Applied A-Level Mathematics
Mechanics
Year 2

M O M E N T S
I I

Ten nails balanced on the head of a single nails.
The most nails balanced on the head of a single nail is 349 and was achieved by Alexander Bendikov in Sevastopol, Russia, on 26th May 2018
(Guinness World Records, April 2020)

M O M E N T S   I I

Lesson 1

A-Level Applied Mathematics
Mechanics : Moments II : Year 2

1.1 Introduction

In this second look at the topic of Moments, ideas previously explored are developed further using the key ideas of resolving forces and friction. The mechanical systems considered are still simple but, even so, are often encountered in everyday life. These include such items as the metalwork supporting a pub's hanging sign, a ladder against a wall, and a plank of wood resting against an oil barrel that's on its side. In such situations, unlike those considered previously, the main rod may not be horizontal or the forces acting on the rod may not be at right angles to the rod.

1.2 A Hanging Basket

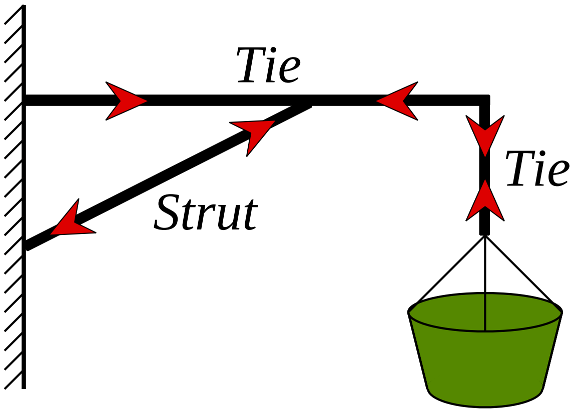
Most people, when looking at a hanging basket, focus on the plant in the basket.
To a mechanical engineer, however, the supporting structure is of more interest.
Notice that the main element in the metalwork is a horizontal rod to which the basket is attached at the end furthest from the wall. Expecting this rod to simply stick out of the vertical wall and, without bending, hold up a heavy basket would require a heavier and thicker rod than is being employed here. There is a second rod, called a strut, that is supporting the main rod, from below. It's because this second rod has internal forces that are thrusting outward at each end that it's called a strut.
Previous work involving, for example, a tow bar between a car and a caravan, dealt with a tie, the tow bar; a rod with an internal inward force at either end; a tension.

1.3 Struts and Ties

This is a strut.
Its internal forces are pushing outward at either end.
The rod is resisting being compressed, that is, it's resisting being made shorter.
The forces within the rod are referred to as thrusts.

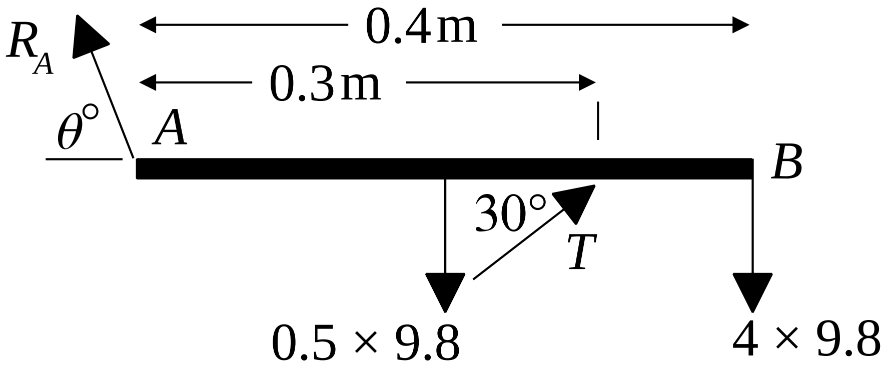
This is a tie.
Its internal forces are pulling inward at either end.
The rod is resisting being stretched, that is, it's resisting being made longer.
The forces within the rod are referred to as tensions.

Here is a diagram of the hanging basket showing some of the struts and ties.



1.4 Force at an Angle

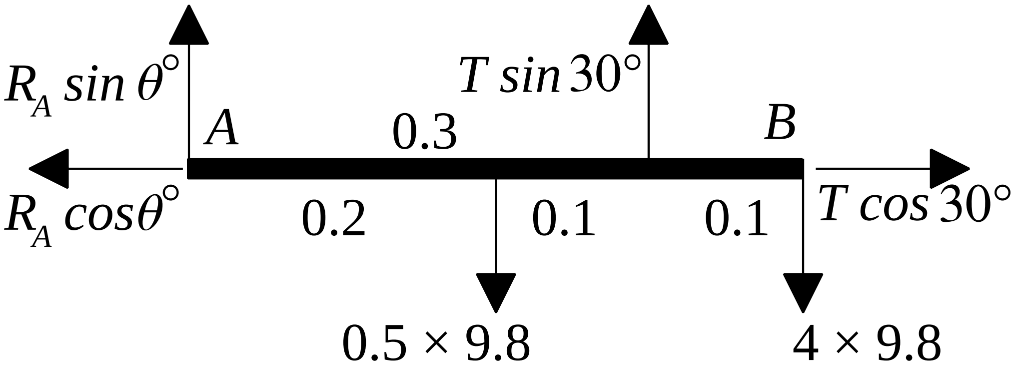
With the difference between a strut and a tie understood a diagram showing the forces acting on the horizontal rod holding up the hanging basket can now be drawn.



Notice the force from the strut is acting at an angle, as is the force exerted on the rod by the wall, R_A. The angle between strut and rod is easy to physically measure, it's 30°, and the length to where the strut meets the rod is also easily measured, it's 0.3 metres along the 0.4 metre rod from *A*.

For a mechanical engineer, the problem now is to determine the thrust in the strut and the force of the wall on the rod. It is good practice for the joint at the wall not to be under unnecessarily large forces and so it is assumed that the joint there can be modelled as a smooth hinge. The rod is assumed to be uniform.

The solution begins having already resolved the thrust, T, and the reaction R_A into component parts perpendicular and parallel to the rod, thus obtaining the following working diagram,



Determine **(i)** The thrust, T, in the strut.

(ii) The magnitude of the reaction, R_A, of the wall on the rod.

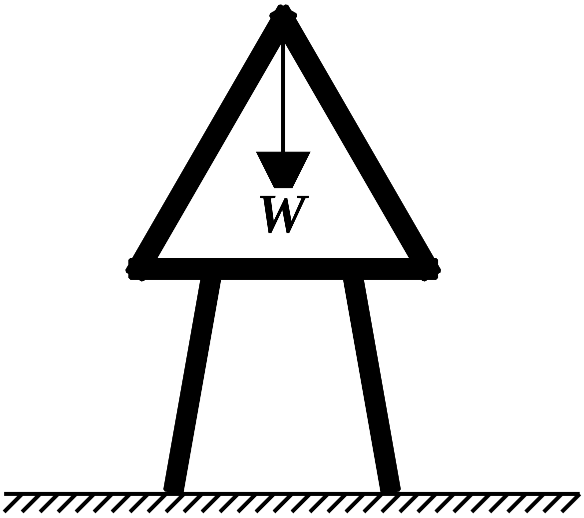
(iii) The angle, θ, the reaction R_A makes to the horizontal.

Teaching Video : http://www.NumberWonder.co.uk/Video/v9084(1).mp4

1.5 Exercise

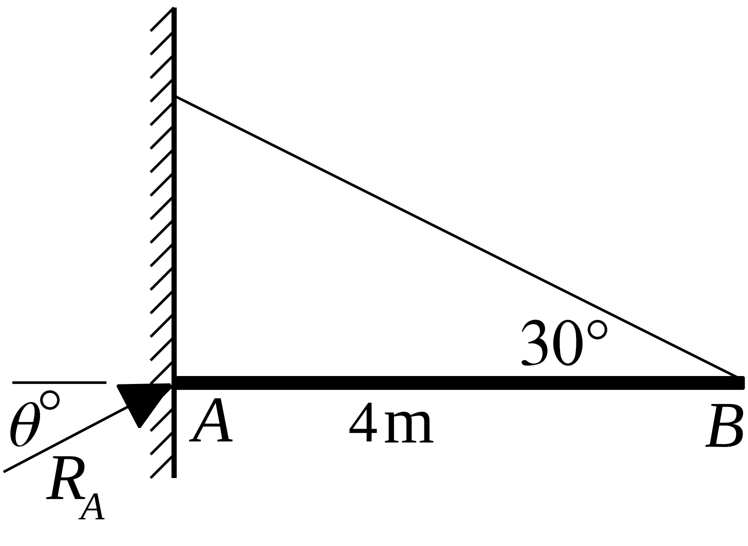
Question 1

Five wooden beams are joined together to form a structure from which a weight, *W*, can be raised and lowered. Where each beam is connected to another a single bolt is used, effectively forming a smooth hinge. The structure is concreted into the ground. On the diagram next to each of the five beams label if it is acting as a strut or a tie.



