3.6 Answers

3.6.1 Solution to 3.4 Example

As $f(z)$ has real coefficients, two solutions are $z = 3 \pm \mathrm{i}$

$\therefore$ Two factors are,

$$\begin{aligned} & (z - 3 - \mathrm{i})\,(z - 3 + \mathrm{i}) \\ = \; & ((z - 3) - \mathrm{i})\,((z - 3) + \mathrm{i}) \\ = \; & (z - 3)^2 - \mathrm{i}^2 \\ = \; & z^2 - 6z + 9 + 1 \\ = \; & z^2 - 6z + 10 \end{aligned}$$

$$\begin{array}{r|rrrrr} & & & 2z^2 & -\;2z & +\;1 \\ \hline z^2 - 6z + 10 & 2z^4 & -\;14z^3 & +\;33z^2 & -\;26z & +\;10 \\ & 2z^4 & -\;12z^3 & +\;20z^2 & & \\ \cline{2-4} & & -\;2z^3 & +\;13z^2 & -\;26z & \\ & & -\;2z^3 & +\;12z^2 & -\;20z & \\ \cline{3-5} & & & z^2 & -\;6z & +\;10 \\ & & & z^2 & -\;6z & +\;10 \\ \cline{4-6} & & & & & 0 \end{array}$$

$$2z^4 - 14z^3 + 33z^2 - 26z + 10 = 0$$

$$\left(z^2 - 6z + 10\right)\left(2z^2 - 2z + 1\right) = 0$$

$$\text{Either } z^2 - 6z + 10 = 0 \text{ or } 2z^2 - 2z + 1 = 0$$

$$z = 3 \pm \mathrm{i} \quad \text{or} \quad z = \frac{2 \pm \sqrt{(-2)^2 - 4 \times 2 \times 1}}{2 \times 2}$$

$$z = \frac{2 \pm \sqrt{-4}}{4}$$

$$z = \frac{1}{2} \pm \frac{1}{2}\mathrm{i}$$

[5 marks]

3.6.2 Solutions to 3.5 Exercise

Answer 1

As $z = 3$ is a root, $(z - 3)$ must be a factor.
Using the technique of Polynomial Division from the Year 1 course,

$$
\begin{array}{r|l}
 & \quad\quad\quad z^2 - 2z + 5 \\
\hline
z - 3 & z^3 - 5z^2 + 11z - 15 \\
 & \underline{z^3 - 3z^2} \\
 & \quad\quad - 2z^2 + 11z \\
 & \quad\quad \underline{- 2z^2 + 6z} \\
 & \quad\quad\quad\quad 5z - 15 \\
 & \quad\quad\quad\quad \underline{5z - 15} \\
 & \quad\quad\quad\quad\quad\quad 0 \leftarrow \text{ as expected}
\end{array}
$$

So, we can now factorise the cubic into one linear and one quadratic factor,

$$z^3 - 5z^2 + 11z - 15 = 0$$

$$\left(z - 3 \right)\left(z^2 - 2z + 5 \right) = 0$$

Either $\quad z - 3 = 0 \quad$ or $\quad z^2 - 2z + 5 = 0$

We already know about the root from the linear factor so the task now is to get the two complex roots from the quadratic factor. Your calculator will let you check the answers but, of course, you need to show how to get them.

I'm going to use the well known formula for solving quadratic equations.

$$z^2 - 2z + 5 = 0$$

$$
\begin{aligned}
z &= \frac{2 \pm \sqrt{(-2)^2 - 4 \times 1 \times 5}}{2 \times 1} \\
&= \frac{2 \pm \sqrt{-16}}{2} \\
&= \frac{2 \pm \sqrt{16 \times (-1)}}{2} \\
&= \frac{2 \pm 4\mathrm{i}}{2} \\
&= 1 \pm 2\mathrm{i} \quad \text{are the two complex roots of the equation}
\end{aligned}
$$

