11.2 Answers to 11.1 Test

Answer 1

(C) Real Numbers, $x, y \in \mathbb{R}$

[1 mark]

Answer 2

(a) As all of the coefficients of the cubic are real numbers, the complex roots come in conjugate pairs. Thus the third root is $-3 - 2\text{i}$

[1 mark]

(b) The roots are $x_1 = 4$, $x_2 = -3 + 2\text{i}$ and $x_3 = -3 - 2\text{i}$

So the factors are,

$$(x - 4)(x + 3 - 2\text{i})(x + 3 + 2\text{i})$$

$$= (x - 4)\left((x + 3)^2 - (2\text{i})^2\right)$$

$$= (x - 4)\left(x^2 + 6x + 9 - 4\text{i}^2\right)$$

$$= (x - 4)\left(x^2 + 6x + 13\right)$$

Multiplication grid :

	x^2	$6x$	13
x	x^3	$6x^2$	$13x$
-4	$-4x^2$	$-24x$	-52

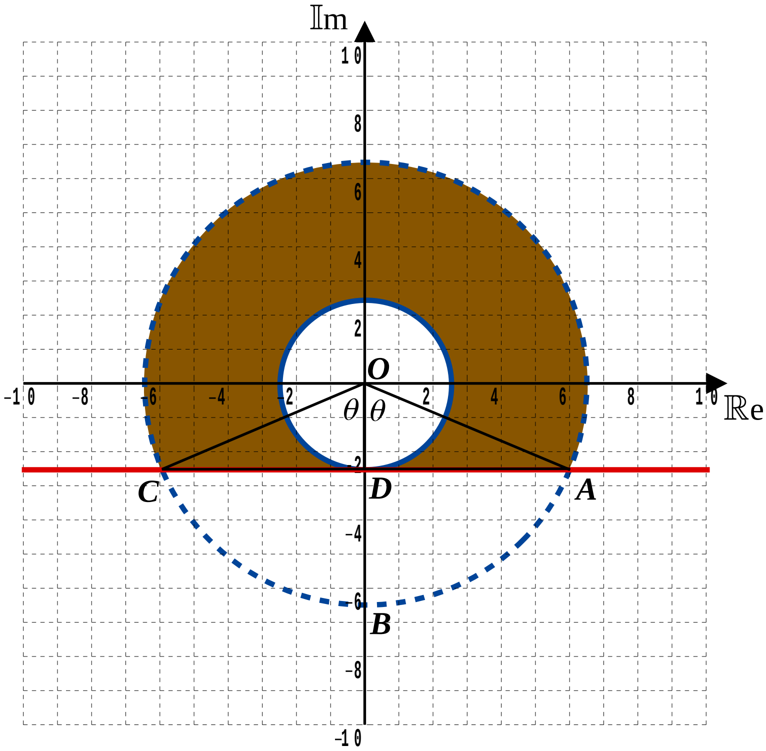
$$= x^3 + 2x^2 - 11x - 52$$

$$\therefore \quad a = 2 \quad \text{and} \quad b = -11$$

[5 marks]

Answer 3

(i) The $|z|$ is a circle centre (0, 0) : Write is as $|z - (0 + 0i)|$ if it helps.
So this circle's radius varies between 2.5 and 6.5 giving an annulus (ring).
$|z| = |z + 5i|$ could be written as $|z - (0 + 0i)| = |z - (0 - 5i)|$
to make it clear that it is the perpendicular bisector of the line segment
between (0, 0) and (0, – 5) ; 'obviously' the line with equation $y = -2.5$.