Question 2

A nonuniform rod *AB* has a weight of 100 N and length 4 metres.
It is freely hinged to a vertical wall at *A*, and held in horizontal equilibrium by a cable attached at an angle of 30° to end *B*.
The tension in the cable is 80 N.



From an inspection of the hinge at end *A*, it is clear that the wall is exerting a force on the rod in the approximate direction indicated on the diagram by R_A.

(i) When the 80 N force in the cable is described as a “tension” (rather than a “thrust”) does this tell you that the supporting cable is a strut or a tie ?

(ii) Draw a “working diagram” showing all significant forces that are acting on the rod. Any forces that are not parallel or perpendicular to the rod should be resolved into component parts which are.

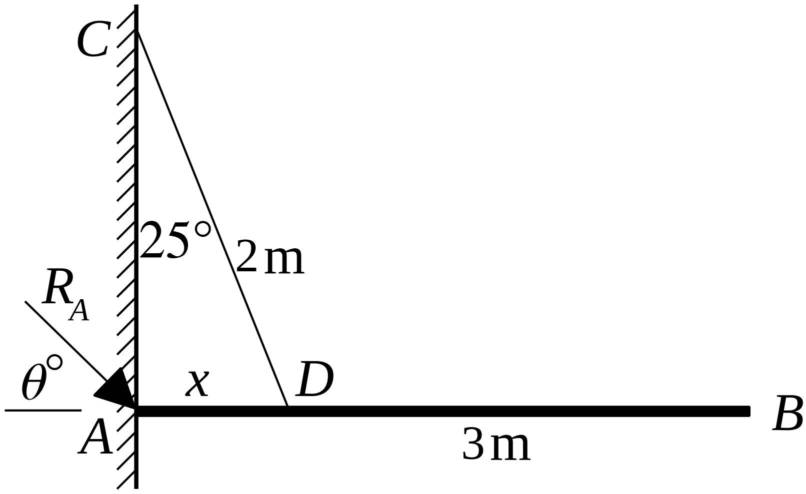
(iii) By taking moments about A, find the distance of the centre of mass of the nonuniform rod from the wall.

(iv) By balancing forces vertically obtain one equation in terms of R_A and θ.

(v) By balancing forces horizontally obtain another equation in terms of R_A and θ.

(vi) Solve your part (iv) and (v) equations simultaneously to find R_A and θ.

Question 3



A beam *AB* of mass 18 kg and length 3 m is smoothly hinged to a vertical wall at *A*.
The beam is held in equilibrium in a horizontal position by a wire of length 2 m.
One end of the wire is fixed to a point *C* on the wall vertically above *A*.
The other end of the wire is fixed to a point *D* on the beam such that angle *ACD* is 25°.
The beam is modelled as a uniform rod and the wire as a light inextensible string.
From looking at the hinge at end *A*, it is clear that the wall is exerting a force on the rod in the approximate direction indicated on the diagram by R_A.

(i) Draw a "working diagram" showing all significant forces that are acting on the rod. Any forces that are not parallel or perpendicular to the rod should be resolved into component parts which are.

(ii) Use simple trigonometry to calculate the length *AD*, marked x in the diagram.

(iii) By taking moments about *A*, find the tension in the tie *DC*.

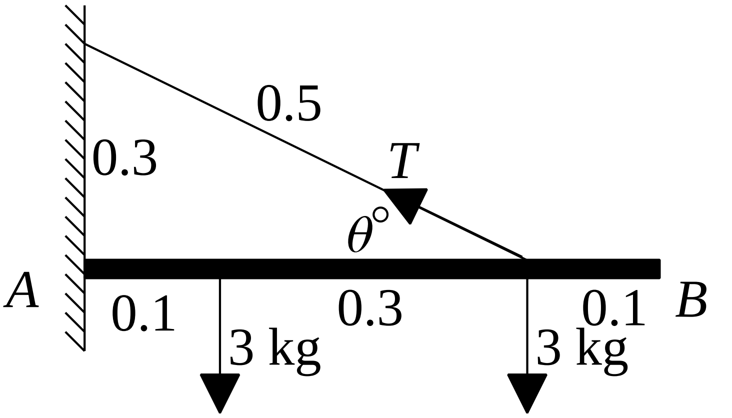
(iv) By balancing forces vertically obtain an equation in terms of R_A and θ.

(v) By balancing forces horizontally obtain another equation in terms of R_A and θ.

(vi) Solve your part (iv) and (v) equations simultaneously to find R_A and θ.

Question 4

The Bath Sweet Shop has a 6 kg sign suspended centrally from two points 0.3 metres apart from an 0.5 metre metal rod. This horizontal uniform rod has a mass of 2 kg and is supported by a tie which is also of length 0.5 metres. The rod is attached to the wall by a smooth hinge which is 0.3 metres vertically below where the tie is attached to the wall. The tie is attached to the rod 0.4 metres along the rod where it makes an angle of θ to the rod. Some, but not all, of this information is shown on the diagram below,



Determine

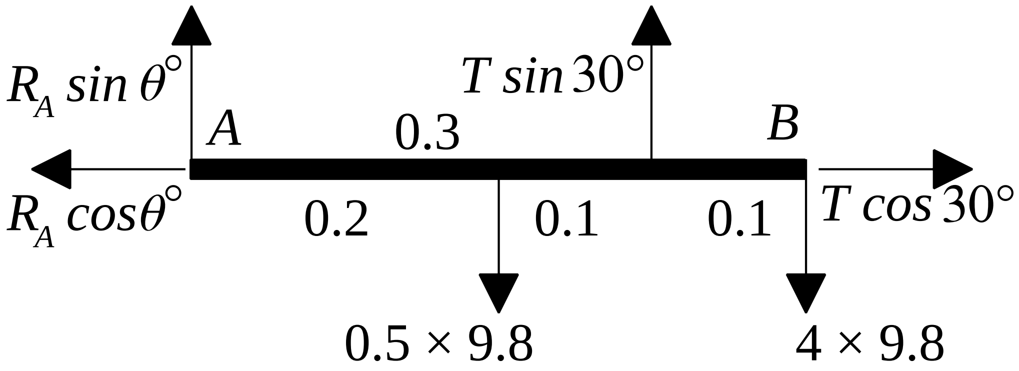
(i) The angle marked θ.

(ii) The tension T in the tie supporting the rod.

(iii) The magnitude and the direction of the force applied by the wall on the rod at end A.