[5 marks]

Answer 2

(i)

$$f(z) = z^3 - 6z^2 + 21z - 26$$

$$\begin{aligned} f(2) &= 2^3 - 6 \times 2^2 + 21 \times 2 - 26 \\ &= 8 - 24 + 42 - 26 \\ &= 0 \qquad \square \end{aligned}$$

[1 mark]

(ii) As $z = 2$ is a root, $(z - 2)$ must be a factor.

$$\begin{array}{r|l} & \quad\;\; 4z^2 - 4z + 13 \\ \cline{2-2} z - 2 & z^3 - 6z^2 + 21z - 26 \\ & z^3 - 2z^2 \\ \cline{2-2} & \quad - 4z^2 + 21z \\ & \quad - 4z^2 - 8z \\ \cline{2-2} & \qquad\qquad 13z - 26 \\ & \qquad\qquad 13z - 26 \\ \cline{2-2} & \qquad\qquad\qquad\;\; 0 \leftarrow \text{as expected} \end{array}$$

So, we can now factorise the cubic into one linear and one quadratic factor,

$$z^3 - 5z^2 + 11z - 15 = 0$$

$$(z - 2)(z^2 - 4z + 13) = 0$$

Either $\quad z - 2 = 0 \quad$ or $\quad z^2 - 4z + 13 = 0$

We already know about the root from the linear factor so the task now is to get the two complex roots from the quadratic factor.

Using the formula for solving quadratic equations.

$$z^2 - 4z + 13 = 0$$

$$\begin{aligned} z &= \frac{4 \pm \sqrt{(-4)^2 - 4 \times 1 \times 13}}{2 \times 1} \\ &= \frac{4 \pm \sqrt{-36}}{2} \\ &= \frac{4 \pm \sqrt{36 \times (-1)}}{2} \\ &= \frac{4 \pm 6i}{2} \\ &= 2 \pm 3i \end{aligned}$$

are the two complex roots of the equation

[4 marks]

Answer 3

As $z_1 = 1 + 2\mathrm{i}$ is a root, and the polynomial has integer (and $\therefore$ real) coefficients, the roots must occur in conjugate pairs and so $z_2 = 1 - 2\mathrm{i}$ must also be a root.

Finding the product of the two associated factors...

$$\begin{aligned}(z - 1 - 2\mathrm{i})\,(z - 1 + 2\mathrm{i}) &= (z - 1)^2 - 4\,\mathrm{i}^2 \\ &= z^2 - 2z + 1 + 4 \\ &= z^2 - 2z + 5\end{aligned}$$

So we now know that,

$$\left(z^2 - 2z + 5\right)\left(Az + B\right) = 2z^3 - 3z^2 + 8z + 5$$

$$\therefore \quad Az + B = 2z + 1$$

$$2z^3 - 3z^2 + 8z + 5 = 0$$

$$\left(z^2 - 2z + 5\right)\left(2z + 1\right) = 0$$

Thus : $\quad z_1 = 1 + 2\mathrm{i} \qquad z_2 = 1 - 2\mathrm{i} \qquad$ and $\qquad z_3 = -\dfrac{1}{2}$

[5 marks]

Answer 4

(a) As $x_1 = \mathrm{i}$ is a root, and the polynomial has integer (and $\therefore$ real) coefficients, the roots must occur in conjugate pairs and so $x_2 = -\mathrm{i}$ must also be a root.

