3 + 2\sin t, \qquad y = 4 + 2\cos 2t, \qquad 0 \leqslant t < 2\pi$$

(a) Show that all points of C satisfy $y = 6 - (x - 3)^2$

[2 marks]

(b) **(i)** Sketch the curve C

(ii) Explain briefly why C does not include all points of

$$y = 6 - (x - 3)^2, \qquad x \in \mathbb{R}$$

[3 marks in total for (b)]

The line with equation $x + y = k$, where k is a constant, intersects C at two distinct points.

(c) State the range of values of k, writing your answer in set notation.

[5 marks]

5.5 Answers to 5.4 Exercise

Answer 1

There are several ways of doing this question.
The following is most like the method used in previous questions where it's easy to get one equation for $\cos t$ and another for $\sin t$ these then being combined by use of the trigonometric identity $\cos^2 x + \sin^2 x = 1$

$$x = 10\cos t \quad \Rightarrow \quad \cos t = \frac{x}{10}$$

$$y = 4\sqrt{2}\,\sin t \quad \Rightarrow \quad \sin t = \frac{y}{4\sqrt{2}}$$

$$\cos^2 t + \sin^2 t = 1$$

$$\left(\frac{x}{10}\right)^2 + \left(\frac{y}{4\sqrt{2}}\right)^2 = 1$$

$$\frac{x^2}{100} + \frac{y^2}{32} = 1$$

Multiply through by 100×32

$$32x^2 + 100y^2 = 100 \times 32$$

Divide through by 4

$$8x^2 + 25y^2 = 800$$

$$8(66 - y^2) + 25y^2 = 800 \qquad \text{From the } C_2 \text{ equation}$$

$$528 - 8y^2 + 25y^2 = 800$$

$$17y^2 = 272$$

$$y^2 = 16$$

$$y = \pm 4$$

Using the C_2 equation as the easiest route to get x,

$$x^2 + y^2 = 66$$

$$x^2 = 66 - 16$$

$$x = \pm\sqrt{50}$$

$$= \pm 5\sqrt{2}$$

As S is in the fourth quadrant, $S(5\sqrt{2}, -4)$

[6 marks]

Answer 2

(a) $x = 4\,tan\,t \quad \Rightarrow \quad \frac{dx}{dt} = 4\,sec^2\,t \;\Rightarrow\; \frac{dt}{dx} = \frac{1}{4}\,cos^2\,t$

$y = 5\sqrt{3}\;sin\,2t \quad \Rightarrow \frac{dy}{dt} = 10\sqrt{3}\;cos\,2t$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= 10\sqrt{3}\;cos\,2t \;\times\; \frac{1}{4}\,cos^2\,t$$

$$= \frac{5\sqrt{3}}{2}\,cos\,2t\,cos^2\,t$$

As the coordinates of P are known,

$$4\sqrt{3} = 4\,tan\,t$$

$$t = arctan\left(\sqrt{3}\right)$$

$$= \frac{\pi}{3}$$

$$\frac{dy}{dx} = \frac{5\sqrt{3}}{2}\,cos\left(\frac{2\pi}{3}\right)\;\left(cos\left(\frac{\pi}{3}\right)\right)^2$$

$$= \frac{5\sqrt{3}}{2}\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)^2$$

$$= -\frac{5\sqrt{3}}{16}$$

[4 marks]

(b) $\frac{dy}{dx} = 0$

$$\frac{5\sqrt{3}}{2}\,cos\,2t\,cos^2\,t = 0$$

Either $cos\,2t = 0$ or $cos^2\,t = 0$

For $cos^2\,t = 0 \;\Rightarrow\; cos\,t = 0 \;\Rightarrow\; t = \frac{\pi}{2}$

but this is outside the domain specified and so is not a solution

That leaves $cos\,2t = 0 \;\Rightarrow\; 2t = \frac{\pi}{2} \;\Rightarrow\; t = \frac{\pi}{4}$

This value of t is now used to yield the point, $Q\left(4,\; 5\sqrt{3}\right)$

[2 marks]

Answer 3

$$\begin{aligned} x &= 4\cos\left(t + \frac{\pi}{6}\right) \\ &= 4\cos t\cos\left(\frac{\pi}{6}\right) - 4\sin t\sin\left(\frac{\pi}{6}\right) \\ &= 2\sqrt{3}\cos t - 2\sin t \qquad \text{but} \qquad y = 2\sin t \\ x &= 2\sqrt{3}\cos t - y \\ x + y &= 2\sqrt{3}\cos t \\ \cos t &= \frac{x+y}{2\sqrt{3}} \end{aligned}$$

$$\cos^2 t + \sin^2 t = 1$$

$$\left(\frac{x+y}{2\sqrt{3}}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$$

$$\frac{(x+y)^2}{12} + \frac{y^2}{4} = 1$$

Multiply through by 12

$$(x+y)^2 + 3y^2 = 12$$

That is $a = 3$ and $b = 12$

[5 marks]

Answer 4

(a) $x = sec\, t \Rightarrow \dfrac{dx}{dt} = sec\, t\, tan\, t$

$y = tan\, t \Rightarrow \dfrac{dy}{dt} = sec^2\, t$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= sec^2\, t \times \frac{1}{sec\, t\, tan\, t}$$

$$= sec\, t \times cot\, t$$

$$= \frac{1}{cos\, t} \times \frac{cos\, t}{sin\, t}$$

$$= \frac{1}{sin\, t}$$

$$= csc\, t \qquad \square$$

[4 marks]

(b) When $t = \dfrac{\pi}{4}$

$x = sec\left(\dfrac{\pi}{4}\right) \Rightarrow x = \dfrac{1}{cos\left(\frac{\pi}{4}\right)} \Rightarrow x = \sqrt{2}$

$y = tan\left(\dfrac{\pi}{4}\right) \Rightarrow y = 1$

$\dfrac{dy}{dx} = csc\left(\dfrac{\pi}{4}\right) \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{sin\left(\frac{\pi}{4}\right)} \Rightarrow \dfrac{dy}{dx} = \sqrt{2}$

$$y = mx + c$$

$$1 = \sqrt{2} \times \sqrt{2} + c$$

$$c = -1$$

The equation of the tangent to C at $\left(\sqrt{2},\ 1\right)$ is thus

$$y = \sqrt{2}\, x - 1$$

[4 marks]

Answer 5

(a) $x = sin^2 t \Rightarrow \dfrac{dx}{dt} = 2 \sin t \cos t$

$$y = 2 \tan t \Rightarrow \frac{dy}{dt} = 2 \sec^2 t$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= 2 \sec^2 t \times \frac{1}{2 \sin t \cos t}$$

$$= \sec^2 t \csc t \sec t$$

$$= \sec^3 t \csc t$$

$$= \frac{1}{\cos^3 t \sin t}$$

[4 marks]

(b) When $t = \dfrac{\pi}{3}$

$$x = \sin^2\left(\frac{\pi}{3}\right) \Rightarrow x = \frac{3}{4}$$

$$y = 2 \tan\left(\frac{\pi}{3}\right) \Rightarrow y = 2\sqrt{3}$$

$$\frac{dy}{dx} = \frac{1}{\cos^3\left(\frac{\pi}{3}\right) \sin\left(\frac{\pi}{3}\right)} \Rightarrow \frac{dy}{dx} = 8 \times \frac{2}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{16\sqrt{3}}{3}$$

$$y = mx + c$$

$$2\sqrt{3} = \frac{16\sqrt{3}}{3} \times \frac{3}{4} + c$$

$$2\sqrt{3} = 4\sqrt{3} + c$$

$$c = -2\sqrt{3}$$

The equation of the tangent to C at $\left(\dfrac{3}{4},\ 2\sqrt{3}\right)$ is thus

$$y = \frac{16\sqrt{3}}{3} x - 2\sqrt{3}$$

This will cut the x-axis when $y = 0$

$$\frac{16\sqrt{3}}{3} x = 2\sqrt{3}$$

$$x = \frac{3}{8}$$

That is, at $\left(\dfrac{3}{8},\ 0\right)$

[6 marks]

Answer 6

(a)

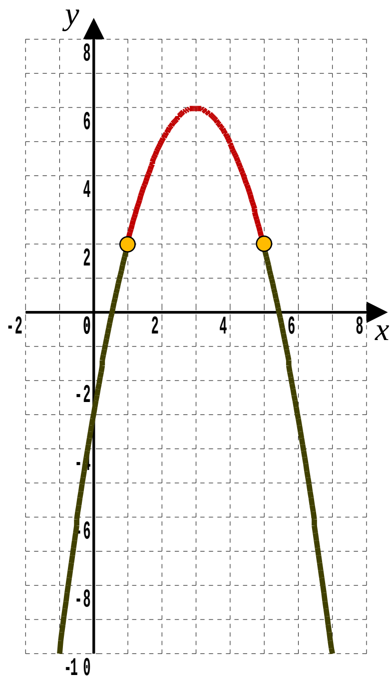
$$\begin{aligned}
\text{RHS} &= 6 - (x - 3)^2 \\
&= 6 - (3 + 2\sin t - 3)^2 \\
&= 6 - \left(4\sin^2 t\right) \\
&= 4 + (2) - 4\sin^2 t \qquad \text{As the parametric for } y \text{ has an isolated 4} \\
&= 4 + \left(2\cos^2 t + 2\sin^2 t\right) - 4\sin^2 t \\
&= 4 + 2\cos^2 t - 2\sin^2 t \\
&= 4 + 2\left(\cos^2 t - \sin^2 t\right) \\
&= 4 + 2\cos 2t \\
&= y \\
&= \text{LHS} \qquad \square
\end{aligned}$$

[2 marks]

(b) **(i)** It's an upside down parabola due to the $-x^2$ term

It has a maximum at (3, 6) from the given completed square form

(ii) As $-1 \leqslant \sin t \leqslant 1 \Rightarrow 1 \leqslant x \leqslant 5$



It's the red part of the graph that is wanted.
The two green parts cannot be reached by
the parametric equations.

[3 marks]

(c) $x + y = k \Rightarrow y = -x + k$

This is a line loping downwards at 45°

The lower bound on k will be when the line goes through the endpoint when $x = 5$.

$$y = 6 - (5 - 3)^2$$
$$= 2$$
$$y = -x + k$$
$$2 = -5 + k$$
$$k = 7$$

For the upper bound, we need then the curve has a gradient of – 1 corresponding to the line being a tangent to the curve.

$$y = 6 - (x - 3)^2$$
$$= 6 - \left(x^2 - 6x + 9\right)$$
$$= -x^2 + 6x - 3$$
$$\frac{dy}{dx} = -2x + 6$$

Thus, for $-2x + 6 = -1$

$$-2x = -7$$
$$x = \frac{7}{2}$$
$$y = 6 - \left(\frac{7}{2} - 3\right)^2$$
$$= 6 - \frac{1}{4}$$
$$= \frac{23}{4}$$
$$y = -x + k$$
$$\frac{23}{4} = -\frac{7}{2} + k$$
$$k = \frac{37}{4}$$

Thus $7 \leqslant k < \frac{37}{4}$

[5 marks]

Lesson 6

A-Level Pure Mathematics : Year 2
Differentiation IV

6.1 Practice Makes Progression (Homework)

Any solution based entirely on graphical or numerical methods is not acceptable
Marks Available : 26

Question 1

A-Level Specimen Exam Question from June 2000, Paper P3, Q4 edited (Edexcel)

A curve is given by parametric equations

$$x = 4\sin^3 t, \qquad y = \cos 2t, \qquad 0 \leqslant t \leqslant \frac{\pi}{4}$$

(a) Show that $\dfrac{dy}{dx} = -\dfrac{1}{3\sin t}$

[4 marks]

(b) Find an equation of the normal to the curve where $t = \dfrac{\pi}{6}$

[4 marks]

Question 2

A-Level Examination Question from June 2007, Paper C4, Q6 edited (Edexcel)

A curve has parametric equations

$$x = tan^2 t, \qquad y = sin\, t, \qquad 0 \leqslant t \leqslant \frac{\pi}{2}$$

(a) Show $\dfrac{dy}{dx} = \dfrac{cos^k\, t}{2\, sin\, t}$ where k is an integer whose value is to be found

[3 marks]

(b) Find the exact gradient of the curve at the point where $t = \dfrac{\pi}{4}$

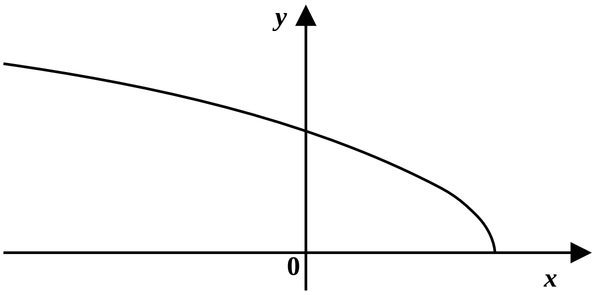
[3 marks]

(c) Find a Cartesian equation of the curve in the form $y^2 = f(x)$

[4 marks]

Question 3

A-Level Examination Question from June 2009, Paper C4, Q5 edited (Edexcel)



The graph is of the curve with parametric equations

$$x = 2\cos 2t, \qquad y = 6\sin t, \qquad 0 \leqslant t \leqslant \frac{\pi}{2}$$

(a) Find the gradient of the curve at the point where $t = \dfrac{\pi}{3}$

[4 marks]

(b) Find a Cartesian equation of the curve in the form

$$y = f(x), \qquad -2 \leqslant x \leqslant 2$$

[4 marks]

6.2 Answers

Answer 1

(a) $x = 4\sin^3 t \Rightarrow \dfrac{dx}{dt} = 12\sin^2 t\cos t$

$y = \cos 2t \Rightarrow \dfrac{dy}{dt} = -2\sin 2t \Rightarrow \dfrac{dy}{dt} = -4\sin t\cos t$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= \frac{-4\sin t\cos t}{12\sin^2 t\cos t}$$

$$= -\frac{1}{3\sin t}$$

[4 marks]

