2.3 Answers

2.3.1 Solutions to 2.1 Complex Number Division

Division :

$$\frac{z}{w} = \frac{a + b\mathrm{i}}{c + d\mathrm{i}} \times \frac{c - d\mathrm{i}}{c - d\mathrm{i}}$$

$$= \frac{ac - ad\mathrm{i} + bc\mathrm{i} - bd\,\mathrm{i}^2}{c^2 - cd\mathrm{i} + cd\mathrm{i} - d\,\mathrm{i}^2}$$

$$= \frac{(ac + bd) + (bc - ad)\mathrm{i}}{c^2 + d^2}$$

$$= \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}\,\mathrm{i}$$

As a, b, c and d are all real numbers, each of $\frac{ac + bd}{c^2 + d^2}$ and $\frac{bc - ad}{c^2 + d^2}$ will

will also real numbers which we could call u and v.

$$\therefore \frac{z}{w} = u + v\mathrm{i}$$ which is in the form of a complex number □

[3 marks]

2.3.2 Solution to 2.2 Exercise

Answer 1

(i)

$$v^* = 4 - 2\mathrm{i}$$

[1 mark]

(ii)

$$\begin{aligned} w &= \frac{u}{v} \\ &= \frac{5 - 3\mathrm{i}}{4 + 2\mathrm{i}} \times \frac{4 - 2\mathrm{i}}{4 - 2\mathrm{i}} \\ &= \frac{20 - 10\mathrm{i} - 12\mathrm{i} + 6\,\mathrm{i}^2}{16 - 8\mathrm{i} + 8\mathrm{i} - 4\,\mathrm{i}^2} \\ &= \frac{14 - 22\mathrm{i}}{20} \\ &= \frac{7}{10} - \frac{11}{10}\,\mathrm{i} \end{aligned}$$

[3 marks]

Answer 2

$$\begin{aligned} \frac{u}{u^*} &= \frac{3 + \mathrm{i}}{3 - \mathrm{i}} \times \frac{3 + \mathrm{i}}{3 + \mathrm{i}} \\ &= \frac{9 + 6\mathrm{i} + \mathrm{i}^2}{9 - 3\mathrm{i} + 3\mathrm{i} - \mathrm{i}^2} \\ &= \frac{8 + 6\mathrm{i}}{10} \\ &= \frac{4}{5} + \frac{3}{5}\,\mathrm{i} \end{aligned}$$

[4 marks]

Answer 3

(i)

$$\frac{z_1 z_2}{z_3} = \frac{(1+i)(2+i)}{(3+i)}$$

$$= \frac{2+3i+i^2}{(3+i)}$$

$$= \frac{(1+3i)}{(3+i)} \times \frac{(3-i)}{(3-i)}$$

$$= \frac{3-i+9i-3i^2}{9-3i+3i-i^2}$$

$$= \frac{6+8i}{10} = \frac{3}{5} + \frac{4}{5}i$$

[3 marks]

(ii)

$$\frac{(z_2)^2}{z_1} = \frac{(2+i)(2+i)}{(1+i)}$$

$$= \frac{4+4i+i^2}{1+i}$$

$$= \frac{3+4i}{1+i} \times \frac{1-i}{1-i}$$

$$= \frac{3-3i+4i-4i^2}{1-i+i-i^2}$$

$$= \frac{7}{2} + \frac{1}{2}i$$

[3 marks]

(iii)

$$\frac{2z_1 + 5z_3}{z_2} = \frac{2(1+i)+5(3+i)}{(2+i)}$$

$$= \frac{2+2i+15+5i}{2+i}$$

$$= \frac{17+7i}{2+i} \times \frac{2-i}{2-i}$$

$$= \frac{34-17i+14i-7i^2}{4-2i+2i-i^2}$$

$$= \frac{41-3i}{5} = \frac{41}{5} - \frac{3}{5}i$$

[3 marks]

Answer 4

$$\frac{2+i}{2-i} - \frac{2-i}{2+i} = \frac{2+i}{2-i} \times \frac{2+i}{2+i} - \frac{2-i}{2+i} \times \frac{2-i}{2-i}$$

$$= \frac{4+2i+2i+i^2}{4+2i-2i-i^2} - \frac{4-2i-2i+i^2}{4+2i-2i-i^2}$$

$$= \frac{3+4i}{5} - \frac{3-4i}{5}$$

$$= \frac{8}{5}i$$

[4 marks]