1.6 Answers

1.6.1 Solution to 1.4 Example



(i) Taking moments about *A*,

$$T \sin 30 \times 0.3 = 0.5 \times 9.8 \times 0.2 + 4 \times 9.8 \times 0.4$$

$$0.5T = 0.98 + 15.68$$

$$T = 33.3 \text{ N}$$

(ii) Balancing forces horizontally,

$$R_A \cos\theta = T \cos 30$$

$$R_A \cos\theta = 33.3 \times 0.866$$

$$= 28.8 \text{ N}$$

Balancing forces vertically,

$$R_A \sin\theta + T \sin 30° = 0.5 \times 9.8 + 4 \times 9.8$$

$$R_A \sin\theta + 33.3 \times 0.5 = 4.9 + 39.2$$

$$R_A \sin\theta = 27.5 \text{ N}$$

Solving the latter two equations simultaneously,

$$\frac{R_A \sin\theta}{R_A \cos\theta} = \frac{27.5}{28.8}$$

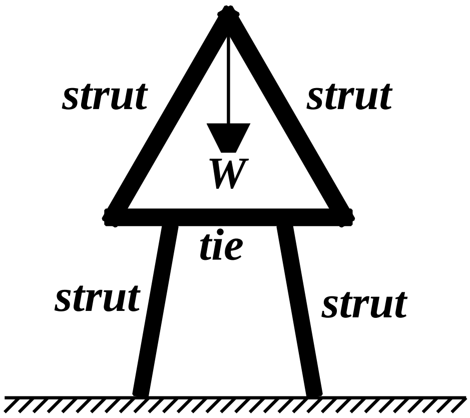
$$\tan\theta = 0.955$$

$$\theta = 43.7°$$

$$\therefore R_A = 39.8 \text{ N}$$

1.6.2 Solutions to 1.5 Exercise

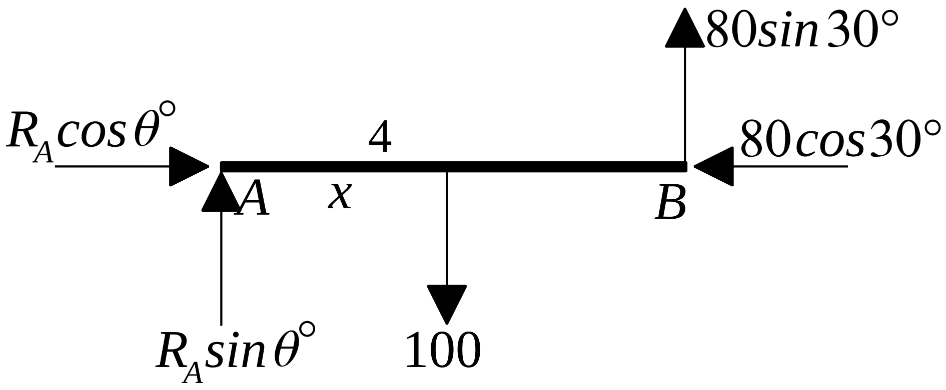
Answer 1



Answer 2

(i) a tie

(ii)



(iii) Taking moments about A,

$$100x = 80\,sin\,30 \times 4$$

$$x = 1.6 \text{ metres}$$

(iv) Balancing forces vertically,

$$R_A\,sin\,\theta + 80\,sin\,30 = 100$$

$$R_A\,sin\,\theta = 60$$

(v) Balancing forces horizontally,

$$R_A\,cos\,\theta = 80\,cos\,30$$

$$= 69.3$$

(vi)

$$\frac{R_A\,sin\,\theta}{R_A\,cos\,\theta} = \frac{60}{69\;3}$$

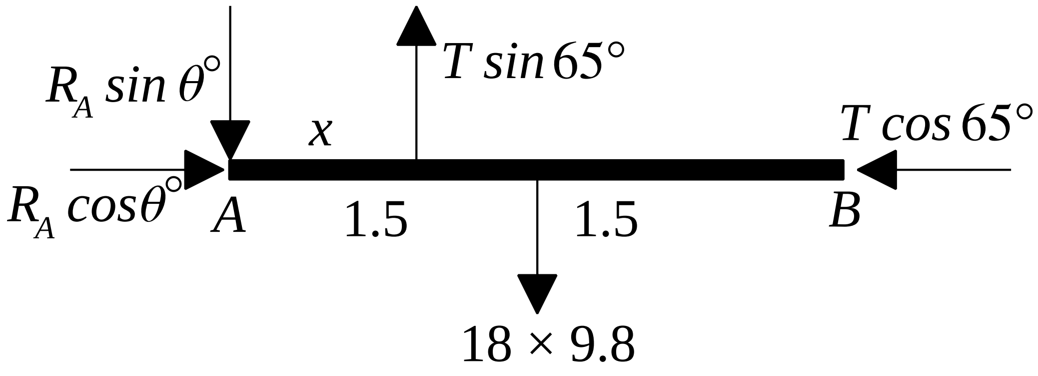
$$tan\,\theta = 0.866$$

$$\theta = 40.9°$$

$$\therefore\; R_A = 91.7\text{ N}$$

Answer 3

(i)



(ii)

$$\sin 25 = \frac{x}{2}$$

$$x = 0.85$$

(iii) Taking moments about *A*,

$$T \sin 65 \times x = 18 \times 9.8 \times 1.5$$

$$T \sin 65 \times 0.85 = 265$$

$$T = 343 \text{ N}$$

(iv) Balancing forces vertically,

$$R_A \sin\theta + 18 \times 9.8 = T \sin 65$$

$$R_A \sin\theta + 176 = 343 \times 0.906$$

$$R_A \sin\theta = 135$$

(v) Balancing forces horizontally,

$$R_A \cos\theta = T \cos 65$$

$$= 343 \times 0.423$$

$$= 145$$

(vi)

$$\frac{R_A \sin\theta}{R_A \cos\theta} = \frac{135}{145}$$

$$\tan\theta = 0.931$$

$$\theta = 43.0°$$

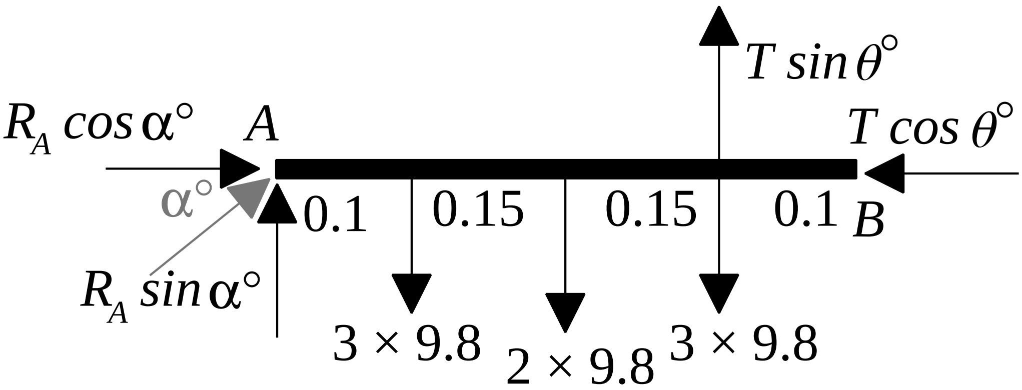
$$\therefore R_A = 198 \text{ N}$$

Answer 4

(i) By simple trigonometry,

$$\cos\theta = \frac{4}{5} \qquad \sin\theta = \frac{3}{5} \qquad \tan\theta = \frac{3}{4} \qquad \theta = 36.9°$$

(ii) The working diagram,



Taking moments about *A*,

$$T\sin\theta = 3 \times 9.8 \times 0.1 + 2 \times 9.8 \times 0.25 + 3 \times 9.8 \times 0.4$$

$$T \times \frac{3}{5} = 2.94 + 4.9 + 11.76$$

$$0.6\,T = 19.6$$

$$T = 32.7\text{ N}$$

(iii) Balancing forces vertically,

$$R_A \sin\alpha + T\sin\theta = 3 \times 9.8 + 2 \times 9.8 + 3 \times 9.8$$

$$R_A \sin\alpha + 32.7 \times \frac{3}{5} = 29.4 + 19.6 + 29.4$$

$$R_A \sin\alpha = 66.8$$

Balancing forces horizontally,

$$R_A \cos\alpha = T\cos\theta$$

$$= 32.7 \times \frac{4}{5}$$

$$= 26.2$$

Combining equations,

$$\frac{R_A \sin\alpha}{R_A \cos\alpha} = \frac{66.8}{26.2}$$

$$\tan\alpha = 2.55$$

$$\alpha = 68.6°$$

$$\therefore\ R_A = 71.7\text{ N}$$

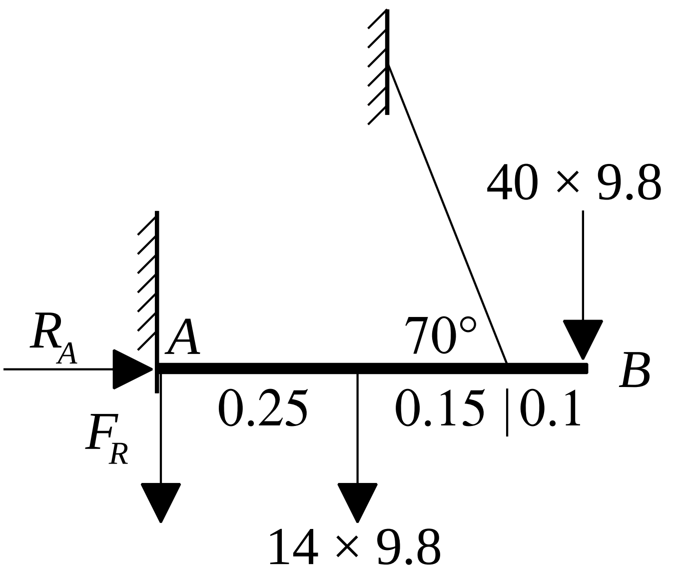
Lesson 2

2.1 Unhinged

A rock climber is abseiling down a rock face. On the way down she stops to have her photograph taken. For dramatic effect she takes her hands off the rope and hangs motionless in equilibrium. Although her boots are about to slip on the vertical section of rock face against which they press, a spectacular photograph is obtained.