(b) When $t = \dfrac{\pi}{6}$

$x = 4\sin^3\left(\dfrac{\pi}{6}\right) \Rightarrow x = 4 \times \left(\dfrac{1}{2}\right)^3 \Rightarrow x = \dfrac{1}{2}$

$y = \cos\left(\dfrac{\pi}{3}\right) \Rightarrow y = \dfrac{1}{2}$

$\dfrac{dy}{dx} = -\dfrac{1}{3\sin\left(\frac{\pi}{6}\right)} \Rightarrow \dfrac{dy}{dx} = -\dfrac{1}{3} \times \dfrac{2}{1} \Rightarrow \dfrac{dy}{dx} = -\dfrac{2}{3}$

For the normal, the gradient will be $\dfrac{3}{2}$

$$y = mx + c$$

$$\frac{1}{2} = \frac{3}{2} \times \frac{1}{2} + c$$

$$c = -\frac{1}{4}$$

$\therefore$ Normal's equation is $y = \dfrac{3}{2}x - \dfrac{1}{4}$

[4 marks]

Answer 2

(a) $x = tan^2 t \Rightarrow \dfrac{dx}{dt} = 2 \tan t \sec^2 t$

$y = \sin t \Rightarrow \dfrac{dy}{dt} = \cos t$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= \frac{\cos t}{2 \tan t \sec^2 t}$$

$$= \frac{\cos t \cos t \cos^2 t}{2 \sin t}$$

$$= \frac{\cos^4 t}{2 \sin t} \qquad \therefore k = 4$$

[3 marks]

(b) As $\sin\left(\dfrac{\pi}{4}\right) = \cos\left(\dfrac{\pi}{4}\right)$ the gradient will be

$$\frac{1}{2} \times \left(\frac{\sqrt{2}}{2}\right)^3 = \frac{1}{2} \times \frac{2\sqrt{2}}{8}$$

$$= \frac{\sqrt{2}}{8}$$

[3 marks]

(c) **METHOD 1** $\cot^2 t + 1 = \csc^2 t$

$$\sin t = y \qquad \tan^2 t = x$$

$$\frac{1}{\sin t} = \frac{1}{y} \qquad \frac{1}{\tan^2 t} = \frac{1}{x}$$

$$\csc t = \frac{1}{y} \qquad \cot^2 t = \frac{1}{x}$$

$$\csc^2 t = \frac{1}{y^2}$$

$$\therefore \frac{1}{x} + 1 = \frac{1}{y^2}$$

$$\frac{1 + x}{x} = \frac{1}{y^2}$$

$$y^2 = \frac{x}{1 + x}$$

[4 marks]

(c) **METHOD 2** (Harry Marshall's solution)

$$y = \sin t \Rightarrow y^2 = \sin^2 t \qquad \text{so } \sin^2 t = y^2$$

$$y^2 = 1 - \cos^2 t \text{ so } \cos^2 t = 1 - y^2$$

$$\begin{aligned}
x &= \tan^2 t \\
&= \frac{\sin^2 t}{\cos^2 t} \\
&= \frac{y^2}{1 - y^2} \\
x\left(1 - y^2\right) &= y^2 \\
x - xy^2 &= y^2 \\
y^2 + xy^2 &= x \\
y^2(1 + x) &= x \\
y^2 &= \frac{x}{1 + x}
\end{aligned}$$

[4 marks]

Answer 3

(a) $x = 2\cos 2t \Rightarrow \dfrac{dx}{dt} = -4\sin 2t = -8\sin t\cos t$

$y = 6\sin t \Rightarrow \dfrac{dy}{dt} = 6\cos t$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= \frac{6\cos t}{-8\sin t\cos t}$$

$$= -\frac{3}{4\sin t}$$

When $t = \dfrac{\pi}{3}$ $\quad \dfrac{dy}{dx} = -\dfrac{3}{4\sin\left(\frac{\pi}{3}\right)}$

$$= -\frac{3 \times 2}{4\sqrt{3}}$$

$$= -\frac{3}{2\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$

$$= -\frac{\sqrt{3}}{2}$$

[4 marks]

(b)

$$x = 2\cos^2 t - 2\sin^2 t$$

$$= 2\left(1 - \sin^2 t\right) - 2\sin^2 t$$

$$= 2 - 4\sin^2 t$$

$$= 2 - 4\left(\frac{y}{6}\right)^2$$

$$4\left(\frac{y}{6}\right)^2 = 2 - x$$

$$\frac{y^2}{9} = 2 - x$$

$$y^2 = 9(2 - x)$$

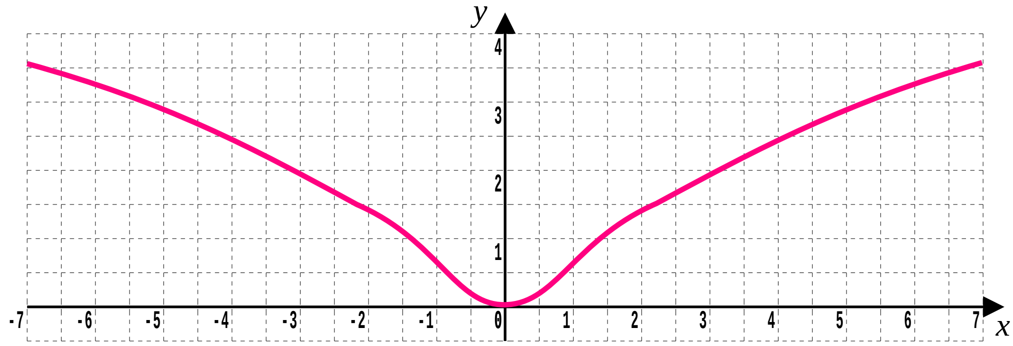
$$y = 3\sqrt{2 - x}$$

No $-\sqrt{\ }$ as the graph provided shows the other branch is not present for the given domain of t

[4 marks]

7.1 Implicit Differentiation

The graph is of the equation $y^3 = x^2 - y$ which, although not particularly complicated, is not writable in the form $y = f(x)$. As a consequence its not a routine task to find an expression for the gradient of this curve using the techniques studied thus far.



Implicit differentiation is a method of sweeping through an equation as it stands with a view to obtaining an expression for $\dfrac{dy}{dx}$ in terms of x and y.

7.2 Example

Obtain an equation of the form $\dfrac{dy}{dx} = f(x, y)$ for the curve $y^3 = x^2 - y$

Teaching Video: http://www.NumberWonder.co.uk/v9081/7.mp4

[4 marks]

7.3 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 30

Question 1

(i) Obtain an equation of the form $\dfrac{dy}{dx} = f(x, y)$ for the curve,

$$y^3 + 3y - x^3 - x = 130$$

[5 marks]

(ii) Verify that the point $P(2,\ 5)$ is on the curve.

[2 marks]

(iii) Show that the gradient at the point $P(2,\ 5)$ is $\dfrac{1}{6}$

[2 marks]

(iv) Determine the equation of the tangent to the curve at the point P
Give your answer in the form $ax + by + c = 0$ where a, b and c are integer constants.

[3 marks]

Question 2

Use implicit differentiation to show that $y^4 = sin(x) + sin(y)$

has the derivative $\dfrac{dy}{dx} = \dfrac{cos(x)}{4y^3 - cos(y)}$

[4 marks]

Question 3

A-Level Examination Question from January 2009, Paper C4, Q1 (Edexcel)

A curve C has the equation $y^2 - 3y = x^3 + 8$

(a) Find $\dfrac{dy}{dx}$ in terms of x and y

[4 marks]

(b) Hence find the gradient of C at the point where $y = 3$

[3 marks]

Question 4

A-Level Examination Question from June 2006, Paper C4, Q1 (Edexcel)

A curve C is described by the equation

$$3x^2 - 2y^2 + 2x - 3y + 5 = 0$$

Find an equation of the normal to C at the point (0, 1), giving your answer in the form $ax + by + c = 0$, where a, b and c are integers.

[7 marks]

7.4 Answers

7.4.1 Solution to 7.2 Example (Teaching Video)

$$
\begin{aligned}
y^3 &= x^2 - y \\
3y^2 \frac{dy}{dx} &= 2x - \frac{dy}{dx} \\
3y^2 \frac{dy}{dx} + \frac{dy}{dx} &= 2x \\
\frac{dy}{dx}\left(3y^2 + 1\right) &= 2x \\
\frac{dy}{dx} &= \frac{2x}{3y^2 + 1}
\end{aligned}
$$

7.4.1 Solutions to 7.3 Exercise

Answer 1

(i)

$$y^3 + 3y - x^3 - x = 130$$

$$3y^2\frac{dy}{dx} + 3\frac{dy}{dx} - 3x^2 - 1 = 0$$

$$3y^2\frac{dy}{dx} + 3\frac{dy}{dx} = 3x^2 + 1$$

$$\frac{dy}{dx}\left(3y^2 + 3\right) = 3x^2 + 1$$

$$\frac{dy}{dx} = \frac{3x^2 + 1}{3y^2 + 3}$$

[5 marks]

(ii)

$$\begin{aligned} \text{LHS} &= y^3 + 3y - x^3 - x \\ &= 5^3 + 3 \times 5 - 2^3 - 2 \\ &= 125 + 15 - 8 - 2 \\ &= 130 \\ &= \text{RHS} \qquad \therefore \left(2,\ 5\right) \text{ is on the curve } \square \end{aligned}$$

[2 marks]

(iii)

$$\begin{aligned} \frac{dy}{dx} &= \frac{3x^2 + 1}{3y^2 + 3} \\ &= \frac{3 \times 2^2 + 1}{3 \times 5^2 + 3} \\ &= \frac{13}{78} \\ &= \frac{1}{6} \qquad \square \end{aligned}$$

[2 marks]

(iv)

$$y = mx + c$$

$$5 = \frac{1}{6} \times 2 + c$$

$$c = \frac{14}{3}$$

$$y = \frac{1}{6}x + \frac{14}{3}$$

$$x - 6y + 28 = 0$$

[3 marks]

Answer 2

$$y^4 = sin(x) + sin(y)$$

$$4y^3 \frac{dy}{dx} = cos(x) + cos(y) \frac{dy}{dx}$$

$$4y^3 \frac{dy}{dx} - cos(y) \frac{dy}{dx} = cos(x)$$

$$\frac{dy}{dx}\left(4y^3 - cos(y)\right) = cos(x)$$

$$\frac{dy}{dx} = \frac{cos(x)}{4y^3 - cos(y)}$$

[4 marks]

Answer 3

(a)

$$y^2 - 3y = x^3 + 8$$

$$2y^1 \frac{dy}{dx} - 3\frac{dy}{dx} = 3x^2$$

$$\frac{dy}{dx}(2y - 3) = 3x^2$$

$$\frac{dy}{dx} = \frac{3x^2}{2y - 3}$$

[4 marks]

(b)

When $y = 3$, $\quad 3^2 - 3 \times 3 = x^3 + 8$

$$9 - 9 = x^3 + 8$$

$$x^3 = -8$$

$$x = -2$$

$$\frac{dy}{dx} = \frac{3x^2}{2y - 3}$$

$$= \frac{3 \times (-2)^2}{2 \times 3 - 3}$$

$$= \frac{12}{3}$$

$$= 4$$

[3 marks]

Answer 4

$$3x^2 - 2y^2 + 2x - 3y + 5 = 0$$

$$6x - 4y\frac{dy}{dx} + 2 - 3\frac{dy}{dx} = 0$$

$$4y\frac{dy}{dx} + 3\frac{dy}{dx} = 6x + 2$$

$$\frac{dy}{dx}(4y + 3) = 6x + 2$$

$$\frac{dy}{dx} = \frac{6x + 2}{4y + 3}$$

When $x = 0$ and $y = 1$, $\dfrac{dy}{dx} = \dfrac{6 \times 0 + 2}{4 \times 1 + 3}$

$$= \frac{2}{7}$$

For the normal, $m_N = -\dfrac{7}{2}$

$$y = mx + c$$

$$1 = -\frac{7}{2} \times 0 + c$$

$$c = 1$$

$$y = -\frac{7}{2}x + 1$$

Multiply through by 2

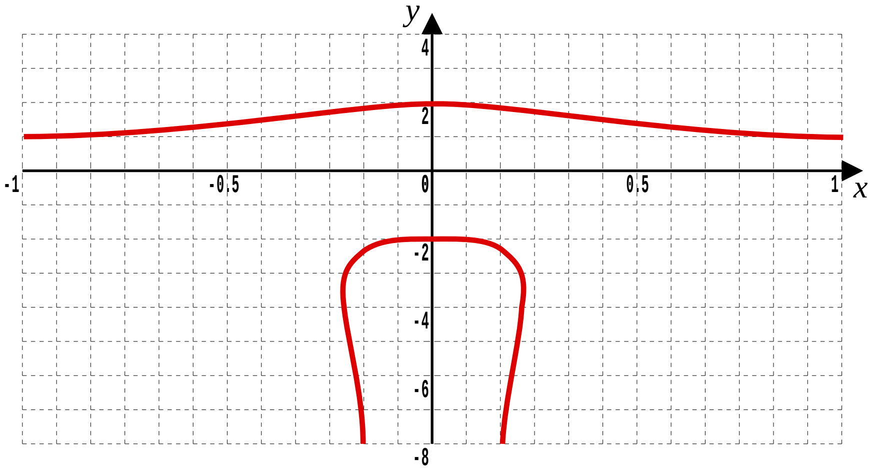
$$7x + 2y - 2 = 0$$

[7 marks]

8.1 Product Rule Implicitly

The graph is of the equation

$$4x^2y^3 = 4 + x^2 - y^2$$



The equation is a tangle of x and y, each varying and depending on the other.
To find the gradient of this curve will require implicit differentiation but the term on the left hand side is a product.
Full marks for thinking "No problem, I can use the product rule" !