[5 marks]

(ii)

$$|AD| = \sqrt{6.5^2 - 2.5^2}$$
$$= 6$$
$$\theta = arccos\left(\frac{2.5}{6.5}\right)$$
$$= 1.176$$

Area Segment *ABCD* = Area Sector *OABC* – Area Triangle *OADC*

$$= \frac{1}{2} \times 6.5^2 \times 2 \times 1.176 - \frac{1}{2} \times 12 \times 2.5$$
$$= 49.69 - 15$$
$$= 34.69$$

Shaded Area = Area Large Circle – Area Small Circle – Area Segment *ABCD*

$$= \pi \times 6.5^2 - \pi \times 2.5^2 - 34.69$$
$$= 132.73 - 19.63 - 34.69$$
$$= 78.4$$

[5 marks]

Answer 4

(i) Various ways through this are to be found...

$$\frac{2z + 3}{z + 5 - 2\text{i}} = 1 + \text{i}$$

$$(2z + 3) = (z + 5 - 2\text{i})(1 + \text{i})$$

Multiplication grid :

	z	$+5$	-2i
1	z	$+5$	-2i
$+\text{i}$	$z\text{i}$	$+5\text{i}$	-2i^2

$$\begin{aligned}
2z + 3 &= z + 5 - 2\text{i} + z\text{i} + 5\text{i} - 2\text{i}^2 \\
&= z + 7 + 3\text{i} + z\text{i} \\
2z - z - z\text{i} &= 7 + 3\text{i} - 3 \\
z - z\text{i} &= 4 + 3\text{i} \\
z(1 - \text{i}) &= 4 + 3\text{i} \\
z &= \frac{4 + 3\text{i}}{1 - \text{i}} \times \frac{1 + \text{i}}{1 + \text{i}} \\
&= \frac{4 + 4\text{i} + 3\text{i} + 3\text{i}^2}{1 - \text{i}^2} \\
&= \frac{1 + 7\text{i}}{2} \\
&= \frac{1}{2} + \frac{7}{2}\text{i}
\end{aligned}$$

[5 marks]

(ii)

$$\begin{aligned}
|w| &= 15 \\
|(3 + \lambda\text{i})(2 + \text{i})| &= 15 \\
|6 + 3\text{i} + 2\lambda\text{i} + \lambda\text{i}^2| &= 15 \\
|(6 - \lambda) + (3 + 2\lambda)\text{i}| &= 15 \\
(6 - \lambda)^2 + (3 + 2\lambda)^2 &= 225 \\
36 - 12\lambda + \lambda^2 + 9 + 12\lambda + 4\lambda^2 &= 225 \\
5\lambda^2 &= 180 \\
\lambda^2 &= 36 \\
\lambda &= \pm 6
\end{aligned}$$

[4 marks]

Answer 5

(a)

$$|z - 6i| = 2\,|z - 3|$$
$$|x + yi - 6i| = 2\,|x + yi - 3|$$
$$|x + (y - 6)i| = 2\,|(x - 3) + yi|$$
$$x^2 + (y - 6)^2 = 4\left((x - 3)^2 + y^2\right)$$
$$x^2 + y^2 - 12y + 36 = 4\left(x^2 - 6x + 9 + y^2\right)$$
$$x^2 + y^2 - 12y + 36 = 4x^2 - 24x + 36 + 4y^2$$
$$3x^2 + 3y^2 - 24x + 12y = 0$$

