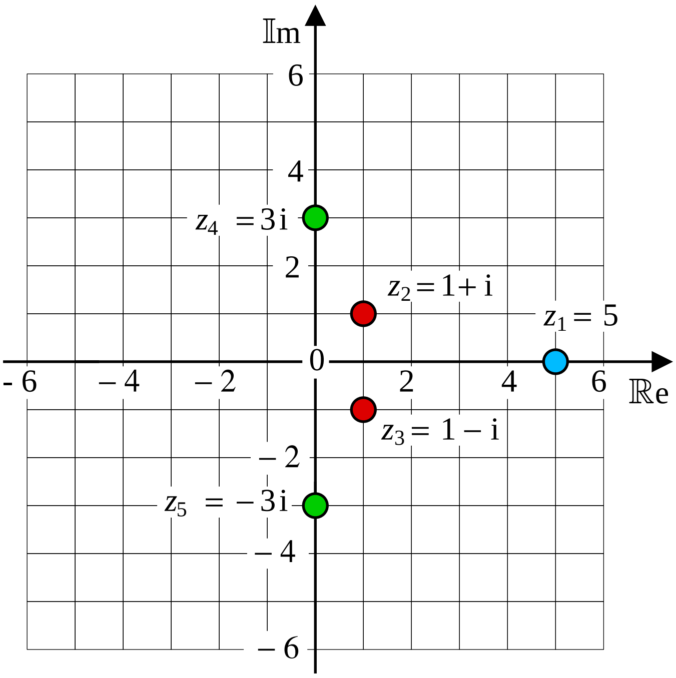
4.4 Answers

4.4.1 Solution to 4.2 A Polynomial From Its Roots



The roots on the Argand diagram have been coloured to show that z_1 is a real root (blue), z_2 and z_3 are a conjugate pair (red), and z_4 and z_5 are a second conjugate pair (green).

Writing out the factors in order of difficulty, and grouped in conjugate pairs where possible,

$$(z - 5)(z - 3\text{i})(z + 3\text{i})(z - 1 - \text{i})(z - 1 + \text{i})$$

$$= \left(z - 5\right)\left(z^2 + 9\right)\left((z - 1)^2 - \text{i}^2\right)$$

$$= \left(z^3 - 5z^2 + 9z - 45\right)\left(z^2 - 2z + 2\right)$$

	z^3	$-5z^2$	$+9z$	-45
z^2	z^5	$-5z^4$	$9z^3$	$-45z^2$
$-2z$	$-2z^4$	$10z^3$	$-18z^2$	$90z$
$+2$	$2z^3$	$-10z^2$	$18z$	-90

$$= z^5 - 7z^4 + 21z^3 - 73z^2 + 108z - 90$$

Such answers can be checked online in Wolfram Alpha where typing this quintic in will result in the five roots being displayed.

[5 marks]

4.4.2 Solutions to 4.3 Exercise

Answer 1

(i)

$$(z - 7 - 5\mathrm{i})\ (z - 7 + 5\mathrm{i})$$

[1 mark]

(ii)

$$\begin{aligned} & (z - 7 - 5\mathrm{i})\ (z - 7 + 5\mathrm{i}) \\ &= \left((z - 7)^2 - (5\mathrm{i})^2 \right) \\ &= \left(z^2 - 14z + 49 - 25\,\mathrm{i}^2 \right) \\ &= z^2 - 14z + 74 \qquad \square \end{aligned}$$

[2 marks]

(iii)

$$2z^2 - 28z + 148 = 0$$

$$2\left(z^2 - 14z + 74 \right) = 0$$

Either 2 = 0 which it clearly does not,

or $z^2 - 14z + 74 = 0$ which we already know has roots $7 \pm 5\mathrm{i}$

$\therefore\ 2z^2 - 28z + 148 = 0$ has the same roots as $z^2 - 14z + 74 = 0$ $\square$

[2 marks]