The Product Rule

$$\text{If } f = uv \quad \text{then} \quad f' = uv' + u'v$$

8.2 Example

Obtain an equation of the form $\dfrac{dy}{dx} = f(x, y)$ for the curve $4x^2y^3 = 4 + x^2 - y^2$

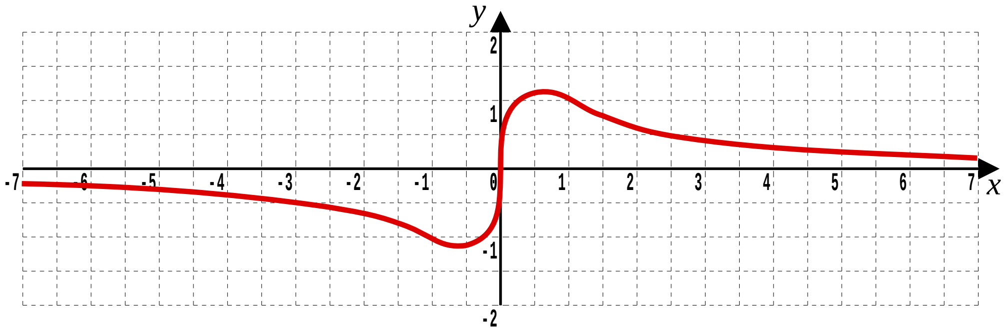
Teaching Video: http://www.NumberWonder.co.uk/v9081/8.mp4

[4 marks]

8.3 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 70

Question 1



(i) Obtain an equation of the form $\dfrac{dy}{dx} = f\,(x,\ y\,)$ for the curve,

$$3\,x^2\,y = 4x - y^3$$

[6 marks]

(ii) Verify that the point $Q\,(\,1,\ \ 1\,)$ is on the curve.

[2 marks]

(iii) Show that the gradient at the point $Q\,(\,1,\ \ 1\,)$ is $-\dfrac{1}{3}$

[2 marks]

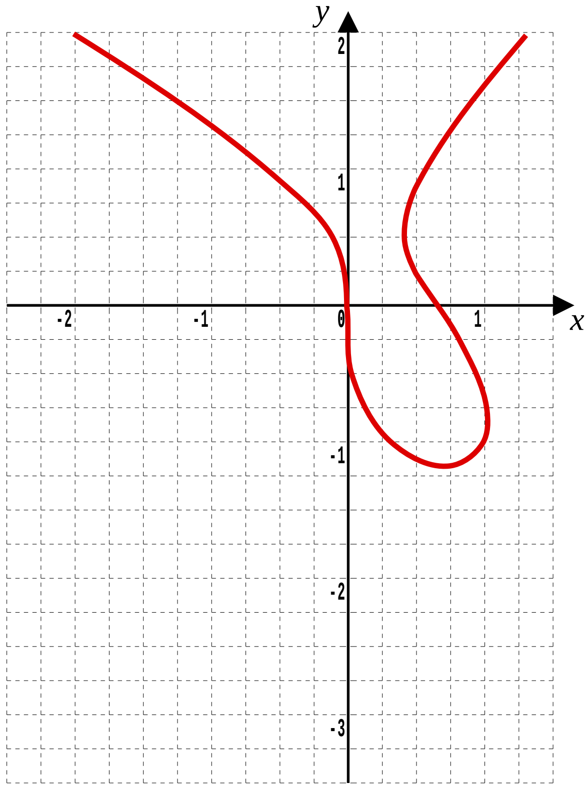
(iv) Determine the equation of the tangent to the curve at the point Q
Give your answer in the form $ax + by + c = 0$ where a, b and c are integer constants.

[2 marks]

Question 2

The graph is of the equation

$$2xy = y^3 + 2x - 3x^2$$



(i) Obtain an equation of the form $\dfrac{dy}{dx} = f(x, y)$ for the curve,

$$2xy = y^3 + 2x - 3x^2$$

[6 marks]

(ii) From scrutiny of the graph it looks as if $R\left(-2,\ 2\right)$ is a point with integer coordinates that is on the graph.

Verify that $R\left(-2,\ 2\right)$ is indeed on the curve.

[2 marks]

(iii) From looking at the graph, find another point with integer coordinates other than $\left(0,\ 0\right)$ that is on the curve.
Use mathematics to verify that your point is indeed on the curve.

[3 marks]

(iv) Find the gradient at your part (iii) point.

[2 marks]

(v) Determine the equation of the tangent to the curve at your part (iii) point.

[2 marks]

(vi) Draw your tangent onto the graph, paying particular attention to where it crosses the *y*-axis.

[3 marks]

Question 3

A-Level Examination Question from January 2012, Paper C4, Q1 (Edexcel)

The curve C has the equation

$$2x + 3y^2 + 3x^2 y = 4x^2$$

The point P on the curve has coordinates $(-1,\ 1)$

(a) Find the gradient of the curve at P

[5 marks]

(b) Hence find the equation of the normal to C at P, giving your answer in the form $ax + by + c = 0$, where a, b and c are integers.

[3 marks]

Question 4

A-Level Examination question from January 2006, Paper C4, Q1 (Edexcel)

The curve C is described by the equation

$$3x^2 + 4y^2 - 2x + 6xy - 5 = 0$$

Find an equation of the tangent to C at the point $(1, -2)$, giving your answer in the form $ax + by + c = 0$, where a, b and c are integers

[7 marks]

Question 5

A-Level Examination Question from June 2005, Paper C4, Q2 (Edexcel)

A curve C has equation

$$x^2 + 2xy - 3y^2 + 16 = 0$$

Find the coordinates of the points on the curve where $\frac{dy}{dx} = 0$

[7 marks]

Question 6

A-Level Examination Question from January 2008, Paper C4, Q5 (Edexcel)

A curve C is described by the equation

$$x^3 - 4y^2 = 12xy$$

(a) Find the coordinates of the two points on the curve where $x = -8$

[3 marks]

(b) Find the gradient of the curve at each of these points

[6 marks]

Question 7

A-Level Examination Question from June 2008, Paper C4, Q4 (Edexcel)

A curve has equation

$$3x^2 - y^2 + xy = 4$$

The points P and Q lie on the curve.

The gradient of the tangent to the curve is $\frac{8}{3}$ at P and at Q

(a) Use implicit differentiation to show that $y - 2x = 0$ at P and at Q

[6 marks]

(b) Find the coordinates of P and Q

[3 marks]

8.4 Answers

8.4.1 Solution to 8.2 Example (Teaching Video)

$$4x^2y^3 = 4 + x^2 - y^2$$

$$(4x^2)(y^3) = 4 + x^2 - y^2$$

$$(4x^2) \times 3y^2\frac{dy}{dx} + 8x \times (y^3) = 2x - 2y\frac{dy}{dx}$$

$$12x^2y^2\frac{dy}{dx} + 2y\frac{dy}{dx} = 2x - 8xy^3$$

Divide through by 2

$$6x^2y^2\frac{dy}{dx} + y\frac{dy}{dx} = x - 4xy^3$$

$$\frac{dy}{dx}(6x^2y^2 + y) = x - 4xy^3$$

$$\frac{dy}{dx} = \frac{x - 4xy^3}{6x^2y^2 + y}$$

$$= \frac{x(1 - 4y^3)}{y(6x^2y + 1)}$$

[4 marks]

8.4.2 Solutions to 8.3 Exercise

Answer 1

(i)

$$3x^2 y = 4x - y^3$$

$$3x^2 \frac{dy}{dx} + 6xy = 4 - 3y^2 \frac{dy}{dx}$$

$$\frac{dy}{dx}\left(3x^2 + 3y^2\right) = 4 - 6xy$$

$$\frac{dy}{dx} = \frac{4 - 6xy}{3x^2 + 3y^2}$$

[6 marks]

(ii)

$$\text{LHS} = 3x^2 y \qquad \text{RHS} = 4x - y^3$$

$$= 3 \times 1^2 \times 1 \qquad = 4 \times 1 - 1^3$$

$$= 3 \qquad = 3$$

$$= \text{LHS}$$

$\therefore\ Q(1,\ 1)$ is on the curve □

[2 marks]

(iii)

$$\frac{dy}{dx} = \frac{4 - 6xy}{3x^2 + 3y^2}$$

$$= \frac{4 - 6 \times 1 \times 1}{3 \times 1^2 + 3 \times 1^2}$$

$$= \frac{-2}{6}$$

$$= -\frac{1}{3} \quad □$$

[2 marks]

(iv)

$$y = mx + c$$

$$1 = -\frac{1}{3} \times 1 + c$$

$$c = \frac{4}{3}$$

$$y = -\frac{1}{3}x + \frac{4}{3}$$

$$x + 3y - 4 = 0$$

[2 marks]

Answer 2

(i)

$$2xy = y^3 + 2x - 3x^2$$

$$2x\frac{dy}{dx} + 2y = 3y^2\frac{dy}{dx} + 2 - 6x$$

$$\frac{dy}{dx}\left(2x - 3y^2\right) = 2 - 6x - 2y$$

$$\frac{dy}{dx} = \frac{2 - 6x - 2y}{2x - 3y^2}$$

[6 marks]

(ii)

$$\begin{aligned} \text{LHS} &= 2xy \\ &= 2 \times (-2) \times 2 \\ &= -8 \end{aligned} \qquad \begin{aligned} \text{RHS} &= y^3 + 2x - 3x^2 \\ &= 2^3 + 2 \times (-2) - 3 \times (-2)^2 \\ &= 8 - 4 - 12 \\ &= -8 \\ &= \text{LHS} \end{aligned}$$

$\therefore\ R\left(-2,\ 2\right)$ is on the curve $\square$

[2 marks]

(iii) The only possibility is (1, – 1)

$$\begin{aligned} \text{LHS} &= 2xy \\ &= 2 \times 1 \times (-1) \\ &= -2 \end{aligned} \qquad \begin{aligned} \text{RHS} &= y^3 + 2x - 3x^2 \\ &= (-1)^3 + 2 \times 1 - 3 \times 1^2 \\ &= -1 + 2 - 3 \\ &= -2 \\ &= \text{LHS} \end{aligned}$$

$\therefore\ \left(1, -1\right)$ is on the curve

[2 marks]

(iv) At (1, – 1) $\dfrac{dy}{dx} = \dfrac{2 - 6x - 2y}{2x - 3y^2}$

$$= \frac{2 - 6 \times 1 - 2 \times (-1)}{2 \times 1 - 3 \times (-1)^2}$$

$$= 2$$

[2 marks]

(v)

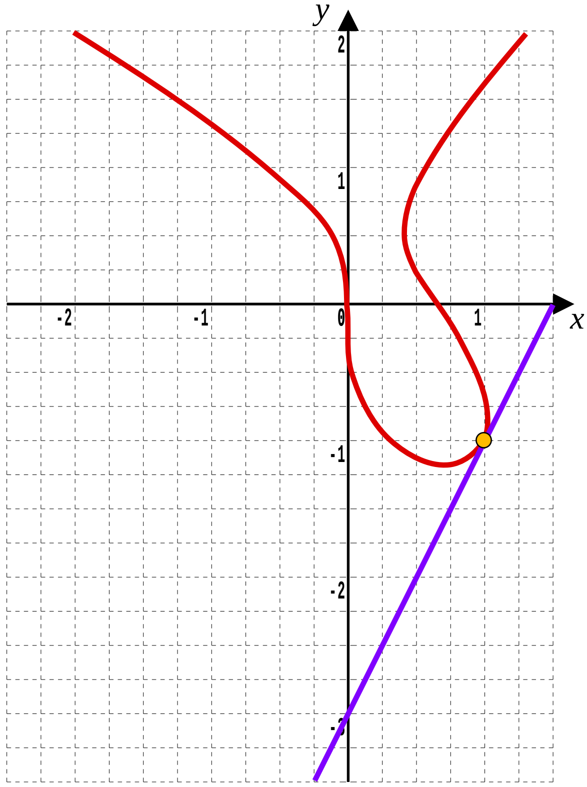
$$y = mx + c$$

$$-1 = 2 \times 1 + c \Rightarrow c = -3$$

$$y = 2x - 3$$

[2 marks]

(vi)



[3 marks]

Answer 3

(a)

$$2x + 3y^2 + 3x^2 y = 4x^2$$

Differentiate with respect to x

$$2 + 6y\frac{dy}{dx} + 3x^2\frac{dy}{dx} + 6xy = 8x$$

$$\frac{dy}{dx}\left(6y + 3x^2\right) = 8x - 6xy - 2$$

$$\frac{dy}{dx} = \frac{8x - 6xy - 2}{6y + 3x^2}$$

At $P(-1,\ 1)$ this becomes, $\frac{dy}{dx} = \frac{-8 + 6 - 2}{6 + 3}$

$$= -\frac{4}{9}$$

[5 marks]

(b)

Thus, $m_N = \frac{9}{4} \Rightarrow y = mx + c$

$$1 = \frac{9}{4} \times (-1) + c$$

$$c = \frac{13}{4}$$

$$y = \frac{9}{4}x + \frac{13}{4}$$

$$9x - 4y + 13 = 0$$

[3 marks]

Answer 4

$$3x^2 + 4y^2 - 2x + 6xy - 5 = 0$$

Differentiate with respect to x

$$6x + 8y\frac{dy}{dx} - 2 + 6x\frac{dy}{dx} + 6y = 0$$

$$\frac{dy}{dx}(8y + 6x) = 2 - 6x - 6y$$

Divide through by 2

$$\frac{dy}{dx}(4y + 3x) = 1 - 3x - 3y$$

$$\frac{dy}{dx} = \frac{1 - 3x - 3y}{4y + 3x}$$

At $(1, -2)$ this becomes, $\frac{dy}{dx} = \frac{1 - 3 + 6}{-8 + 3}$

$$= -\frac{4}{5}$$

$$y = mx + c$$

$$-2 = -\frac{4}{5} \times 1 + c$$

$$c = -\frac{6}{5}$$

$$\therefore\ y = -\frac{4}{5}x - \frac{6}{5}$$

$$4x + 5y + 6 = 0$$

[7 marks]

Answer 5

$$x^2 + 2xy - 3y^2 + 16 = 0$$

Differentiate with respect to x

$$2x + 2x\frac{dy}{dx} + 2y - 6y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(6y - 2x) = 2x + 2y$$