Finding the product of the two associated factors...

$$\begin{aligned}(x - \mathrm{i})\,(x + \mathrm{i}) &= x^2 - \mathrm{i}^2 \\ &= x^2 + 1\end{aligned}$$

So we now know that,

$$\begin{aligned}x^4 + 6x^3 + 26x^2 + 6x + 25 &= \left(x^2 + 1\right)\left(Ax^2 + Bx + C\right) \\ &= \left(x^2 + 1\right)\left(x^2 + 6x + 25\right)\end{aligned}$$

~ Alternatively, a polynomial long division could be used ~

[4 marks]

(b)

$$\left(x^2 + 6x + 25\right) = 0$$

$$x = \frac{-6 \pm \sqrt{6^2 - 4 \times 1 \times 25}}{2 \times 1}$$

$$x = \frac{-6 \pm \sqrt{-1 \times 64}}{2}$$

$$x = -3 \pm 4\mathrm{i}$$

The four roots of the quartic are thus

$x_1 = \mathrm{i} \qquad x_2 = -\mathrm{i} \qquad x_3 = -3 + 4\mathrm{i} \qquad$ and $\qquad x_4 = -3 - 4\mathrm{i}$

[2 marks]

Answer 5

(a) By inspection,

$$z^4 - 6z^3 + 34z^2 - 54z + 225 = (z^2 + 9)(z^2 + az + b)$$
$$= (z^2 + 9)(z^2 - 6z + 25)$$

~ Alternatively, a polynomial long division could be used ~

[2 marks]

(b)

$$z^4 - 6z^3 + 34z^2 - 54z + 225 = 0$$
$$(z^2 + 9)(z^2 - 6z + 25) = 0$$

Either $z^2 + 9 = 0$ or $z^2 - 6z + 25 = 0$

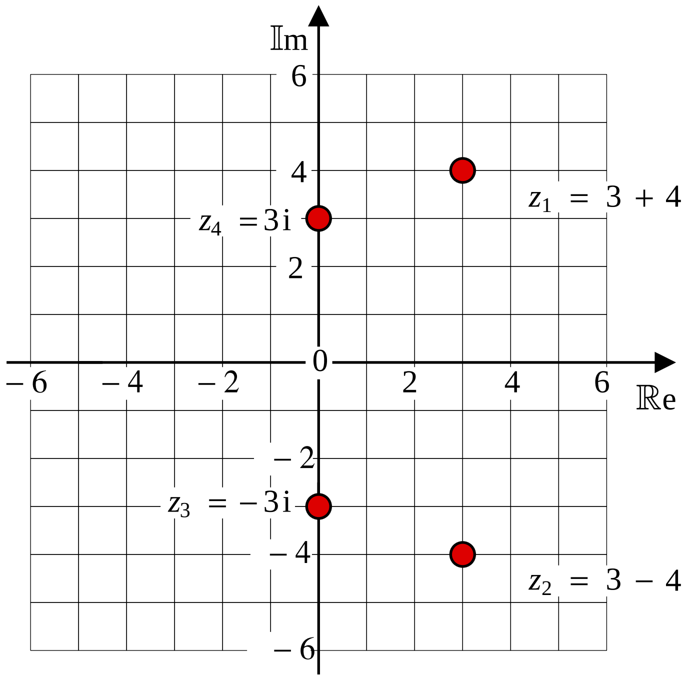
$$z^2 = -9 \qquad z = \frac{6 \pm \sqrt{(-6)^2 - 4 \times 1 \times 25}}{2 \times 1}$$

$$z = \pm\sqrt{-1 \times 9} \qquad = \frac{6 \pm \sqrt{-1 \times 64}}{2}$$

$$z = \pm 3\mathrm{i} \qquad = 3 \pm 4\mathrm{i}$$

[4 marks]

(c)



[2 marks]

Answer 6

(a)

$$2x^3 - 9x^2 + kx - 13 = 0 \qquad k \in \mathbb{Z}$$

As $x = \dfrac{1}{2}$ is a root,

$$\begin{aligned} 2 \times \left(\frac{1}{2}\right)^3 - 9 \times \left(\frac{1}{2}\right)^2 + k \times \left(\frac{1}{2}\right) - 13 &= 0 \\ \frac{1}{4} - \frac{9}{4} + \frac{k}{2} - 13 &= 0 \\ 1 - 9 + 2k - 52 &= 0 \\ 2k &= 60 \\ k &= 30 \end{aligned}$$