Answer 2

From the roots,

$$z_1 = 1 \qquad z_2 = \mathrm{i} \qquad z_3 = -1 \qquad \text{and} \qquad z_4 = -\mathrm{i}$$

the factors can be written out directly, taking care to put the conjugate pair next to each other (not critical in this case),

$$\begin{aligned} & (z - 1)\ (z + 1)\ (z - \mathrm{i})\ (z + \mathrm{i}) \\ &= \left(z^2 - 1 \right) \left(z^2 - \mathrm{i}^2 \right) \\ &= \left(z^2 - 1 \right) \left(z^2 + 1 \right) \\ &= z^4 - 1 \end{aligned}$$

[4 marks]

Answer 3

(i) The conjugate pair are z_2 and z_3

(ii) The Fundamental Theorem of Algebra.

(iii) The two complex roots are a conjugate pair.

(iv) From the roots,

$$z_1 = 3 \qquad z_2 = -4 + 2\text{i} \qquad \text{and} \qquad z_3 = -4 - 2\text{i}$$

the factors can be written out directly, taking care to put the conjugate pair next to each other,

$$(z - 3)(z + 4 - 2\text{i})(z + 4 + 2\text{i})$$

$$= (z - 3)\left((z + 4)^2 - 4\text{i}^2\right)$$

$$= (z - 3)\left(z^2 + 8z + 20\right)$$

Multiplication grid :

	z^2	$8z$	20
z	z^3	$8z^2$	$20z$
-3	$-3z^2$	$-24z$	-60

$$= z^3 + 5z^2 - 4z - 60$$

[1, 1, 1, 4 marks]

Answer 4

(a)

$$z_3 = 1 + 5\text{i}$$

[1 mark]

(b) From the roots,

$$z_1 = 2 \qquad z_2 = 1 - 5\text{i} \qquad \text{and} \qquad z_3 = 1 + 5\text{i}$$

the factors can be written out directly, taking care to put the conjugate pair next to each other,

$$(z - 2)(z - 1 + 5\text{i})(z - 1 - 5\text{i})$$

$$= (z - 2)\left((z - 1)^2 - 25\text{i}^2\right)$$

$$= (z - 2)\left(z^2 - 2z + 26\right)$$

Multiplication grid :

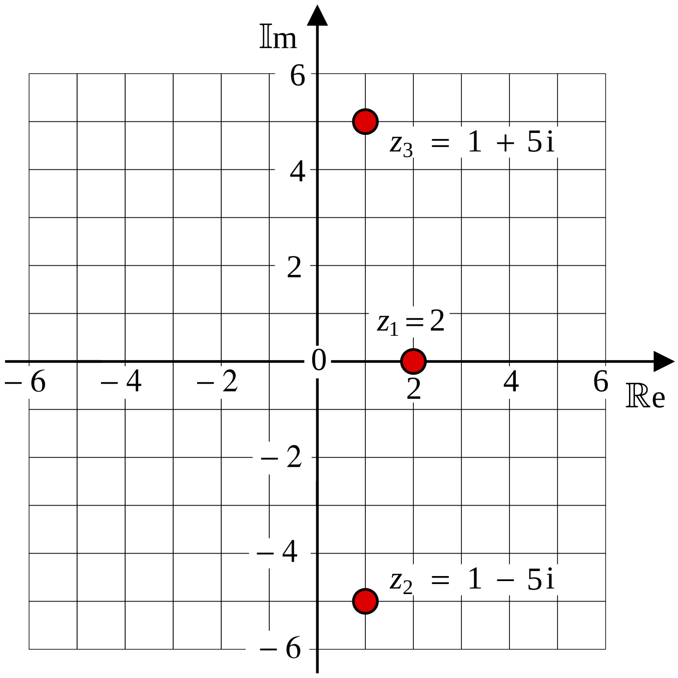
	z^2	$-2z$	26
z	z^3	$-2z^2$	$26z$
-2	$-2z^2$	$4z$	-52

$$= z^3 - 4z^2 + 30z - 52$$

$$\therefore\ p = -4 \qquad \text{and} \qquad q = -52$$

[5 marks]