Answer 5

$$w^{-1} = \frac{1}{w}$$

$$= \frac{1}{\sqrt{3}+\sqrt{2}\,i} \times \frac{\sqrt{3}-\sqrt{2}\,i}{\sqrt{3}-\sqrt{2}\,i}$$

$$= \frac{\sqrt{3}-\sqrt{2}\,i}{3-\sqrt{6}\,i+\sqrt{6}\,i-2\,i^2}$$

$$= \frac{\sqrt{3}-\sqrt{2}\,i}{5}$$

$$= \frac{\sqrt{3}}{5} - \frac{\sqrt{2}}{5}i$$

[4 marks]

Answer 6

$$\frac{1}{i} + \frac{1}{1+i} + \frac{1}{1-i}$$

$$= \frac{1}{i} \times \frac{i}{i} + \frac{1}{1+i} \times \frac{1-i}{1-i} + \frac{1}{1-i} \times \frac{1+i}{1+i}$$

$$= \frac{i}{i^2} + \frac{1-i}{1-i+i-i^2} + \frac{1+i}{1+i-i-i^2}$$

$$= -i + \frac{1-i}{2} + \frac{1+i}{2}$$

$$= \frac{-2i+1-i+1+i}{2}$$

$$= \frac{2-2i}{2}$$

$$= 1-i$$

[4 marks]

Answer 7

$$\text{LHS} = (1 + 2i)^5$$

$$\begin{aligned}
&= \ \ 1 \times (1)^5 \times (2i)^0 \\
&+ \ \ 5 \times (1)^4 \times (2i)^1 \\
&+ 10 \times (1)^3 \times (2i)^2 \\
&+ 10 \times (1)^2 \times (2i)^3 \\
&+ \ \ 5 \times (1)^1 \times (2i)^4 \\
&+ \ \ 1 \times (1)^0 \times (2i)^5
\end{aligned}$$

$$= 1 + 10i + 40\,i^2 + 80\,i^3 + 80\,i^4 + 32\,i^5$$

$$= 1 + 10i - 40 - 80i + 80 + 32i$$

$$= 41 - 38i$$

$$= \text{RHS} \qquad \square$$

[4 marks]

Answer 8

(i)

$$z = \frac{3 + q\mathrm{i}}{q - 5\mathrm{i}} \times \frac{q + 5\mathrm{i}}{q + 5\mathrm{i}}$$

$$= \frac{3q + 15\mathrm{i} + q^2\mathrm{i} + 5q\,\mathrm{i}^2}{q^2 + 5q\,\mathrm{i} - 5q\,\mathrm{i} - 25\,\mathrm{i}^2}$$

$$= \frac{-2q + \left(15 + q^2\right)\mathrm{i}}{25 + q^2}$$

$$= -\frac{2q}{25 + q^2} + \frac{15 + q^2}{25 + q^2}\,\mathrm{i}$$

It was stated that the real part of this is $\frac{1}{13}$

$$\therefore\ -\frac{2q}{25 + q^2} = \frac{1}{13}$$

$$-26q = 25 + q^2$$

$$q^2 + 26q + 25 = 0$$

$$(q + 1)(q + 25) = 0$$

Either $(q + 1) = 0$ or $(q + 25) = 0$

Either $q = -1$ or $q = -25$

[5 marks]

(ii)

$$z = -\frac{2q}{25 + q^2} + \frac{15 + q^2}{25 + q^2}\,\mathrm{i}$$

Either $z = \frac{1}{13} + \frac{15 + (-1)^2}{25 + (-1)^2}\,\mathrm{i}$ or $z = \frac{1}{13} + \frac{15 + (-25)^2}{25 + (-25)^2}\,\mathrm{i}$

Either $z = \frac{1}{13} + \frac{16}{26}\,\mathrm{i}$ or $z = \frac{1}{13} + \frac{640}{650}\,\mathrm{i}$

Either $z = \frac{1}{13} + \frac{8}{13}\,\mathrm{i}$ or $z = \frac{1}{13} + \frac{64}{65}\,\mathrm{i}$

[3 marks]

Answer 9

$$\frac{z+4}{z-3} = \frac{4-\mathrm{i}\sqrt{2}+4}{4-\mathrm{i}\sqrt{2}-3}$$

$$= \frac{8-\mathrm{i}\sqrt{2}}{1-\mathrm{i}\sqrt{2}} \times \frac{1+\mathrm{i}\sqrt{2}}{1+\mathrm{i}\sqrt{2}}$$

$$= \frac{8+8\sqrt{2}\,\mathrm{i}-\sqrt{2}\,\mathrm{i}-2\,\mathrm{i}^2}{1-\mathrm{i}\sqrt{2}+\mathrm{i}\sqrt{2}-2\,\mathrm{i}^2}$$

$$= \frac{10+7\sqrt{2}\,\mathrm{i}}{3}$$

$$= \frac{10}{3} + \frac{7\sqrt{2}}{3}\,\mathrm{i}$$

[5 marks]

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