Divide through by 2

$$\frac{dy}{dx}(3y - x) = x + y$$

$$\frac{dy}{dx} = \frac{x + y}{3y - x}$$

For $\frac{dy}{dx} = 0, \quad x + y = 0 \Rightarrow y = -x$

Replace y in the original equation with $-x$ and solve for x

$$x^2 + 2x(-x) - 3(-x)^2 + 16 = 0$$

$$x^2 - 2x^2 - 3x^2 + 16 = 0$$

$$4x^2 = 16$$

$$x = \pm 2$$

So the points where $\frac{dy}{dx} = 0$ are $(-2, 2), (2, -2)$

[7 marks]

Answer 6

(a)

$$x^3 - 4y^2 = 12xy$$

When $x = -8$, $(-8)^3 - 4y^2 = 12(-8)y$

$$-512 - 4y^2 = -96y$$

Divide through by -4

$$128 + y^2 = 24y$$

$$y^2 - 24y + 128 = 0$$

$(y - 16)(y - 8) = 0$ or use the Q-Formula

Either $y = 16$ or $y = 8$

$\therefore$ Coordinates are, $(-8, 8)$, $(-8, 16)$

[3 marks]

(b)

$$x^3 - 4y^2 = 12xy$$

Differentiate with respect to x

$$3x^2 - 8y\frac{dy}{dx} = 12x\frac{dy}{dx} + 12y$$

$$\frac{dy}{dx}(12x + 8y) = 3x^2 - 12y$$

$$\frac{dy}{dx} = \frac{3x^2 - 12y}{12x + 8y}$$

For $(-8, 8)$, $\dfrac{dy}{dx} = \dfrac{3 \times (-8)^2 - 12 \times 8}{12 \times (-8) + 8 \times 8}$

$$= \frac{192 - 96}{-96 + 64}$$

$$= -3$$

For $(-8, 16)$, $\dfrac{dy}{dx} = \dfrac{3 \times (-8)^2 - 12 \times 16}{12 \times (-8) + 8 \times 16}$

$$= \frac{192 - 192}{-96 + 128}$$

$$= 0$$

[6 marks]

Answer 7

(a)

$$3x^2 - y^2 + xy = 4$$

Differentiate with respect to x

$$6x - 2y\frac{dy}{dx} + x\frac{dy}{dx} + y = 0$$

$$\frac{dy}{dx}(2y - x) = 6x + y$$

$$\frac{dy}{dx} = \frac{6x + y}{2y - x}$$

At P and Q, $\frac{dy}{dx} = \frac{8}{3}$

$$\frac{6x + y}{2y - x} = \frac{8}{3}$$

$$3(6x + y) = 8(2y - x)$$

$$18x + 3y = 16y - 8x$$

$$26x = 13y$$

$$y = 2x$$

$$y - 2x = 0 \qquad \square$$

[6 marks]

(b)

At P and at Q, $y = 2x$

$$3x^2 - (2x)^2 + x(2x) = 4$$

$$3x^2 - 4x^2 + 2x^2 = 4$$

$$x^2 = 4$$

$$x = \pm 2$$

Thus, $(-2, -4)$, $(2, 4)$

[3 marks]

9.1 Exponentials Implicitly

Implicit differentiation questions can involve the more "fancy functions" such as the exponentials, logarithms or trigonometric. The focus in this lesson is upon the exponential function.

It's already known that if $y = e^x$ then $\dfrac{dy}{dx} = e^x$

In other word the exponential function $y = e^x$ is its own derivative.

However, this is a special result of the following more general rule:

The Derivative of an Exponential

Given that a is a constant,

$$\text{if } y = a^x \text{ then } \frac{dy}{dx} = a^x \ln a$$

Proof

$$
\begin{aligned}
y &= a^x \\
&= e^{\ln a^x} && \text{as } e^x \text{ and } \ln x \text{ are each the inverse function of the other} \\
&= e^{x \ln a} && \text{using a rule of logs} \\
\frac{dy}{dx} &= e^{x \ln a} \ln a && \text{by the Chain Rule : } \ln a \text{ is "just a number"} \\
&= e^{\ln a^x} \ln a && \text{using a rule of logs} \\
&= a^x \ln a && \text{as } e^x \text{ and } \ln x \text{ are each the inverse function of the other}
\end{aligned}
$$

□

Notes : 1) This rule is not given in the examination formulae booklet

2) The rule that if $y = e^x$ then $\dfrac{dy}{dx} = e^x$ is a special case of this more general rule with $a = e$, because $\ln e = 1$

9.2 Example

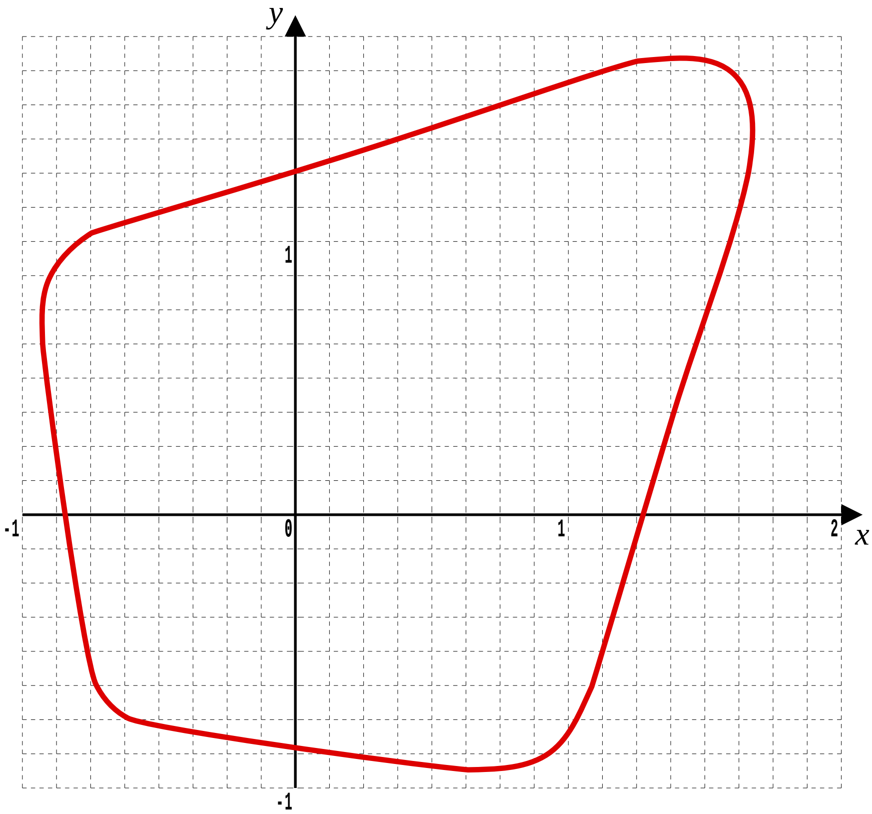
Differentiate $y = 2^x$

Solution : Using the rule for The Derivative of an Exponential with $a = 2$

$$\frac{dy}{dx} = 2^x \ln 2 \qquad \text{And that's it done !}$$

9.3 A Beautiful Curve

The graph is of the equation $3^y 3^x = x^6 + y^6$



Obtain an equation of the form $\dfrac{dy}{dx} = f(x, y)$ for the curve $3^y 3^x = x^6 + y^6$

Teaching Video: http://www.NumberWonder.co.uk/v9081/9.mp4

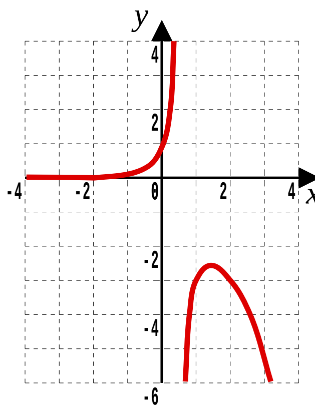
[6 marks]

9.4 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 40

Question 1

The graph is of the equation $3^x = y - 2xy$



(i) Use implicit differentiation with respect to x to obtain an equation of the form $\dfrac{dy}{dx} = f(x, y)$ for the curve $3^x + 2xy = y$

[7 marks]

(ii) Verify that the point $(2, -3)$ is on the curve

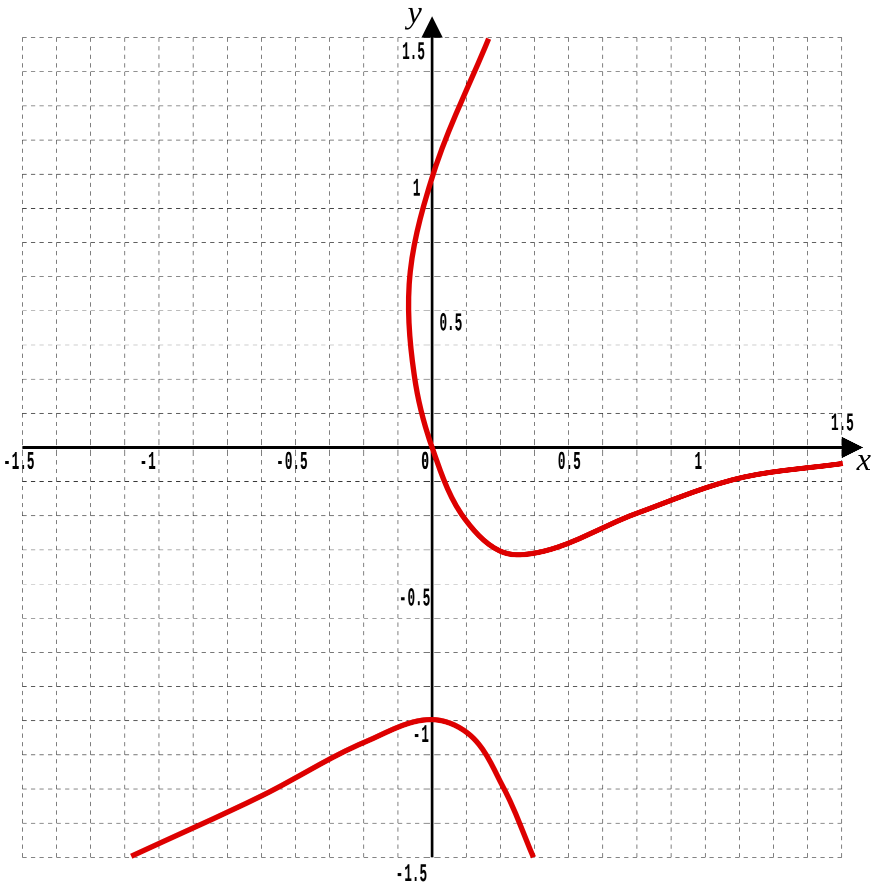
[2 marks]

(iii) Determine the exact value of the gradient at the point $(2, -3)$

[2 marks]

Question 2

The graph is of the equation $y\,e^{3x} + 3x = y^3$



(i) Use implicit differentiation with respect to x to obtain an equation of the form $\dfrac{dy}{dx} = f(x, y)$ for the curve $y\,e^{3x} + 3x = y^3$

[7 marks]

(ii) Find the equation of the normal to the curve at the origin.
Add this normal to the graph.

[3 marks]

(iii) Find the equation of the normal to the curve at the point $(0, \ 1)$
Add this normal to the graph.

[3 marks]

(iv) Use algebra to determine the coordinates of where the normal of part (ii) intersects the normal of part (iii).
Mark this point on the graph.

[3 marks]

Question 3

A-Level Examination Question from January 2018, IAL, Paper C34, Q1 (Edexcel)

A curve C has equation

$$3^x + xy = x + y^2 \qquad y > 1$$

The point P with coordinates $(4,\ 11)$ lies on C

Find the exact value of $\frac{dy}{dx}$ at the point P

Give your answer in the form $a + b\ln 3$, where a and b are rational numbers

[6 marks]

Question 4

A-Level Examination Question from June 2010, Paper C4, Q3 (Edexcel)

The curve C has equation

$$2^x + y^2 = 2xy$$

Find the exact value of $\dfrac{dy}{dx}$ at the point on C with coordinates $(3,\ 2)$

[7 marks]

9.5 Answers

9.5.1 Solution to 9.3 A Beautiful Curve (Teaching Video)

$$3^y\,3^x = x^6 + y^6$$

$$(3^y)(3^x) = x^6 + y^6$$

Differentiate with respect to x

$$(3^y)\,3^x \ln 3 + 3^y \ln 3 \frac{dy}{dx}(3^x) = 6x^5 + 6y^5\frac{dy}{dx}$$

$$\frac{dy}{dx}\left(3^y\,3^x \ln 3 - 6y^5\right) = 6x^5 - 3^y\,3^x \ln 3$$

$$\frac{dy}{dx} = \frac{6x^5 - 3^y\,3^x \ln 3}{3^y\,3^x \ln 3 - 6y^5}$$

[7 marks]

9.5.2 Solutions to 9.4 Exercise

Answer 1

(i)

$$3^x + 2xy = y$$

Differentiate with respect to x

$$3^x \ln 3 + 2x\frac{dy}{dx} + 2y = \frac{dy}{dx}$$

$$\frac{dy}{dx}(1 - 2x) = 3^x \ln 3 + 2y$$

$$\frac{dy}{dx} = \frac{3^x \ln 3 + 2y}{1 - 2x}$$

[7 marks]

(ii)

$$\begin{aligned} \text{LHS} &= 3^x + 2xy \\ &= 3^2 + 2 \times 2 \times (-3) \\ &= 9 - 12 \\ &= -3 \end{aligned}$$

$$\begin{aligned} \text{RHS} &= y \\ &= -3 \\ &= \text{LHS} \end{aligned}$$

$\therefore\ (2, -3)$ is on the curve

[2 marks]

(iii)

$$\begin{aligned} \frac{dy}{dx} &= \frac{3^x \ln 3 + 2y}{1 - 2x} \\ &= \frac{3^2 \ln 3 + 2 \times (-3)}{1 - 2 \times 2} \\ &= 2 - 3\ln 3 \end{aligned}$$

[2 marks]