At the moment the photograph is being taken, her situation can be modelled as follows,

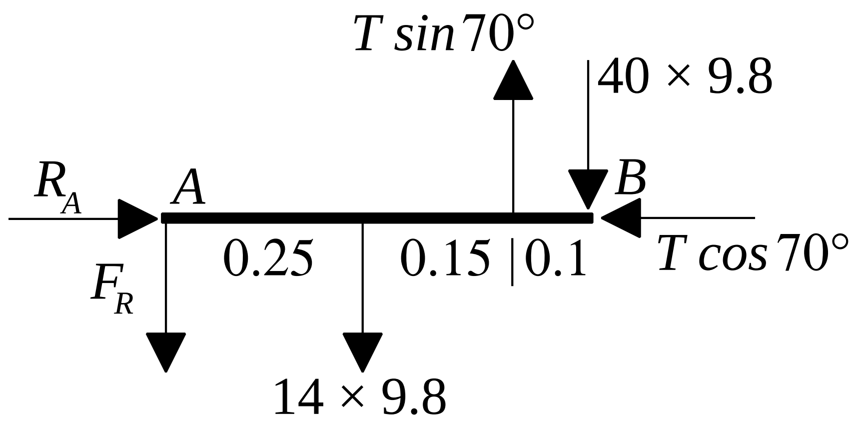
- Her legs as a single uniform horizontal rod *AB* of mass 14 kg and length 0.5 m
- The remainder of her body as a mass of 40 kg acting at *B*
- The supporting rope as being attached to the rod 0.1 m from end *B*
- The overall reaction of the rock on her shoes as being composed of two component parts, a reaction perpendicular to the vertical rock face, R_A, and a friction force parallel to the vertical rock face, F_R



The difference between this situation and those of Lesson 1 is that, rather than having a hinge at one end of the rod where is meets a vertical support, it is simply friction that is holding that end of the rod up.

The interest is in finding,
(i) The tension in the climber's rope, T,
(ii) The friction force, F_R, that's stopping her boots sliding up the rock face,
(iii) The normal reaction, R_A, on her boots,
(iv) The coefficient of friction, μ, between her boots and the rock face.
In order to relate this question to those of Lesson 1, the overall force exerted by the rock face on end A of the rod, her boots, will also be found.

To begin answering these questions, the following working diagram is the starting point. As before, the standard technique of resolving the tension in the climber's rope, T, has been applied to obtain its component parts, $T \sin 70°$ and $T \cos 70°$. Notice that the $T \cos 70°$ force has been translated along its “line of action” to place it at a convenient place on the diagram. That this can be done is a routine technique, used repeatedly in this topic.



Teaching Video : http://www.NumberWonder.co.uk/Video/v9084(2).mp4

(i) What is the tension in the climber's rope, T ?

(ii) Find the friction force, F_R, that's stopping her boots sliding up the rock face,

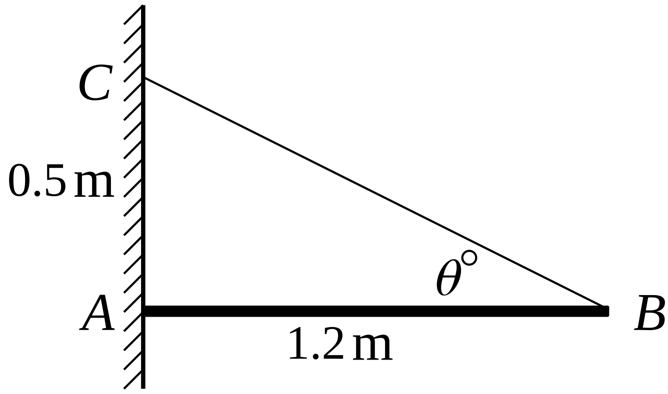
(iii) What is the normal reaction, R_A, on her boots ?

(iv) Determine coefficient of friction, μ, between her boots and the rock face.

(v) Calculate the overall force exerted by the rock face on end A of the rod.

2.2 Exercise

Question 1



A uniform rod *AB* of length 1.2 metres and mass 30 kg rests with one end, *A*, touching a rough horizontal wall. The rod is kept horizontal by a light inextensible string *BC* where *C* lies on the vertical wall directly above *A*. The tension in the tie *BC* is *T*. The plane *ABC* is perpendicular to the wall and $\angle ABC$ is $\theta°$.

(i) Draw a "working diagram" showing all significant forces acting on the rod including friction, F_R, parallel to the wall and a normal reaction, R_A, perpendicular to the wall. On this diagram show the tension, *T*, resolved into component parts parallel and perpendicular to the rod.

[4 marks]

(ii) Use simple trigonometry to write down the exact values of *tan* θ, *cos* θ and *sin* θ

[2 mark]

(iii) Show that the tension in the string is 39g Newtons.

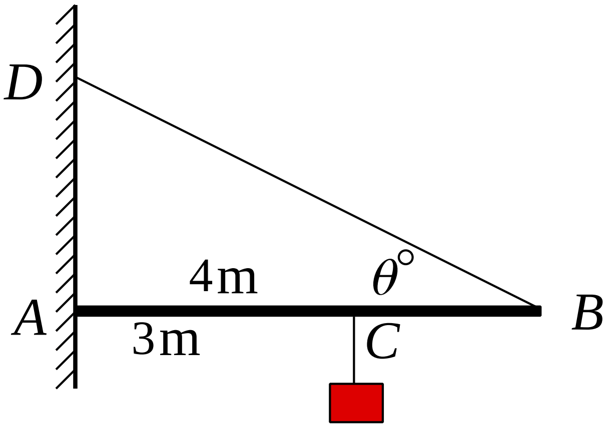
[4 marks]

(iv) Given that the rod is in limiting equilibrium, calculate the coefficient of friction between the rod and the wall.

[6 marks]

Question 2

A-Level examination Question from January 2007, M2, Q5 (edited)



A horizontal uniform rod AB has mass m and length 4 metres. The end A rests against a rough vertical wall. A particle of mass $2m$ is attached to the rod at the point C, where AC = 3 metres. One end of a light inextensible string BD is attached to the rod at B and the other end is attached to the wall at a point D, where D is vertically above A. The rod is in equilibrium in a vertical plane perpendicular to the wall.

The string is inclined at an angle θ to the horizontal, where $\tan\theta = \frac{3}{4}$, as shown.

(a) Find, in terms of m and g, the tension in the string.

[5 marks]

(b) Show that the horizontal component of the force exerted by the wall on the rod has magnitude $\frac{8}{3}mg$

[3 marks]

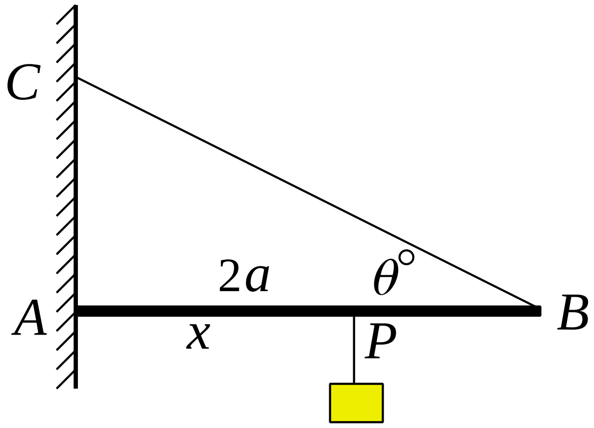
The coefficient of friction between the wall and the rod is μ. Given that the rod is in limiting equilibrium,

(c) find the value of μ

[4 marks]

Question 3

A-Level Examination Question from June 2018, Paper 3, Q9



A plank, AB, of mass M and length $2a$, rest with its end A against a rough wall.
The plank is held in a horizontal position by a rope. One end of the rope is attached to the plank at B and the other end is attached to the wall at the point C, which is vertically above A.
A small block of mass $3M$ is attached to the plank at the point P, where $AP = x$.
The plank is in equilibrium in a vertical plane which is perpendicular to the wall.

The angle between the rope and the plank is θ, where $\tan\theta = \frac{3}{4}$, as shown.