[3 marks]

(b) As $x = \dfrac{1}{2}$ is a root, $(2x - 1)$ must be a factor

$$\begin{array}{r|l} & \quad\; x^2 - 4x + 13 \\ \cline{2-2} 2x - 1 & 2x^3 - 9x^2 + 30x - 13 \\ & \underline{x^3 - x^2} \\ & \quad - 8x^2 + 30x \\ & \quad \underline{- 8x^2 + 4x} \\ & \qquad\qquad 26x - 13 \\ & \qquad\qquad \underline{26z - 13} \\ & \qquad\qquad\qquad 0 \leftarrow \text{as expected} \end{array}$$

Thus,

$$2x^3 - 9x^2 + 30x - 13 = 0$$

$$(2x - 1)(x^2 - 4x + 13) = 0$$

Either $2x - 1 = 0$ or $x^2 - 4x + 13 = 0$

$$2x = 1 \qquad\qquad x = \frac{4 \pm \sqrt{(-4)^2 - 4 \times 1 \times 13}}{2 \times 1}$$

$$x = \frac{1}{2} \qquad\qquad = \frac{4 \pm \sqrt{-1 \times 36}}{2}$$

$$= 2 \pm 3\mathrm{i}$$

[4 marks]

Answer 7

(a) As the cubic has real number coefficients, and complex roots will occur as a conjugate pair.

So the “third root” must be $5 - 2\mathrm{i}$

[1 mark]

(b) Written as factors, because we know the roots, the cubic is,

$$(z - 2)\,(z - 5 - 2\mathrm{i})\,(z - 5 + 2\mathrm{i})$$

$$= (z - 2)\left((z - 5)^2 - (2\mathrm{i})^2\right)$$

$$= (z - 2)\left(z^2 - 10z + 25 - 4\mathrm{i}^2\right)$$

$$= (z - 2)\left(z^2 - 10z + 29\right)$$

Multiplication grid :

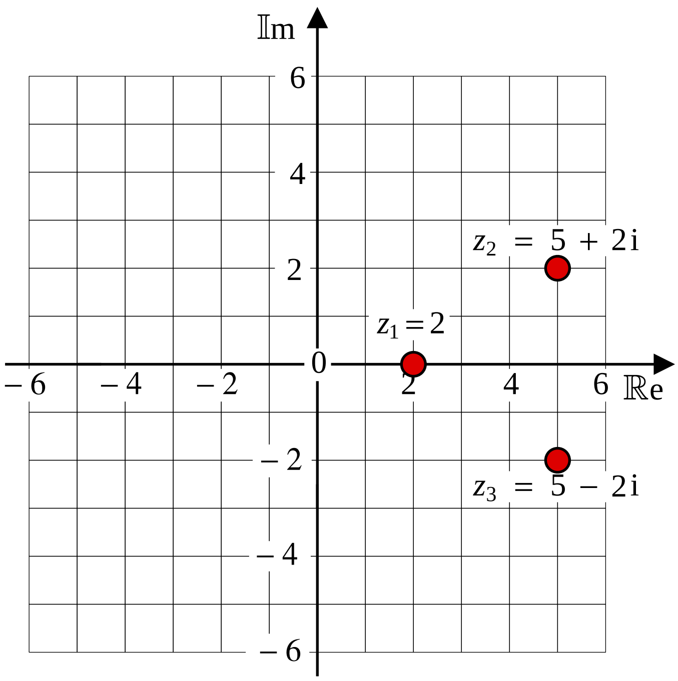
	z^2	$-10z$	29
z	z^3	$-10z^2$	$29z$
-2	$-2z^2$	$20z$	-58

$$= z^3 - 12z + 49z - 58$$

$$\therefore\ c = 49 \quad \text{and} \quad d = -58$$

[5 marks]

(c)



[2 marks]

Answer 8

(a) As $x_1 = 3 - 2i$ is a root, and the polynomial has integer (and $\therefore$ real) coefficients, the roots must occur in conjugate pairs.
So $x_2 = 3 + 2i$ must also be a root.