Answer 2

(i)

$$y e^{3x} + 3x = y^3$$

Differentiate with respect to x

$$y\,3\,e^{3x} + \frac{dy}{dx} e^{3x} + 3 = 3\,y^2 \frac{dy}{dx}$$

$$\frac{dy}{dx}\left(3\,y^2 - e^{3x}\right) = 3y\,e^{3x} + 3$$

$$\frac{dy}{dx} = \frac{3y\,e^{3x} + 3}{3\,y^2 - e^{3x}}$$

[7 marks]

(ii)

$$\text{At } (0,\ 0), \quad \frac{dy}{dx} = \frac{3y\,e^{3x} + 3}{3\,y^2 - e^{3x}}$$

$$= -3$$

$$\therefore\ m_N = \frac{1}{3}$$

$$\therefore\ \text{Equation of normal is } y = \frac{1}{3}x$$

See graph on the following page

[3 marks]

(iii)

$$\text{At } (0,\ 1), \quad \frac{dy}{dx} = \frac{3y\,e^{3x} + 3}{3\,y^2 - e^{3x}}$$

$$= 3$$

$$\therefore\ m_N = -\frac{1}{3}$$

$$\therefore\ \text{Equation of normal is } y = -\frac{1}{3}x + 1$$

See graph on the following page

[3 marks]

(iv)

$$\frac{1}{3}x = -\frac{1}{3}x + 1$$

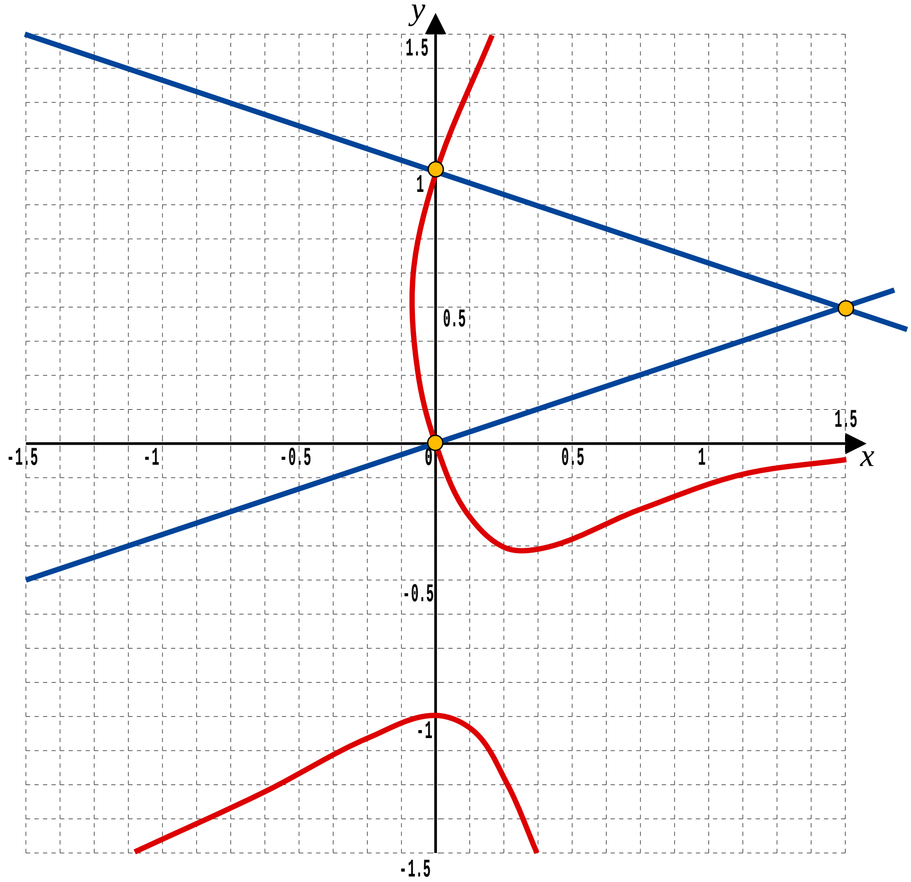
$$\frac{2}{3}x = 1$$

$$x = \frac{3}{2} \text{ and } y = \frac{1}{2}$$

See graph on the following page

[3 marks]

Graph for Answer 2



Answer 3

$$3^x + xy = x + y^2$$

Differentiate with respect to x

$$3^x \ln 3 + x\frac{dy}{dx} + y = 1 + 2y\frac{dy}{dx}$$

$$\frac{dy}{dx}(x - 2y) = 1 - 3^x \ln 3 - y$$

$$\frac{dy}{dx} = \frac{1 - 3^x \ln 3 - y}{x - 2y}$$

At the point $P(4,\ 11)$, $\quad \frac{dy}{dx} = \frac{1 - 3^4 \ln 3 - 11}{4 - 22}$

$$= \frac{-10 - 81 \ln 3}{-18}$$

$$= \frac{5}{9} + \frac{9}{2}\ln 3$$

[6 marks]

Answer 4

$$2^x + y^2 = 2xy$$

$$2^x \ln 2 + 2y \frac{dy}{dx} = 2x \frac{dy}{dx} + 2y$$

$$\frac{dy}{dx}(2y - 2x) = 2y - 2^x \ln 2$$

$$\frac{dy}{dx} = \frac{2y - 2^x \ln 2}{2y - 2x}$$

$$\text{At } (3,\ 2),\quad \frac{dy}{dx} = \frac{2 \times 2 - 2^3 \ln 2}{2 \times 2 - 2 \times 3}$$

$$= \frac{4 - 8 \ln 2}{4 - 6}$$

$$= 4 \ln 2 - 2$$

[7 marks]

Lesson 10

A-Level Pure Mathematics : Year 2
Differentiation IV

10.1 Mr Clever Sits An Exam

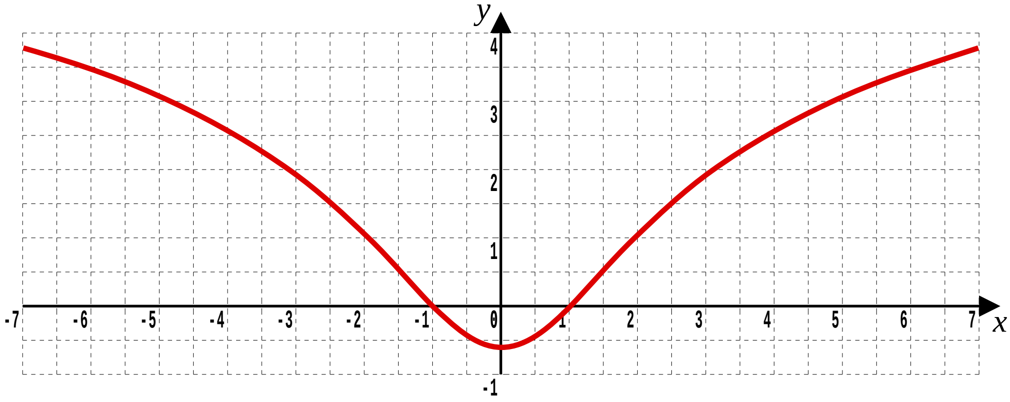
Exam questions on Implicit Differentiation often feature exponential, logarithmic or trigonometric functions. So, get some coffee in your Mr Clever cup, and give these questions a go !

10.2 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 40

Question 1



The graph is of the equation $e^y + y = x^2$

Find the equation of the normal at (1, 0) in the form $y = mx + c$ and, having found it, draw the normal onto the graph.

[8 marks]

Question 2

A-Level Examination Question from June 2009, Paper C4, Q4 (Edexcel)

The curve C has the equation

$$y e^{-2x} = 2x + y^2$$

(a) Find $\frac{dy}{dx}$ in terms of x and y

[5 marks]

The point P on C has coordinates (0, 1)

(b) Find the equation of the normal to C at P, giving your answer in the form $ax + by + c = 0$, where a, b and c are integers.

[4 marks]

Question 3

A-Level Examination Question from June 2011, Paper C4, Q5 (Edexcel)

Find the gradient of the curve with equation

$$\ln y = 2x \ln x, \qquad x > 0, \qquad y > 0$$

at the point on the curve where $x = 2$

Give your answer as an exact value

[7 marks]

Question 4

A-Level Examination Question from January 2010, Paper C4, Q3 (Edexcel)

The curve C has the equation

$$\cos 2x + \cos 3y = 1, \qquad -\frac{\pi}{4} < x < \frac{\pi}{4}, \qquad 0 \leqslant y \leqslant \frac{\pi}{6}$$

(a) Find $\frac{dy}{dx}$ in terms of x and y

[3 marks]

The point P lies on C where $x = \frac{\pi}{6}$

(b) Find the value of y at P

[3 marks]

(c) Find the equation of the tangent to C at P, giving your answer in the form $ax + by + c\pi = 0$, where a, b and c are integers

[3 marks]

Question 5

A-Level Examination Question from January 2007, Paper C4, Q5 (Edexcel)

A set of curves is given by the equation

$$\sin x + \cos y = 0.5$$

(a) Use implicit differentiation to find an expression for $\dfrac{dy}{dx}$

[2 marks]

For $-\pi < x < \pi$ and $-\pi < y < \pi$

(b) Find the coordinates of the points where $\dfrac{dy}{dx} = 0$

[5 marks]

Answer 1

$$e^y + y = x^2$$

Differentiate with respect to x

$$e^y \frac{dy}{dx} + \frac{dy}{dx} = 2x$$

$$\frac{dy}{dx}\left(e^y + 1\right) = 2x$$

$$\frac{dy}{dx} = \frac{2x}{e^y + 1}$$

At $(1,\ 0)$, $\dfrac{dy}{dx} = 1$

$\therefore$ The normal will have gradient -1

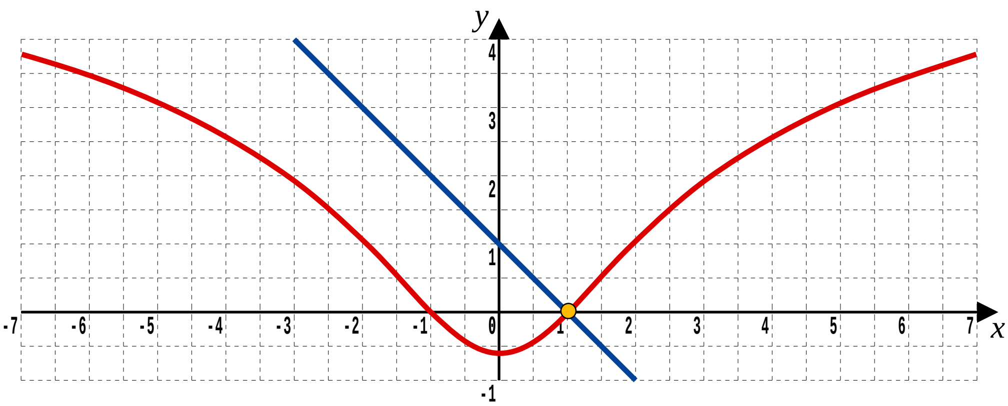
$$y = mx + c$$

$$0 = -1 \times 1 + c$$

$$c = 1$$

Normal at $(1,\ 0)$ has equation $y = -x + 1$

[6 marks]



[2 marks]

Answer 2

(a)

$$y\,e^{-2x} = 2x + y^2$$

Differentiate with respect to x

$$y(-2)e^{-2x} + \frac{dy}{dx}e^{-2x} = 2 + 2y\frac{dy}{dx}$$

$$\frac{dy}{dx}\left(e^{-2x} - 2y\right) = 2 + 2y\,e^{-2x}$$

$$\frac{dy}{dx} = \frac{2 + 2y\,e^{-2x}}{e^{-2x} - 2y}$$

[5 marks]

(b)

At $(0,\ 1)$, $\frac{dy}{dx} = \frac{2 + 2}{1 - 2}$

$= -4$

$\therefore$ The normal will have gradient $\frac{1}{4}$

$$y = mx + c$$

$$1 = \frac{1}{4} \times 0 + c$$

$$c = 1$$

Normal at $(0,\ 1)$ has equation $y = \frac{1}{4}x + 1$

That is, $x - 4y + 4 = 0$

[4 marks]

Answer 3

$$ln\,y = 2x\,ln\,x, \qquad x > 0, \qquad y > 0$$

Differentiate with respect to x

$$\frac{1}{y} \times \frac{dy}{dx} = 2x \times \frac{1}{x} + 2\,ln\,x$$

$$\frac{dy}{dx} = 2y + 2y\,ln\,x$$

$$= 2y\left(1 + ln\,x\right)$$

When $x = 2$, $ln\,y = 2 \times 2 \times ln\,2$

$= 4\,ln\,2$

$= ln\,2^4$

$= ln\,16$

$\therefore\ y = 16$ $\qquad \therefore\ \frac{dy}{dx} = 32\left(1 + ln\,2\right)$

[7 marks]

Answer 4

(a)

$$\cos 2x + \cos 3y = 1, \qquad -\frac{\pi}{4} < x < \frac{\pi}{4}, \qquad 0 \leqslant y \leqslant \frac{\pi}{6}$$

Differentiate with respect to x

$$(-2\sin 2x) + (-3\sin 3y)\frac{dy}{dx} = 0$$

$$(-3\sin 3y)\frac{dy}{dx} = 2\sin 2x$$

$$\frac{dy}{dx} = -\frac{2\sin 2x}{3\sin 3y}$$

[3 marks]

(b)

$$\text{When } x = \frac{\pi}{6},\ \cos\left(\frac{\pi}{3}\right) + \cos 3y = 1$$

$$\cos 3y = 1 - \frac{1}{2}$$

$$= \frac{1}{2}$$

$$3y = \frac{\pi}{3},\ \frac{5\pi}{3},\ \ldots$$

$$y = \frac{\pi}{9} \quad \text{only answer within constraints}$$

[3 marks]