The plank is modelled as a uniform rod, the block is modelled as a particle and the rope is modelled as a light inextensible string.

(a) Using the model, show that the tension in the rope is $\dfrac{5Mg(3x + a)}{6a}$

[3 marks]

The magnitude of the horizontal component of the force exerted on the plank at A by the wall is $2Mg$.

(b) Find x in terms of a.

[2 marks]

The force exerted on the plank at A by the wall acts in a direction which makes an angle β with the horizontal.

(c) Find the value of $\tan \beta$

[5 marks]

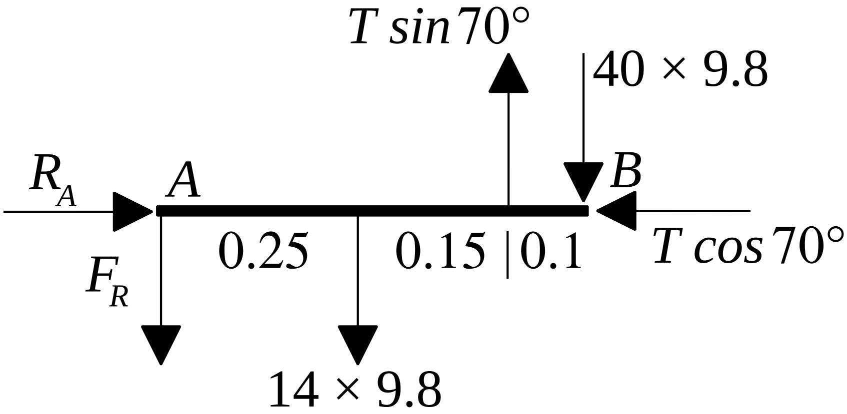
The rope will break if the tension in it exceeds $5Mg$.

(d) Explain how this will restrict the possible positions of P.
You must justify your answer carefully.

[3 marks]

2.3 Answers

2.3.1 Solution to the Introductory 2.1 Example



(i) Taking moments about A,

$$T\,sin\,70 \times 0.4 = 14 \times 9.8 \times 0.25 + 40 \times 9.8 \times 0.5$$

$$0.376\,T = 34.3 + 196$$

$$T = 613\,\mathrm{N}$$

(ii) Balancing forces vertically,

$$F_R + 14 \times 9.8 + 40 \times 9.8 = T\,sin\,70$$

$$F_R + 137 + 392 = 613\,sin\,70$$

$$F_R = 47.0\,\mathrm{N}$$

(iii) Balancing forces horizontally,

$$R_A = T\,cos\,70$$

$$= 613 \times cos70$$

$$= 210\,\mathrm{N}$$

(iv) Being told “her boots are about to slip” implied limiting equilibrium,

$$F_R = \mu\,R_A$$

$$\mu = \frac{47}{210}$$

$$= 0.224$$

(v) Overall force exerted by the rock face on her boots is given by Pythagoras',

$$\sqrt{(F_R)^2 + (R_A)^2} = \sqrt{47^2 + 210^2}$$

$$= 215\,\mathrm{N}$$

and simple trigonometry,

$$\theta = arctan\left(\frac{F_R}{R_A}\right)$$

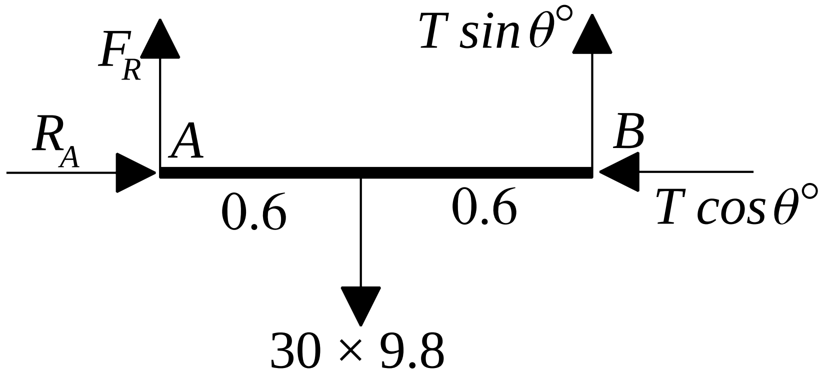
$$= arctan\left(\frac{47}{210}\right)$$

$$= 12.6° \quad \text{below the horizontal}$$

2.3.2 Solutions to 2.2 Exercise

Answer 1

(i)



(ii)

$$\tan\theta = \frac{5}{12} \qquad \cos\theta = \frac{12}{13} \qquad \sin\theta = \frac{5}{13}$$

(iii) Taking moments about *A*,

$$T\sin\theta \times 1.2 = 30 \times g \times 0.6$$

$$T \times \frac{5}{13} \times 1.2 = 18g$$

$$T = \frac{18 \times 13}{5 \times 1.2}\, g$$

$$= 39\,g \text{ Newtons} \qquad \square$$

(iv) Balancing forces horizontally,

$$R_A = T\cos\theta$$

$$= 39\,g \times \frac{12}{13}$$

$$= 36\,g \text{ N}$$

Balancing forces vertically,

$$F_R + T\sin\theta = 30\,g$$

$$F_R = 30\,g - 39\,g \times \frac{5}{13}$$

$$= 15\,g \text{ N}$$

Considering friction,

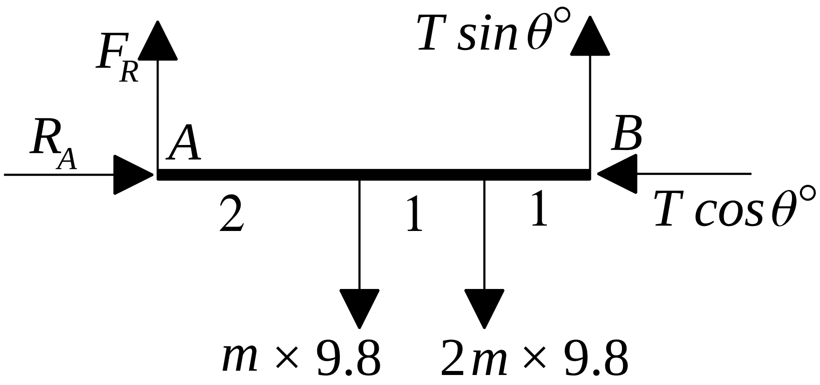
$$F_R = \mu R_A$$

$$\mu = \frac{15\,g}{36\,g}$$

$$= \frac{5}{12} \qquad (0.42)$$

Answer 2

(a) First a working diagram,



$$tan\,\theta = \frac{3}{4} \qquad cos\,\theta = \frac{4}{5} \qquad sin\,\theta = \frac{3}{5}$$

Taking moments about A,

$$T\,sin\,\theta \times 4 = m \times g \times 2 + 2m \times g \times 3$$

$$T \times \frac{3}{5} \times 4 = 2mg + 6mg$$

$$T = \frac{10}{3}\,mg \qquad (3.33mg)$$

(b) Balancing forces horizontally,

$$R_A = T\,cos\,\theta$$

$$= \frac{10}{3}\,mg \times \frac{4}{5}$$

$$= \frac{8}{3}\,mg \qquad \square$$

(c) Balancing forces vertically,

$$F_R + T\,sin\,\theta = m \times g + 2m \times g$$

$$F_R = 3\,mg - \frac{10}{3}\,mg \times \frac{3}{5}$$

$$= mg$$

Considering friction,

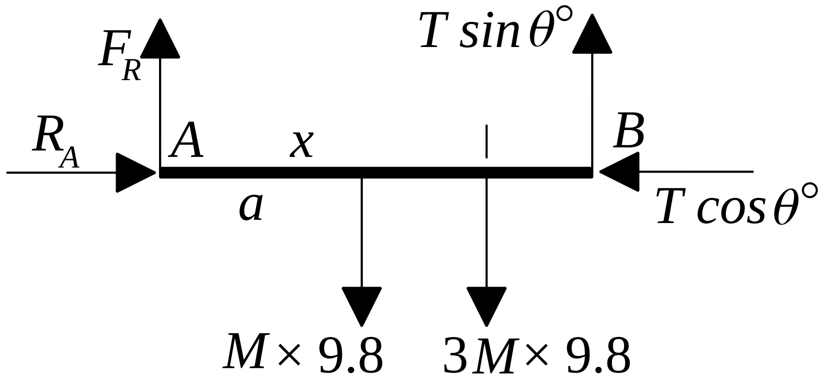
$$F_R = \mu\,R_A$$

$$\mu = \frac{mg}{\frac{8}{3}\,mg}$$

$$= \frac{3}{8} \qquad (0.38)$$

Answer 3

(a) First a working diagram,



$$tan\,\theta = \frac{3}{4} \qquad cos\,\theta = \frac{4}{5} \qquad sin\,\theta = \frac{3}{5}$$