Finding the product of the two associated factors...

$$\begin{aligned}(x - 3 - 2i)(x - 3 + 2i) &= (x - 3)^2 - 4i^2 \\ &= x^2 - 6x + 9 + 4 \\ &= x^2 - 6x + 13\end{aligned}$$

Now to use this quadratic factor to extract the other quadratic factor from the quartic equation,

$$\begin{array}{r|l} & x^2 + 0x + 6 \\ \hline x^2 - 6x + 13 & x^4 - 6x^3 + 19x^2 - 36x + 78 \\ & \underline{x^4 - 6x^3 + 13x^2} \\ & 6x^2 - 36x + 78 \\ & \underline{6x^2 - 36x + 78} \\ & 0 \leftarrow \text{ As expected}\end{array}$$

Thus,

$$x^4 - 6x^3 + 19x^2 - 36x + 78 = 0$$
$$\left(x^2 - 6x + 13\right)\left(x^2 + 6\right) = 0$$

Either $x^2 - 6x + 13 = 0$ which yields x_1 and x_2

Or $x^2 + 6 = 0$

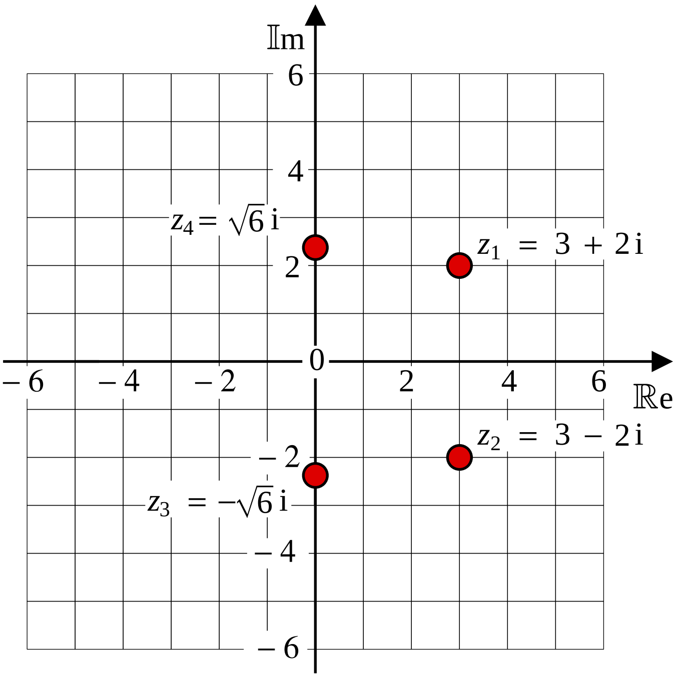
$$x^2 = -6$$
$$x = \pm\sqrt{6}\,i$$

So the solutions to the quartic are,

$x_1 = 3 - 2i \quad x_2 = 3 + 2i \quad x_3 = \sqrt{6}\,i$ and $x_4 = -\sqrt{6}\,i$

[7 marks]

(b)



[2 marks]

Answer 9

$$z^4 - 16 = 0$$

$$\left(z^2\right)^2 - 4^2 = 0$$

$$\left(z^2 - 4\right)\left(z^2 + 4\right) = 0$$

$$\left(z^2 - (2)^2\right)\left(z^2 - (2\text{i})^2\right) = 0$$

$$(z - 2)(z + 2)(z - 2\text{i})(z + 2\text{i}) = 0$$

Thus the roots are,

$z_1 = 2 \qquad z_2 = -2 \qquad z_3 = 2\text{i} \qquad$ and $\qquad z_4 = -2\text{i}$

[4 marks]