(c)

$$\text{When } x = \frac{\pi}{6},\ \frac{dy}{dx} = -\frac{2\sin\left(\frac{\pi}{3}\right)}{3\sin\left(\frac{\pi}{3}\right)}$$

$$= -\frac{2}{3}$$

$$y = mx + c$$

$$\frac{\pi}{9} = -\frac{2}{3} \times \frac{\pi}{6} + c$$

$$c = \frac{2\pi}{9}$$

$$y = -\frac{2}{3}x + \frac{2\pi}{9}$$

That is, $6x + 9y - 2\pi = 0$

[3 marks]

Answer 5

(a)

$$\sin x + \cos y = 0.5$$

$$\cos x - \frac{dy}{dx} \sin y = 0$$

$$\frac{dy}{dx} = \frac{\cos x}{\sin y}$$

[2 marks]

(b)

$$\frac{dy}{dx} = 0 \text{ when } \cos x = 0$$

$$x = \pm \frac{\pi}{2}$$

$$\text{When } x = \frac{\pi}{2},\ \cos y = 0.5 - \sin\left(\frac{\pi}{2}\right)$$

$$= 0.5 - 1$$

$$y = \arccos(-0.5)$$

$$= \pm \frac{2\pi}{3}$$

$$\therefore \left(\frac{\pi}{2}, -\frac{2\pi}{3} \right), \left(\frac{\pi}{2}, \frac{2\pi}{3} \right)$$

$$\text{When } x = -\frac{\pi}{2},\ \cos y = 0.5 - \sin\left(-\frac{\pi}{2}\right)$$

$$= 0.5 + 1$$

$$= 1.5 \qquad \text{which has no solutions}$$

[5 marks]

Lesson 11

A-Level Pure Mathematics : Year 2

Differentiation IV

11.1 Revision

Any solution based entirely on graphical
or numerical methods is not acceptable

Marks Available : 50

Question 1

Differentiate each of the following, simplifying answers as appropriate;

(i) $y = \dfrac{2x^5}{15}$

[2 marks]

(ii) $y = ln\left(2x^3 + 7\right)$

[2 marks]

(iii) $y = \dfrac{5}{4x^2 - 3}$

[2 marks]

(iv) $y = e^{\sqrt{x}}$

[2 marks]

Question 2

The product rule states that $(u\,v)' = u\,v' + u'\,v$

Use the rule to differentiate $y = 7x^2 \cos x$

[3 marks]

Question 3

The quotient rule states that $\left(\frac{u}{v}\right)' = \frac{v\,u' - v'\,u}{v^2}$

Use the rule to differentiate the following, simplifying your answer;

$$y = \frac{\ln(4x)}{x^2}$$

[3 marks]

Question 4

(i) Use derivatives of $\sin x$ and $\cos x$ to prove the derivative of $\tan x$ is $\sec^2 x$

[4 marks]

(ii) Hence, or otherwise, use the chain rule to differentiate;

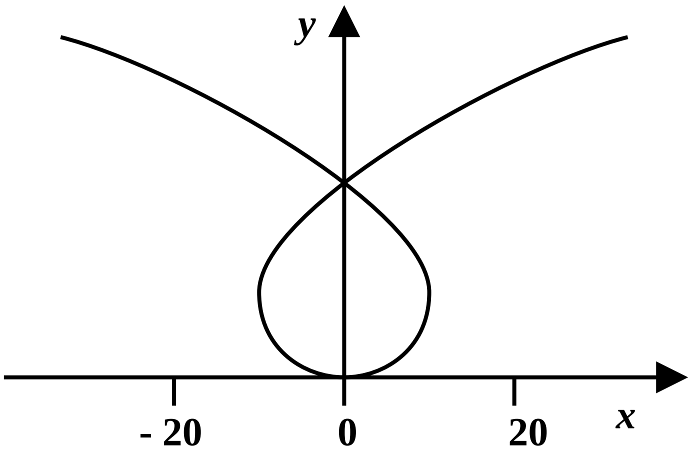
$$y = \tan^2 x$$

[2 marks]

Question 5

The graph is of the parametric equations;

$$x = 12t - t^3 \quad \text{and} \quad y = 3t^2$$



(i) Find, in terms of t,

(a) $\dfrac{dx}{dt}$ **(b)** $\dfrac{dy}{dt}$ **(c)** $\dfrac{dy}{dx}$

[2, 2, 2 marks]

(ii) Write down the coordinates of the point on the curve that corresponds to the parameter t having the value 1

[1 mark]

(iii) What is the gradient of the curve at your part (ii) point ?

[1 mark]

(iv) By making use of your part (ii) and (iii) answers, determine the equation of the tangent to the curve from the point at which $t = 1$

[2 marks]

Question 6

A curve has equation;

$$x^2 + 6xy - y^2 = 90$$

Find an expression for the gradient by means of implicit differentiation.

Write your answer in the form $\frac{dy}{dx} = f(x, y)$

[6 marks]

Question 7

The parametric equations of a curve are;

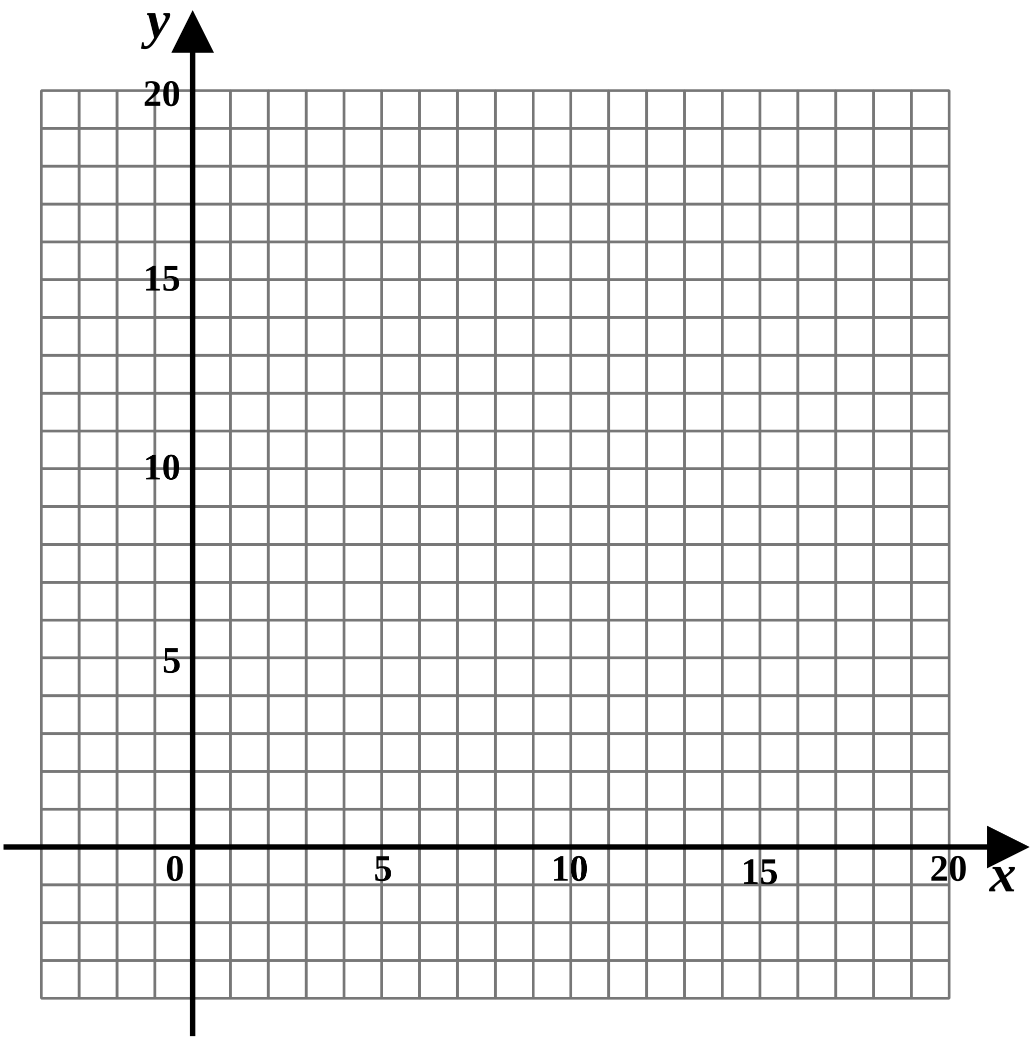
$$x = t^2 + t, \qquad y = t^2 - t$$

(i) Complete the following table by way of working out some points on the graph of this curve.

t	-4	-3	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2	3	4
x											
y											

[3 marks]

(ii) On the graph paper provided below plot the curve



[3 marks]

(iii) Find, in terms of t, an expression for the derivative of this curve.

[4 marks]

(iv) Find, in terms of x and y an expression for the derivative of this curve.

[4 marks]

Answers 11.2

Answer 1

(i) $$y = \frac{2x^5}{15} \Rightarrow \frac{dy}{dx} = \frac{2 \times 5x^4}{15} \Rightarrow \frac{dy}{dx} = \frac{2x^4}{3}$$

[2 marks]

(ii) $$y = ln\left(2x^3 + 7\right) \Rightarrow \frac{dy}{dx} = \frac{6x^2}{2x^3 + 7}$$

[2 marks]

(iii) $$y = \frac{5}{4x^2 - 3} \Rightarrow y = 5\left(4x^2 - 3\right)^{-1}$$

$$\frac{dy}{dx} = -5\left(4x^2 - 3\right)^{-2} \times 8x \Rightarrow \frac{dy}{dx} = -\frac{40x}{\left(4x^2 - 3\right)^2}$$

[2 marks]

(iv) $$y = e^{\sqrt{x}} \Rightarrow y = e^{x^{\frac{1}{2}}}$$

$$\frac{dy}{dx} = e^{x^{\frac{1}{2}}} \times \frac{1}{2}x^{-\frac{1}{2}} \Rightarrow \frac{dy}{dx} = \frac{e^{\sqrt{x}}}{2\sqrt{x}}$$

[2 marks]

Answer 2

$$y = 7x^2 \cos x$$

$$\frac{dy}{dx} = 7x^2\,(-\sin x) + 14x \cos x$$

$$= 7x\left(2\cos x - x \sin x\right)$$

[3 marks]

Answer 3

$$y = \frac{ln(4x)}{x^2}$$

$$\frac{dy}{dx} = \frac{x^2\left(\frac{1}{4x} \times 4\right) - 2x\,ln(4x)}{(x^2)^2}$$

$$= \frac{x - 2x\,ln(4x)}{x^4}$$

$$= \frac{x\,(1 - 2\,ln(4x))}{x^4}$$

$$= \frac{1 - 2\,ln(4x)}{x^3}$$

[3 marks]

Answer 4

(i)

$$\begin{aligned} y &= \tan x \\ &= \frac{\sin x}{\cos x} \\ \frac{dy}{dx} &= \frac{\cos x \cos x - (-\sin x)\ \sin x}{\cos^2 x} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\ &= \frac{1}{\cos^2 x} \\ &= \sec^2 x \qquad \square \end{aligned}$$

[4 marks]

(ii)

$$\begin{aligned} y &= \tan^2 x \\ \frac{dy}{dx} &= 2 \tan x \sec^2 x \end{aligned}$$

[2 marks]

Answer 5

(i) **(a)** $\frac{dx}{dt} = 12 - 3t^2$ **(b)** $\frac{dy}{dt} = 6t$

(c)

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{dt} \times \frac{dt}{dx} \\ &= \frac{6t}{12 - 3t^2} \\ &= \frac{2t}{4 - t^2} \end{aligned}$$

[2, 2, 2 marks]

(ii) When $t = 1,\ x = 11,\ y = 3$ That is, $(11,\ 3)$

[1 mark]

(iii) When $t = 1,\ \frac{dy}{dx} = \frac{2}{3}$

[1 mark]

(iv) $y = mx + c \Rightarrow 3 = \frac{2}{3} \times 11 + c \Rightarrow c = \frac{9 - 22}{3} \Rightarrow c = -\frac{13}{3}$

$$y = \frac{2}{3}x - \frac{13}{3}$$

[2 marks]

Answer 6

$$x^2 + 6xy - y^2 = 90$$

Differentiate with respect to x

$$2x + 6x\frac{dy}{dx} + 6y - 2y\frac{dy}{dx} = 0$$

Divide through by – 2

$$-x - 3x\frac{dy}{dx} - 3y + y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(y - 3x) = x + 3y$$

$$\frac{dy}{dx} = \frac{x + 3y}{y - 3x}$$

[6 marks]

Answer 7

(i)

t	-4	-3	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2	3	4
x	12	6	2	0	$-\frac{1}{4}$	0	$\frac{3}{4}$	2	6	12	20
y	20	12	6	2	$\frac{3}{4}$	0	$-\frac{1}{4}$	0	2	6	12

[3 marks]

(ii) See graph on the following page

[3 marks]

(iii) $\frac{dx}{dt} = 2t + 1$ $\quad \frac{dy}{dt} = 2t - 1$ $\quad \therefore \frac{dy}{dx} = \frac{2t - 1}{2t + 1}$

[4 marks]