Taking moments about A,

$$T\, sin\,\theta \times 2a = M \times g \times a + 3M \times g \times x$$

$$T \times \frac{3}{5} \times 2a = Mga + 3Mgx$$

$$6Ta = 5Mg\,(a + 3x)$$

$$T = \frac{5Mg\,(3x + a)}{6a} \qquad \square$$

(b) Balancing forces horizontally,

$$R_A = T\, cos\,\theta$$

$$2Mg = \frac{5Mg\,(3x + a)}{6a} \times \frac{4}{5}$$

$$30a \times 2Mg = 20Mg\,(3x + a)$$

$$60Mga = 60Mgx + 20Mga$$

$$40Mga = 60Mgx$$

$$60x = 40a$$

$$x = \frac{2}{3}\,a$$

(c) Balancing forces vertically,

$$F_R + T \sin\theta = Mg + 3Mg$$

$$F_R = 4Mg - \frac{5Mg\,(\,3x + a\,)}{6a} \times \frac{3}{5}$$

$$= 4Mg - \frac{Mg\,(\,3 \times \tfrac{2}{3}a + a\,)}{2a}$$

$$= 4Mg - \frac{Mg \times 3a}{2a}$$

$$= \frac{8Mg}{2} - \frac{3Mg}{2}$$

$$= \frac{5Mg}{2}$$

$$\tan\beta = \frac{F_R}{R_A}$$

$$= \frac{5Mg}{2 \times 2Mg}$$

$$= \frac{5}{4}$$

(d)

$$T \leqslant 5Mg$$

$$\frac{5Mg\,(\,3x + a\,)}{6a} \leqslant 5Mg$$

$$\frac{(\,3x + a\,)}{6a} \leqslant 1$$

$$3x + a \leqslant 6a$$

$$3x \leqslant 5a$$

$$x \leqslant \frac{5}{3}a$$

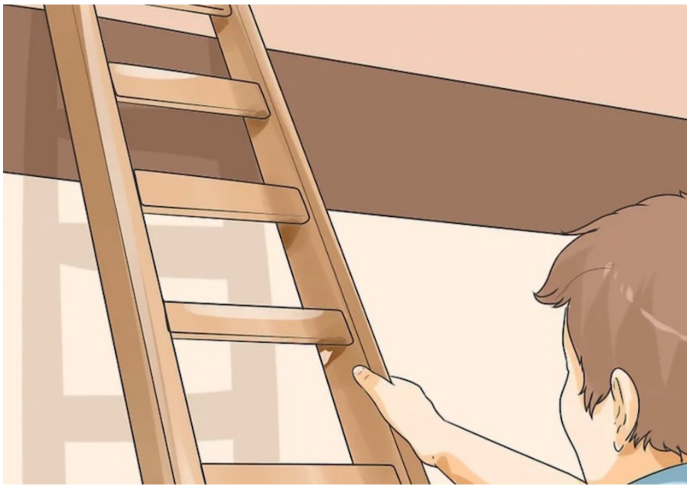
$\therefore$ For the rope not to break, the block must be less than $\dfrac{5a}{3}$ from A

Lesson 3

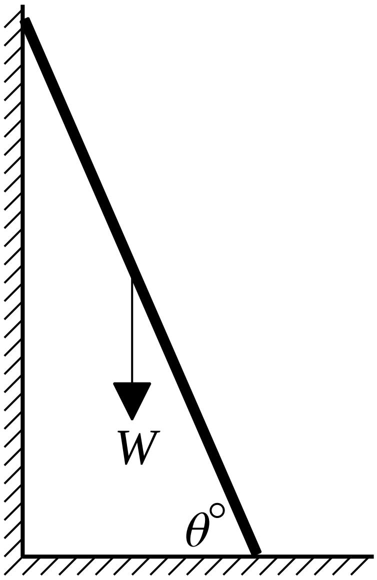
A-Level Applied Mathematics
Mechanics : Moments II : Year 2

3.1 The Non Horizontal Rod

In the previous lessons, the problems considered involved moments in which some of the forces acting on the horizontal rod where not at right angles to the rod. In this lesson the forces will all be either horizontal or vertical but the rod itself will no longer be horizontal. The classic problem in this regard is that of a ladder resting against a wall. Earlier work with friction is called back into play, as it is friction between the foot of a ladder and the ground that allows the ladder to remain in position.

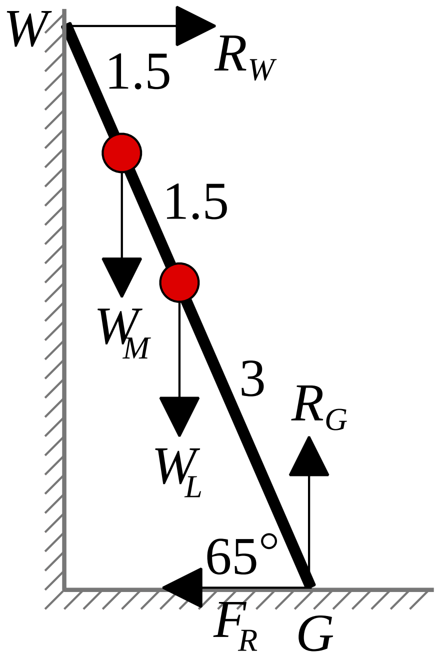


3.2 The Ladder and The Wall



When thinking about a ladder resting against a wall, there is a resemblance to the problem of a block of weight W on a slope at $\theta°$ to the horizontal. So one method of analysis would be to resolve the weight of the ladder, W, into component parts. The component parallel the slope would be $W \sin \theta$ and perpendicular to the slope would be $W \cos \theta$. However, in many ladder problems this is often not the best way to proceed. To see why this is so a typical example will be considered in some depth.

3.3 Example (The Question)



A ladder WG of weight W_L = 120 N and length 6 metres has one end G resting on rough horizontal ground. The other end W rests against a smooth vertical wall. A man of weight W_M = 750 N has gone three-quarters of the way up the ladder but has stopped because he can sense that the ladder is about to slip.
Find the coefficient of friction, μ, between the foot of the ladder and the ground given that the ladder is inclined at 65° to the horizontal.

3.4 Example (The Discussion)

Before trying to solve the problem, a discussion of the diagram above is helpful. The wall is described as "smooth" which is why the reaction where the ladder rests against the wall is perpendicular to the wall.
It is because the question has asked for the coefficient of friction, μ, with the ground that the foot of the ladder has a normal reaction, R_G, and a friction force, F_R. The intention is to determine R_G (which is perpendicular to the ground) and F_R (which is parallel to the ground) and then obtain μ using,

$$F_R = \mu R$$

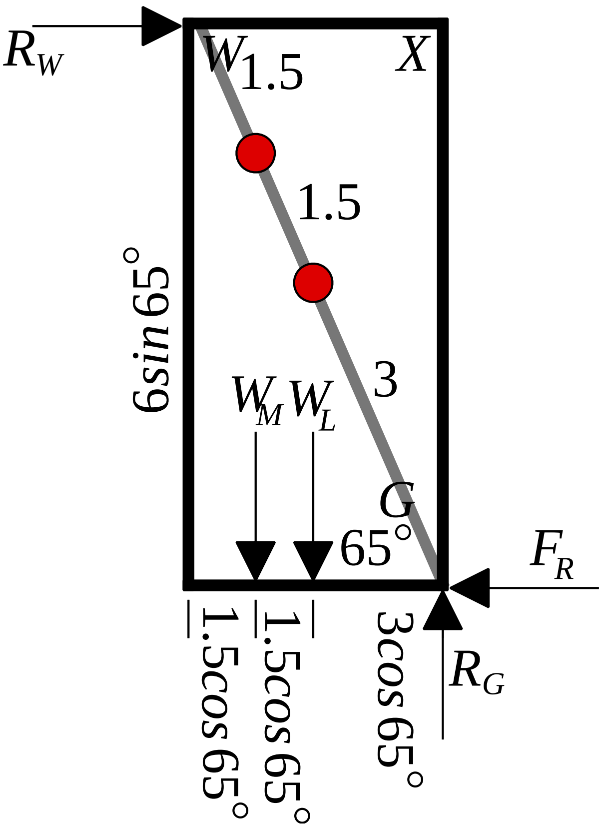
A different question could have asked for the overall action of the ground on the ladder, as the questions in Lesson 1 did. If this were asked for, having already found R_G and F_R, by the method about to be described, this could then be found easily using,

$$R_A = \sqrt{(R_G)^2 + (F_R)^2} \quad \text{and} \quad \tan\theta = \frac{R_G}{F_R}$$ where θ is angle to the horizontal.