(c)



[2 marks]

Answer 5

(a)

$$z = \frac{4}{1 + i} \times \frac{1 - i}{1 - i}$$

$$= \frac{4(1 - i)}{2}$$

$$= 2 - 2i$$

[2 marks]

(b)

$$z^2 = (2 - 2i)^2$$

$$= 4 - 8i + 4i^2$$

$$= -8i$$

[2 marks]

(c) From the roots,

$$z_1 = 2 - 2i \qquad \text{and} \qquad z_2 = 2 + 2i$$

the factors can be written out directly,

$$(z - 2 + 2i)(z - 2 - 2i)$$

$$= (z - 2)^2 - 4i^2$$

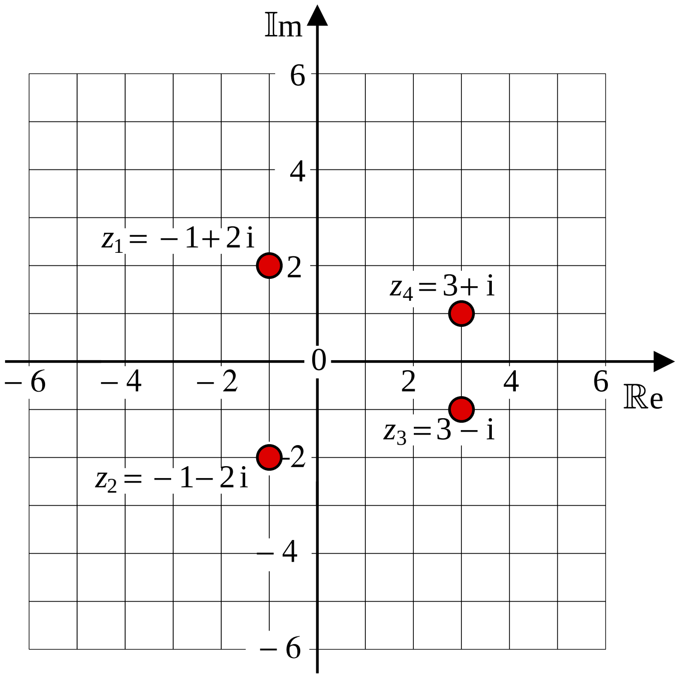
$$= z^2 - 4z + 8$$

$$\therefore\ p = -4 \qquad \text{and} \qquad q = 8$$

[3 marks]

Answer 6

(a)



[4 marks]

(b) From the roots,

$z_1 = -1 + 2i \quad z_2 = -1 - 2i \quad z_3 = 3 - i$ and $z_4 = 3 + i$

the factors can be written out directly, taking care to group each conjugate pair,

$$(z + 1 - 2i) (z + 1 + 2i) (z - 3 + i) (z - 3 - i)$$

$$= \left((z + 1)^2 - (2i)^2 \right) \left((z - 3)^2 - i^2 \right)$$

$$= \left(z^2 + 2z + 1 - 4i^2 \right) \left(z^2 - 6z + 9 - i^2 \right)$$

$$= \left(z^2 + 2z + 5 \right) \left(z^2 - 6z + 10 \right)$$

Multiplication grid :

	z^2	$2z$	5
z^2	z^4	$2z^3$	$5z^2$
$-6z$	$-6z^3$	$-12z^2$	$-30z$
10	$10z^2$	$20z$	50

$$= z^4 - 4z^3 + 3z^2 - 10z + 50$$

Matching this up with, $f(z) = z^4 + az^3 + bz^2 + cz + d$ gives us that

$$a = -4, \quad b = 3. \quad c = -10 \quad \text{and} \quad d = 50$$

[5 marks]