(iv)

$$x = t^2 + t$$

Subtract $\quad y = t^2 - t$

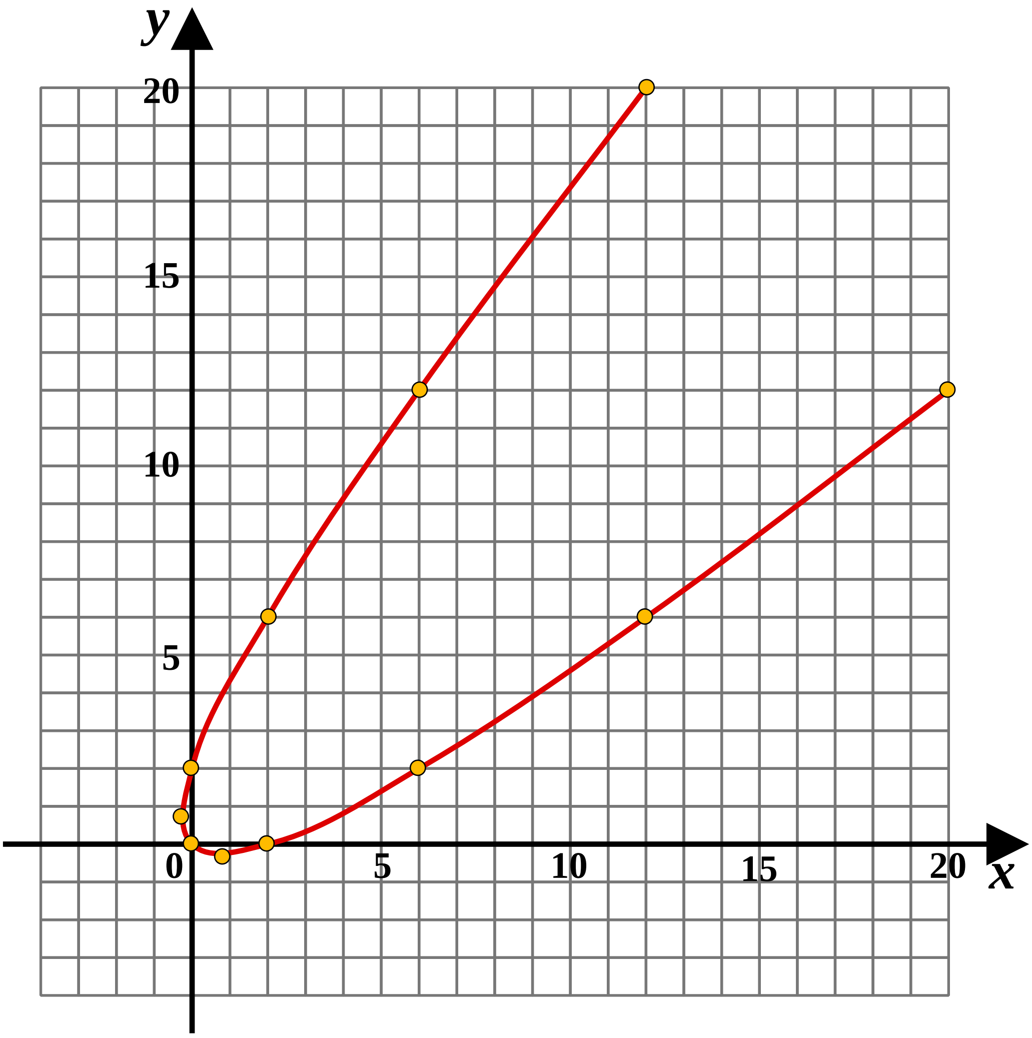
$$x - y = 2t$$

$$\frac{dy}{dx} = \frac{2t - 1}{2t + 1}$$

$$= \frac{x - y - 1}{x - y + 1}$$

[4 marks]

The Question 7, part (ii) graph



Lesson 12

A-Level Pure Mathematics : Year 2

Differentiation IV

12.1 Test

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 50

Question 1

Differentiate each of the following;

(i) $y = \frac{4x^6}{15}$

[2 marks]

(ii) $y = ln\left(x^4 - 1\right)$

[2 marks]

(iii) $y = \frac{7}{3x^2 - 2}$

[2 marks]

(iv) $y = e^{x^2}$

[2 marks]

Question 2

The product rule states that $(uv)' = uv' + u'v$

Use the rule to differentiate $y = 5x^2 \sin x$

[3 marks]

Question 3

The quotient rule states that $\left(\frac{u}{v}\right)' = \frac{vu' - v'u}{v^2}$

Use the rule to differentiate the following, simplifying your answer;

$$y = \frac{e^{4x}}{x^2}$$

[3 marks]

Question 4

(i) Use derivatives of $\sin x$ and $\cos x$ to prove the derivative of $\cot x$ is $-\csc^2 x$

[4 marks]

(ii) Hence, or otherwise, use the chain rule to differentiate;

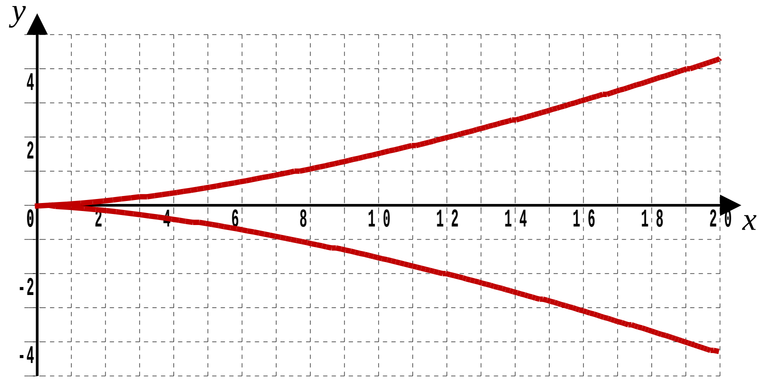
$$y = \cot^2 x$$

[2 marks]

Question 5

The semi-cubical parabola shown below, has parametric equations,

$$x = 12\,t^2, \qquad y = 2\,t^3$$



(i) Find in terms of t,

(a) $\dfrac{dx}{dt}$ **(b)** $\dfrac{dy}{dt}$ **(c)** $\dfrac{dy}{dx}$

[2, 2, 2 marks]

(ii) Write down the coordinates of the point on the curve that corresponds to the parameter t having the value 1

[1 mark]

(iii) What is the gradient of the curve at your part (ii) point ?

[1 mark]

(iv) By making use of your part **(ii)** and **(iii)** answers, determine the equation of the normal to the curve from the point at which $t = 1$

[3 marks]

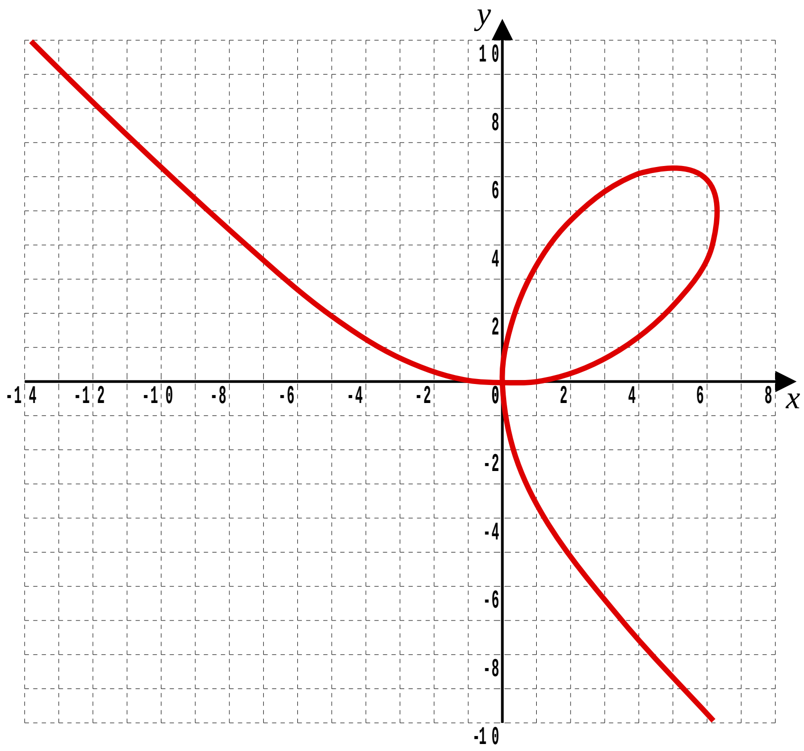
(v) Draw your part (iv) normal onto the graph above

[1 mark]

Question 6

The graph is of a famous curve known as the Folium of Descartes with equation,

$$y^3 + x^3 = 3xy$$



Differentiate Descartes' Folium by means of implicit differentiation.

Write your simplified answer in the form $\dfrac{dy}{dx} = f\,(x,\ y)$

[5 marks]

Question 7

The parametric equations of a curve are;

$$x = t + \frac{1}{t}, \qquad y = t - \frac{1}{t}$$

(i) Complete the following table by way of working out some points on the graph of this curve.

t	-5	-4	-3	-2	-1	$-\frac{1}{2}$	$-\frac{1}{3}$	$-\frac{1}{4}$	$-\frac{1}{5}$
x									
y									

t	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$	1	2	3	4	5
x									
y									

[4 marks]

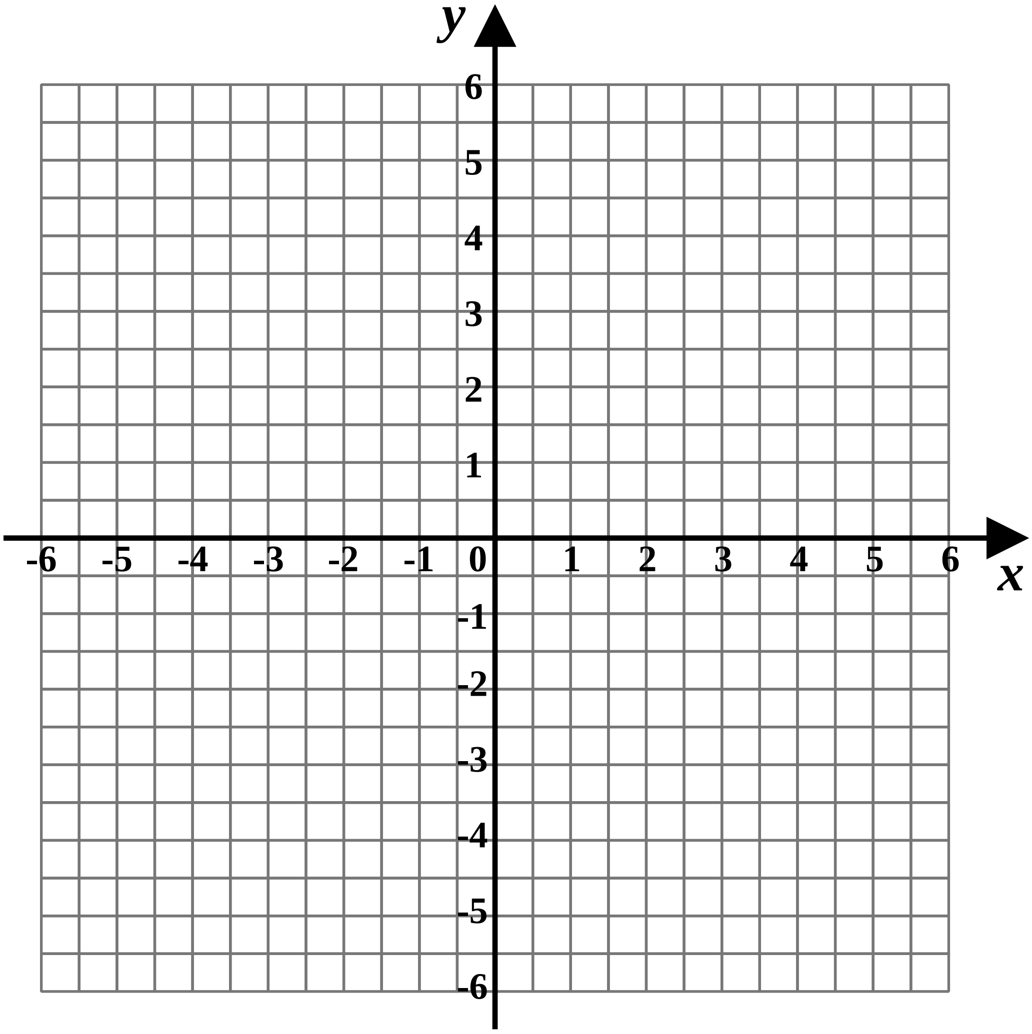
(ii) On the graph paper provided on the following page plot a graph of the curve.

(iii) What is the name of this type of curve ?

[1 mark]

(iv) Find, in terms of t, an expression for the derivative of this curve.

[4 marks]



[4 marks]

12.2 Answers

Answer 1

(i) $y = \dfrac{4x^6}{15} \Rightarrow \dfrac{dy}{dx} = \dfrac{4 \times 6x^5}{15} \Rightarrow \dfrac{dy}{dx} = \dfrac{8x^5}{5}$

[2 marks]

(ii) $y = ln\left(x^4 - 1 \right) \Rightarrow \dfrac{dy}{dx} = \dfrac{4x^3}{x^4 - 1}$

[2 marks]

(iii) $y = \dfrac{7}{3x^2 - 2} \Rightarrow y = 7\left(3x^2 - 2 \right)^{-1}$

$$\frac{dy}{dx} = -7\left(3x^2 - 2 \right)^{-2} \times 6x \Rightarrow \frac{dy}{dx} = -\frac{42x}{\left(3x^2 - 2 \right)^2}$$

[2 marks]

(iv) $y = e^{x^2}$

$$\frac{dy}{dx} = e^{x^2} \times 2x$$

[2 marks]

Answer 2

$$\begin{aligned} y &= 5x^2 \sin x \\ \frac{dy}{dx} &= 5x^2 (\cos x) + 10x \sin x \\ &= 5x \left(x \cos x + 2 \sin x \right) \end{aligned}$$

[3 marks]

Answer 3

$$\begin{aligned} y &= \frac{e^{4x}}{x^2} \\ \frac{dy}{dx} &= \frac{x^2 \left(e^{4x} \times 4 \right) - 2x\, e^{4x}}{(x^2)^2} \\ &= \frac{2x\, e^{4x} (2x - 1)}{x^4} \\ &= \frac{2\, e^{4x} (2x - 1)}{x^3} \end{aligned}$$

[3 marks]

Answer 4

(i)

$$
\begin{aligned}
y &= \cot x \\
&= \frac{\cos x}{\sin x} \\
\frac{dy}{dx} &= \frac{\sin x\,(-\sin x) - \cos x \cos x}{\sin^2 x} \\
&= (-1)\,\frac{\sin^2 x + \cos^2 x}{\sin^2 x} \\
&= -\frac{1}{\sin^2 x} \\
&= -\csc^2 x \qquad \square
\end{aligned}
$$

[4 marks]

(ii)

$$
\begin{aligned}
y &= \cot^2 x \\
\frac{dy}{dx} &= -2 \cot x \csc^2 x
\end{aligned}
$$