An important fact to notice is that all of the forces in this question are naturally either vertical or horizontal. This makes resolving every single force into components that are parallel or perpendicular to the plane an uneccessarily complicated way to do the question. Instead there is a really clever technique, that will be developed next, where the rod itself is replaced by an equivalent structure.

3.5 Example (The Solution)

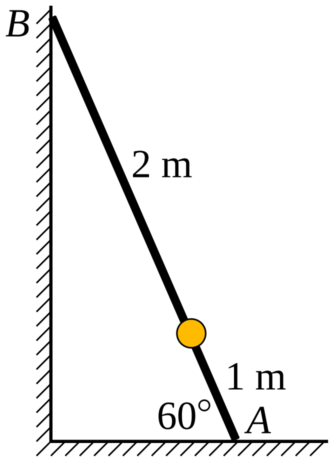
The ladder is replaced by a rigid rectangle shaped light and inextensible rod, one side of which lies along the wall and another along the ground. Each force is projected along its line of action to act on the rectangular rod. Like the ladder from whence it came, The rectangular rod is in equilibrium. So, the sum of moments about any pivot is zero, and the resultant force in any given direction (especially horizontally or vertically) is zero. Here is the new diagram with distances between key points added,



Teaching Video : http://www.NumberWonder.co.uk/Video/v9084(3).mp4

3.6 Exercise

Question 1



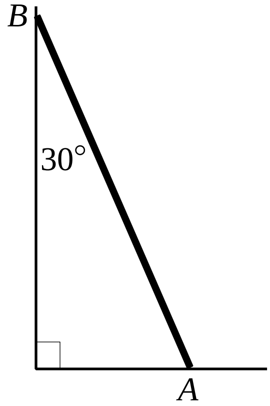
A ladder *AB*, of mass *m* and length 3 metres, has one end *A* resting on rough horizontal ground. The other end *B* rests against a smooth vertical wall. A load of mass 2*m* is fixed on the ladder at the point *C*, where *AC* = 1 metre.
The ladder is modelled as a uniform rod in a vertical plane perpendicular to the wall and the load is modelled as a particle. The ladder rests in limiting equilibrium at an angle of 60° with the ground.

(i) Draw a force diagram in which the ladder has been replaced with a rigid rectangular rod which has one edge along the ground and another along the wall. Mark on all significant forces, projected onto the rectangular rod, and use simple trigonometry to obtain key distances between the forces on the sides of the rectangular rod.

(ii) Find the coefficient of friction between the ladder and the ground.

Question 2

A-Level Examination Question from January 2017, IAL, M2, Q7



A uniform rod AB has mass m and length $2a$. The end A is in contact with rough horizontal ground and the end B is in contact with a smooth vertical wall. The rod rests in equilibrium in a vertical plane perpendicular to the wall and makes an angle of 30° with the wall, as shown. The coefficient of friction between the rod and the ground is μ.

(a) Find, in terms of m and g, the magnitude of the force exerted on the rod by the wall.

[4 marks]

(b) Show that $\mu \geqslant \dfrac{\sqrt{3}}{6}$

[3 marks]

A particle of mass *km* is now attached to the rod at *B*.

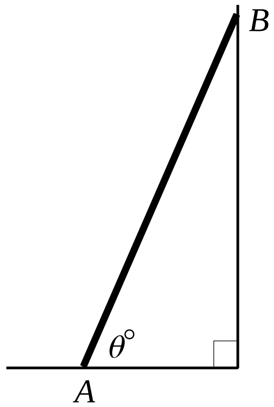
Given that $\mu = \dfrac{\sqrt{3}}{5}$ and that the rod in now in limiting equilibrium,

(c) find the value of *k*

[6 marks]

Question 3

A-Level Specimen Examination Question for the (Cancelled) June 2020 Paper 3, Q2



The ladder AB has length $2a$ and weight W. The ladder rests in equilibrium with end A on rough horizontal ground and end B against a smooth vertical wall.

The ladder rests in a vertical plane perpendicular to the wall. and is inclined at angle θ to the ground. The coefficient of friction between the ladder and the ground is μ.

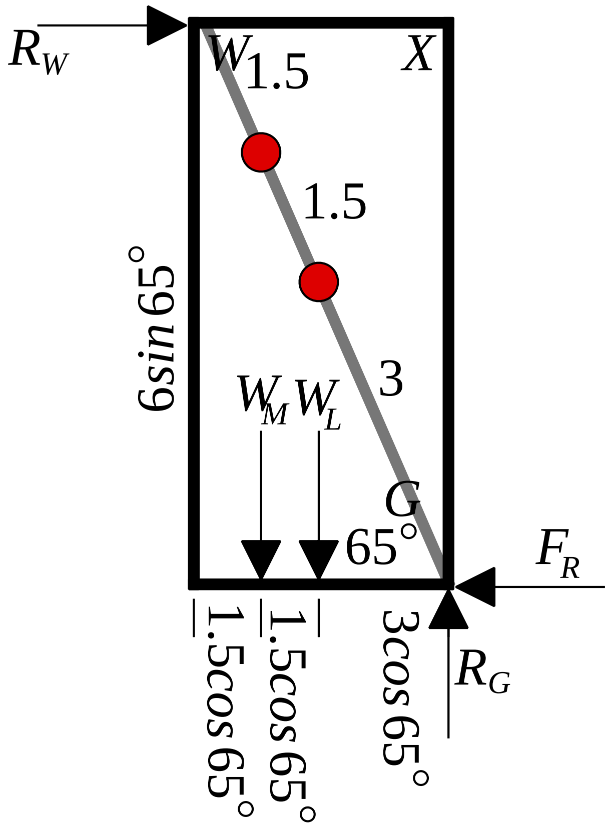
The ladder is on the point of slipping. The ladder is modelled as a uniform rod.

Show that $\mu = \dfrac{1}{2\tan\theta}$

[7 marks]

3.7 Answers

3.7.1 Solution to Introductory Example (2.3, 2.4, 2.5)



Take moments about X,
Note that "the line of action" of R_W and R_G passes through X and so has no turning effect about X,

$$\begin{aligned}
F_R \times 6\,sin\,65 &= W_L \times 3 \times cos\,65 + W_M \times 4.5 \times cos\,65 \\
5.438\,F_R &= 120 \times 3 \times cos\,65 + 750 \times 4.5 \times cos\,65 \\
5.438\,F_R &= 152.1 + 1426 \\
F_R &= 290\text{ N}
\end{aligned}$$

Balancing forces vertically,

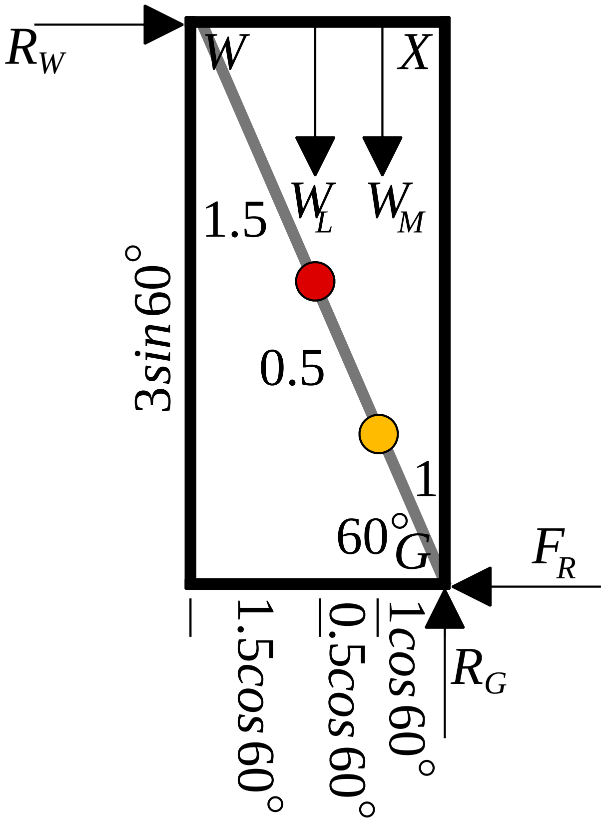
$$\begin{aligned}
R_G &= 120 + 750 \\
&= 870
\end{aligned}$$

$$\begin{aligned}
F_R &= \mu R_G \\
\therefore\ \mu &= \frac{290}{870} \\
&= 0.333
\end{aligned}$$