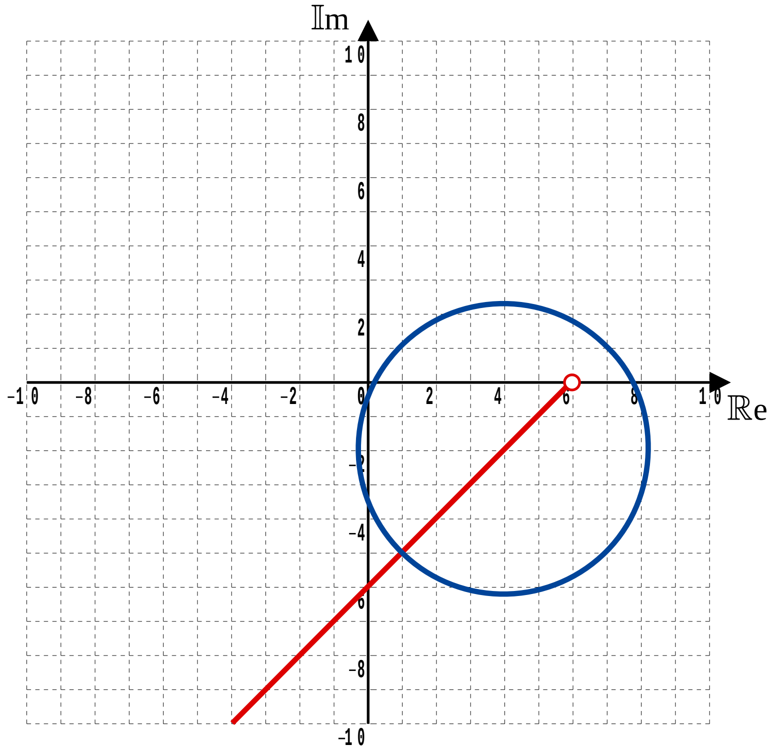
Divide through by 3

$$x^2 - 8x + y^2 + 4y = 0$$
$$\left[x^2 - 8x\right] + \left[y^2 + 4y\right] = 0$$
$$\left[(x - 4)^2 - 16\right] + \left[(y + 2)^2 - 4\right] = 0$$
$$(x - 4)^2 + (y + 2)^2 = 20$$

which is a circle with centre (4, – 2) and radius $2\sqrt{5}$ (4.47)

[6 marks]

(b) $arg\,(z - 6) = -\dfrac{3\pi}{4}$ is a half-line starting (6, 0) heading off $-\ 135°$



[4 marks]

(c)

$$\text{Circle : } (x-4)^2 + (y+2)^2 = 20$$

$$\text{Line : } y = x - 6 \quad \text{with} \quad x < 6$$

$$(x-4)^2 + ((x-6)+2)^2 = 20$$

$$(x-4)^2 + (x-4)^2 = 20$$

$$2(x-4)^2 = 20$$

$$(x-4)^2 = 10$$

$$x - 4 = \pm\sqrt{10}$$

$$x = 4 \pm \sqrt{10}$$

$$x = 4 - \sqrt{10} \quad \text{as } x < 6$$

$$\begin{aligned} y &= x - 6 \\ &= 4 - \sqrt{10} - 6 \\ &= -2 - \sqrt{10} \end{aligned}$$

$$\therefore z = \left(4 - \sqrt{10}\right) + \left(-2 - \sqrt{10}\right)\mathrm{i}$$

[4 marks]

Answer 6

(i) $2 + 3\mathrm{i}$ is a Gaussian prime

[1 mark]

(ii)

$$z^2 - 3 - 4\mathrm{i} = 0$$

$$z^2 = 3 + 4\mathrm{i}$$

$$(a + b\mathrm{i})^2 = 3 + 4\mathrm{i}$$

$$a^2 + 2ab\mathrm{i} + b^2\,\mathrm{i}^2 = 3 + 4\mathrm{i}$$

$$a^2 - b^2 + 2ab\mathrm{i} = 3 + 4\mathrm{i}$$

Two complex numbers (the one on the left and the one on the right) are equal if and only if the real parts are equal and the imaginary parts are equal.

$$\therefore\ a^2 - b^2 = 3 \qquad \text{and} \qquad 2ab = 4$$

$$ab = 2$$

As a, b are integers the only possibility with $|a| > |b|$ (to have $a^2 - b^2$ positive) are, $(a, b) = (-2, -1)$, and $(2, 1)$

Both of these also satisfy $a^2 - b^2 = 3$

Thus, the solutions to

$$z^2 - 5 - 12\mathrm{i} = 0$$

are $-2 - \mathrm{i}$ or $2 + \mathrm{i}$

These could be presented as $\pm (2 + \mathrm{i})$

which has an obvious similarity with previous work with the real numbers

[4 marks]