[2 marks]

Answer 5

(i) **(a)** $\frac{dx}{dt} = 24t$ **(b)** $\frac{dy}{dt} = 6\,t^2$

(c)

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{dt} \times \frac{dt}{dx} \\
&= \frac{6\,t^2}{24t} \\
&= \frac{t}{4}
\end{aligned}
$$

[2, 2, 2 marks]

(ii) When $t = 1,\ x = 12,\ y = 2$ That is, $(12,\ 2)$

[1 mark]

(iii) When $t = 1,\ \frac{dy}{dx} = \frac{1}{4}$

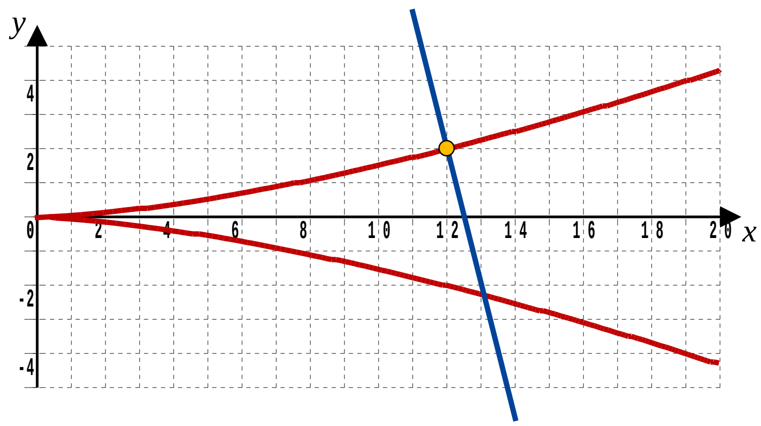
[1 mark]

(iv) $y = mx + c \Rightarrow 2 = -4 \times 12 + c \Rightarrow c = 50$

$y = -4x + 50$

[2 marks]

(v)



[1 mark]

Answer 6

$$y^3 + x^3 = 3xy$$

Differentiate with respect to x

$$3y^2 \frac{dy}{dx} + 3x^2 = 3x\frac{dy}{dx} + 3y$$

Divide through by 3

$$y^2 \frac{dy}{dx} + x^2 = x\frac{dy}{dx} + y$$

$$\frac{dy}{dx}\left(y^2 - x\right) = y - x^2$$

$$\frac{dy}{dx} = \frac{y - x^2}{y^2 - x}$$

[5 marks]

Answer 7

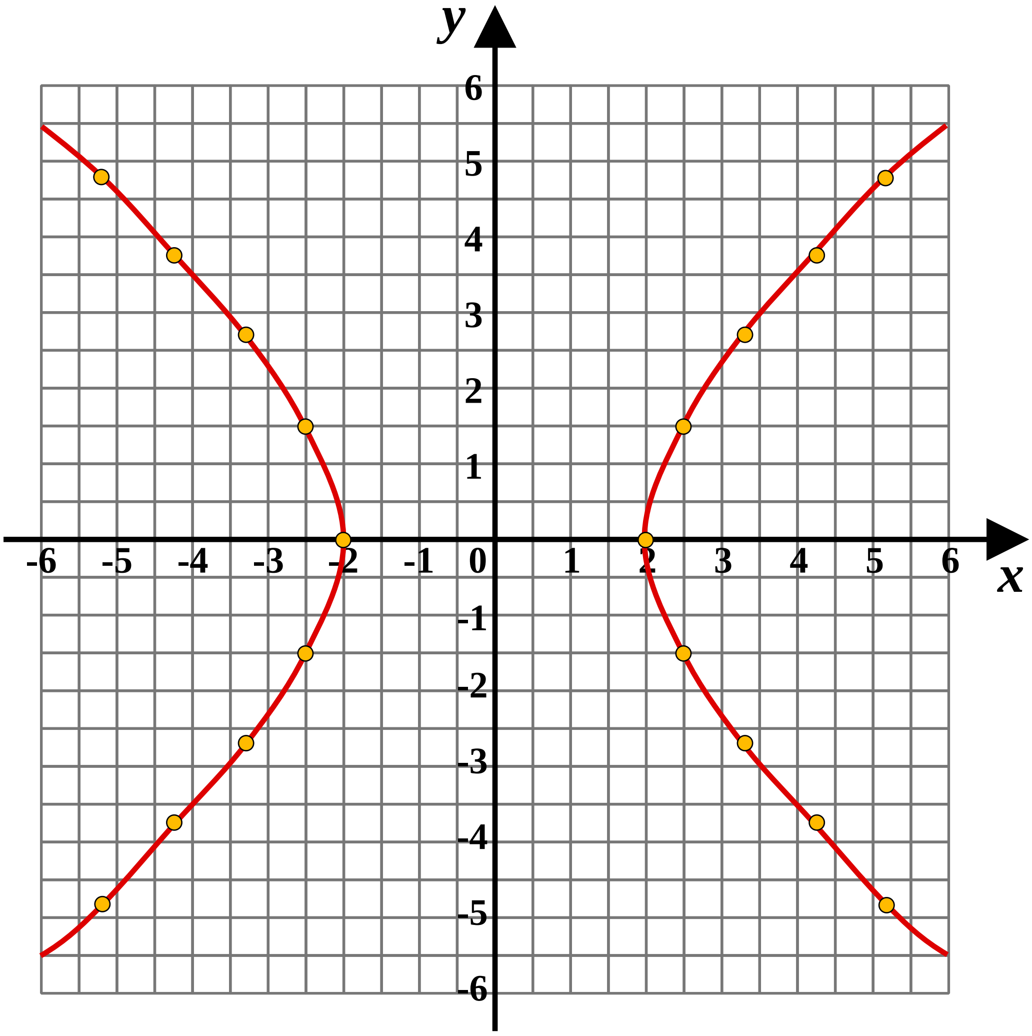
(i)

t	-5	-4	-3	-2	-1	$-\frac{1}{2}$	$-\frac{1}{3}$	$-\frac{1}{4}$	$-\frac{1}{5}$
x	-5.2	-4.25	-3.33	-2.5	-2	-2.5	-3.33	-4.25	-5.2
y	-4.8	-3.75	-2.66	-1.5	0	1.5	2.66	3.75	4.8

t	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$	1	2	3	4	5
x	5.2	4.25	3.33	2.5	2	2.5	3.33	4.25	5.2
y	-4.8	-3.75	-2.66	-1.5	0	1.5	2.66	3.75	4.8

[4 marks]

(ii)



[3 marks]

(iii) A hyperbola, a hyperbolic curve

[1 mark]

(iv)

$$x = t + t^{-1} \Rightarrow \frac{dx}{dt} = 1 - t^{-2}$$

$$y = t - t^{-1} \Rightarrow \frac{dy}{dt} = 1 + t^{-2}$$

$$\frac{dy}{dx} = \frac{1 + t^{-2}}{1 - t^{-2}} \times \frac{t^2}{t^2}$$

$$= \frac{t^2 + 1}{t^2 - 1}$$

[4 marks]

Lesson 13

A-Level Pure Mathematics : Year 2
Differentiation IV

13.1 Differentiation : Later Date Revision

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 40

Question 1

Without looking up the formulae booklet, differentiate each of the following with respect to x. If you can remember a formulae book formula you can use it !

(i) $y = e^{5x}$

(ii) $y = 4\sin 6x$

(iii) $y = \cos^2 3x$

(iv) $y = \tan x$

(v) $y = \dfrac{1}{\cos x}$

(vi) $y = \ln\left(5x^3\right)$

(vii) $y = (\ln(5x))^3$

[14 marks]

Question 2

Use the product rule to find the derivative with respect to x of $y = 3x\, e^{5x}$ giving the answer in the form $\frac{dy}{dx} = A\, e^{5x}(Bx + C)$ where A, B and C are integers to be found.

[3 marks]

Question 3

Use the quotient rule to find the derivative with respect to x of $y = \frac{\sin 3x}{2\, x^2}$

and simplify your answer.

[4 marks]

Question 4

Use the chain rule to find the derivative with respect to x of $y = \left(1 + \cos^2 3x \right)^5$

Write your answer in the form $\frac{dy}{dx} = A \left(1 + \cos^2 3x \right)^B \sin Cx$ where A, B, and C are integers to be found.

[3 marks]

Question 5

A curve is described parametrically by the equations

$$x = t - cos^2 t \qquad\qquad y = sin^2 t$$

(i) Show that, $\dfrac{dy}{dx} = \dfrac{sin\, 2t}{1 + sin\, 2t}$

[4 marks]

(ii) Show that when $t = \dfrac{\pi}{6}$, $\dfrac{dy}{dx} = m\sqrt{3} + n$

for some integer values of *m* and *n* that you should determine.

[4 marks]

Question 6

Find the tangent to the curve $y = \sin x$ when $x = \dfrac{\pi}{3}$

Give your answer in the form $y = mx + c$ and give the exact value for c.

[4 marks]

Question 7

Differentiate implicitly to find $\dfrac{dy}{dx}$ for the curve $4x^3 + 5y^4 + 7xy = 0$

[4 marks]

13.2 Answers (Later Date Revision)

Answer 1

(i) $y = e^{5x}$ $\qquad \dfrac{dy}{dx} = 5\,e^{5x}$

(ii) $y = 4\,sin\,6x$ $\qquad \dfrac{dy}{dx} = 24\,cos\,6x$

(iii) $y = cos^2\,3x$ $\qquad \dfrac{dy}{dx} = -\,6\,cos\,3x\,sin\,3x$

(iv) $y = tan\,x$ $\qquad \dfrac{dy}{dx} = sec^2\,x$ $\qquad$ Use quotient rule if forgotten

(v) $y = \dfrac{1}{cos\,x}$

$= sec\,x$ $\qquad \dfrac{dy}{dx} = sec\,x\,tan\,x$ $\qquad$ Use chain rule if forgotten

(vi) $y = ln\left(5\,x^3\right)$

$= ln\,5 + 3\,ln\,x$ $\qquad \dfrac{dy}{dx} = \dfrac{3}{x}$

(vii) $y = (\,ln\,(5x)\,)^3$ $\qquad \dfrac{dy}{dx} = 3\,(\,ln\,(5x)\,)^2 \times \dfrac{1}{x}$

[14 marks]

Answer 2

$$y = 3x\,e^{5x}$$

$$\frac{dy}{dx} = 3x \times 5\,e^{5x} + 3 \times e^{5x}$$

$$= 3\,e^{5x}\left(5x + 1\right)$$

Thus $A = 3$, $B = 5$ and $C = 1$

[3 marks]

Answer 3

$$y = \frac{sin\,3x}{2\,x^2}$$

$$\frac{dy}{dx} = \frac{2\,x^2\,3\,cos\,3x - 4x\,sin\,3x}{4\,x^4}$$

$$= \frac{2x\,(\,3x\,cos\,3x - 2\,sin\,3x\,)}{4\,x^4}$$

$$= \frac{3x\,cos\,3x - 2\,sin\,3x}{2\,x^3}$$

[4 marks]

Answer 4

$$y = \left(1 + cos^2 3x\right)^5$$

$$\frac{dy}{dx} = 5\left(1 + cos^2 3x\right)^4 \times 2\,cos\,3x \times (-\,sin\,3x) \times 3$$

$$= -15\left(1 + cos^2 3x\right)^4 sin\,6x$$

Thus $A = -15$, $B = 4$ and $C = 6$

[3 marks]

Answer 5

(i)

$$x = t - cos^2 t \quad \Rightarrow \quad \frac{dx}{dt} = 1 - 2\,cos\,t\,(-\,sin\,t)$$

$$= 1 + sin\,2t$$

$$y = sin^2 t \quad \Rightarrow \quad \frac{dy}{dt} = 2\,sin\,t\,cos\,t$$

$$= sin\,2t$$

$$\therefore \frac{dy}{dx} = \frac{sin\,2t}{1 + sin\,2t} \qquad \square$$

[4 marks]

(ii)

When $t = \frac{\pi}{6}$, $\frac{dy}{dx} = \frac{sin\left(\frac{2\pi}{6}\right)}{1 - sin\left(\frac{2\pi}{6}\right)}$

$$= \frac{\left(\frac{\sqrt{3}}{2}\right)}{1 - \left(\frac{\sqrt{3}}{2}\right)} \times \frac{\left(\frac{2}{1}\right)}{\left(\frac{2}{1}\right)}$$

$$= \frac{\sqrt{3}}{2 - \sqrt{3}} \times \frac{2 + \sqrt{3}}{2 + \sqrt{3}}$$

$$= \frac{2\sqrt{3} + 3}{4 - 3}$$

$$= 2\sqrt{3} + 3$$

Thus $m = 2$ and $n = 3$

[4 marks]

Answer 6

$$y = \sin x \qquad \Rightarrow \qquad \frac{dy}{dx} = \cos x$$

When $x = \dfrac{\pi}{3}$, $y = \sin\left(\dfrac{\pi}{3}\right) = \dfrac{\sqrt{3}}{2}$ and $\dfrac{dy}{dx} = \cos\left(\dfrac{\pi}{3}\right) = \dfrac{1}{2}$

For the tangent, $y - mx + c$

$$\frac{\sqrt{3}}{2} = \frac{1}{2} \times \frac{\pi}{3} + c$$

$$c = \frac{\sqrt{3}}{2} - \frac{\pi}{6}$$

$$\therefore\ y = \frac{1}{2}x + \frac{\sqrt{3}}{2} - \frac{\pi}{6}$$

[4 marks]

Answer 7

$$4x^3 + 5y^4 + 7xy = 0$$

$$12x^2 + 20y^3\frac{dy}{dx} + 7x\frac{dy}{dx} + 7y = 0$$

$$20y^3\frac{dy}{dx} + 7x\frac{dy}{dx} = -12x^2 - 7y$$

$$\frac{dy}{dx}\left(20y^3 + 7x\right) = -12x^2 - 7y$$

$$\frac{dy}{dx} = -\frac{12x^2 - 7y}{20y^3 + 7x}$$

[4 marks]