3.7.1 Solution to 2.6 Exercise

Answer 1

(i)



(ii) Take moments about X,

Note that "the line of action" of R_W and R_G passes through X and so has no turning effect about X,

$$F_R \times 3\sin 60 = W_M \times 1 \times \cos 60 + W_L \times 1.5 \times \cos 60$$

$$2.598\, F_R = 2m \times g \times 1 \times \cos 60 + m \times g \times 1.5 \times \cos 60$$

$$2.598\, F_R = mg + 0.75mg$$

$$F_R = 0.6736mg \text{ N}$$

Balancing forces vertically,

$$R_G = W_M + W_L$$

$$= 2mg + mg$$

$$= 3mg$$

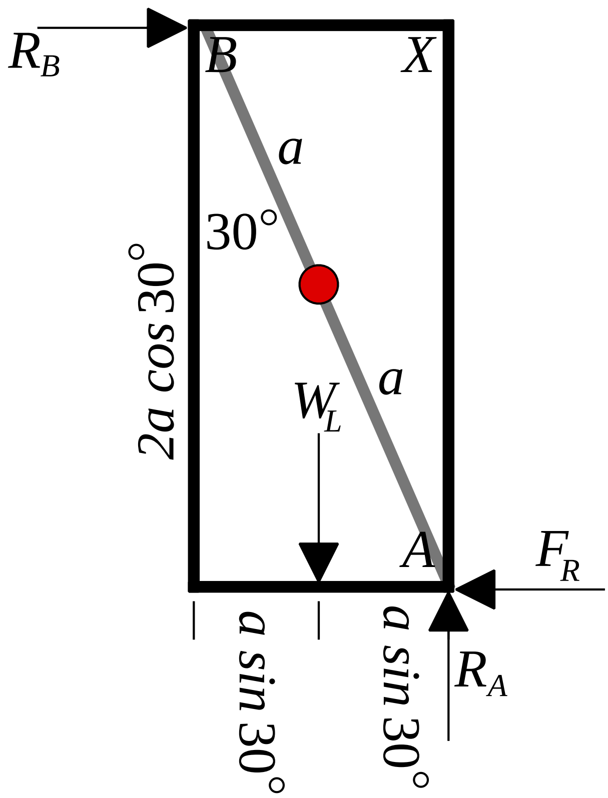
$$F_R = \mu R_G$$

$$\therefore \mu = \frac{0.6736mg}{3mg}$$

$$= 0.22$$

Answer 2

(a) First a working diagram,



Take moments about A,

$$R_B \times 2a\cos 30 = W_L \times a\sin 30$$

$$R_B \times 2 \times a \times \frac{\sqrt{3}}{2} = m \times g \times a \times \frac{1}{2}$$

$$\sqrt{3}\,R_B = \frac{mg}{2}$$

$$R_B = \frac{mg}{2\sqrt{3}}$$

$$= \frac{\sqrt{3}\,mg}{6}$$

(b) Balancing forces horizontally,

$$F_R = R_B$$

$$= \frac{\sqrt{3}\,mg}{6}$$

Balancing forces vertically,

$$R_A = W_L$$

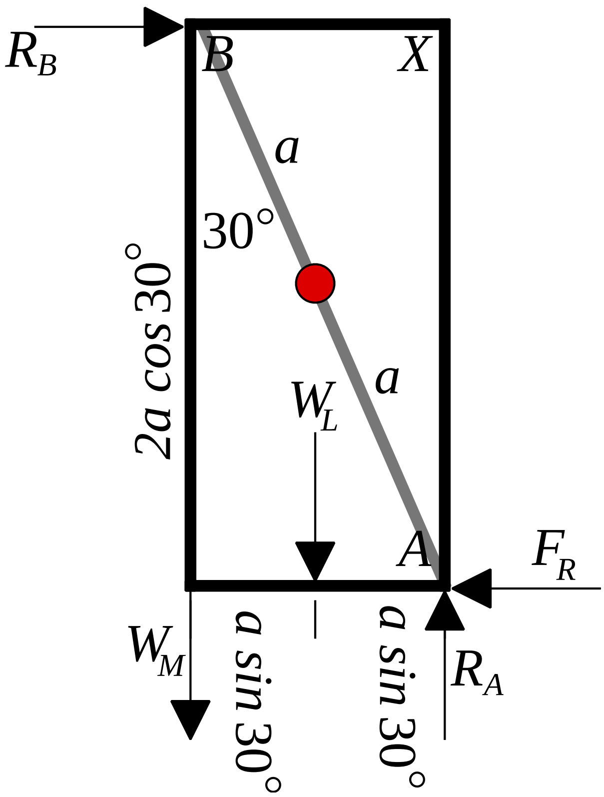
$$= mg$$

$$F_R \leqslant \mu R_A$$

$$\frac{\sqrt{3}\, mg}{6} \leqslant \mu mg$$

$$\mu \geqslant \frac{\sqrt{3}}{6} \qquad \square$$

(c) Amended working diagram,



Balancing forces vertically,

$$\begin{aligned} R_A &= W_L + W_M \\ &= mg + kmg \\ &= mg\,(1 + k) \end{aligned}$$

Considering friction,

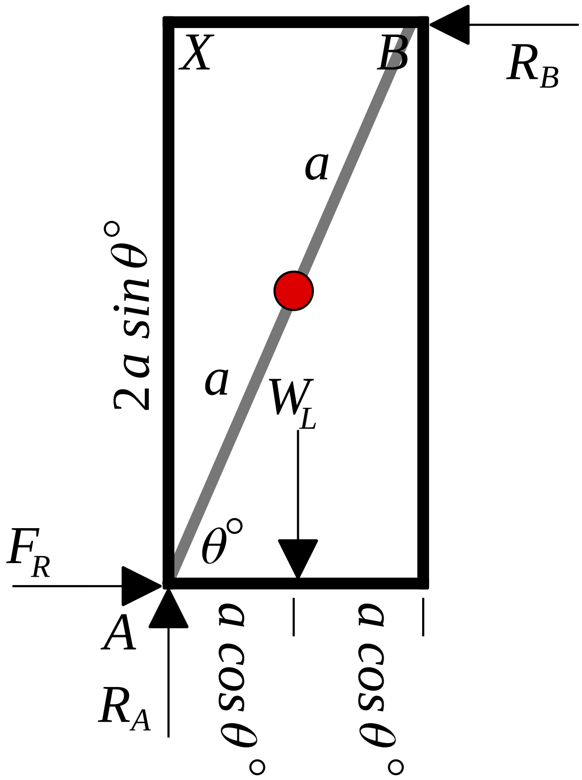
$$\begin{aligned} F_R &= \mu R_A \\ &= \frac{\sqrt{3}\, mg}{5}\,(1 + k) \end{aligned}$$

Take moments about *X*,

$$
\begin{aligned}
F_R \times 2a\cos 30 &= kmg \times 2a\sin 30 + mg \times a\sin 30 \\
\sqrt{3}\,F_R &= kmg + \frac{mg}{2} \\
\sqrt{3} \times \frac{\sqrt{3}\,mg}{5}(1 + k) &= \frac{mg}{2}(2k + 1) \\
6(1 + k) &= 5(2k + 1) \\
6 - 5 &= 10k - 6k \\
4k &= 1 \\
k &= \frac{1}{4}
\end{aligned}
$$

Answer 3

(a) First a working diagram,



Take moments about X,

$$F_R \times 2 \times a \times \sin\theta = W_L \times a \times \cos\theta$$

$$F_R = \frac{W_L\, a \cos\theta}{2\, a \sin\theta}$$

$$F_R = \frac{W_L}{2 \tan\theta}$$

Balance forces vertically,

$$R_A = W_L$$

Consider friction,

$$F_R = \mu R_A$$

$$\frac{W_L}{2 \tan\theta} = \mu \times W_L$$

$$\mu = \frac{1}{2 \tan\theta} \qquad \square$$