5.6 Answers

5.6.1 Solution to 5.3 Example on Finding Principal Arguments

For $z_1 = -5 - 3i$

Think of this as "a clockwise half a turn then anticlockwise a bit"

$$\begin{aligned} arg(z_1) &= -\pi + arctan\left(\frac{3}{5}\right) \\ &= -\pi + 0.540 \\ &= -2.60^c \end{aligned}$$

For $z_2 = 3 - i$

Think of this as "clockwise a bit"

$$\begin{aligned} arg(z_2) &= 0 - arctan\left(\frac{1}{3}\right) \\ &= 0 - 0.322 \\ &= -0.322^c \end{aligned}$$

For $z_3 = 5 + 4i$

Think of this as "anticlockwise a bit"

$$\begin{aligned} arg(z_3) &= arctan\left(\frac{4}{5}\right) \\ &= 0.675^c \end{aligned}$$

For $z_4 = -3 + 4i$

Think of this as "anticlockwise half a turn then clockwise a bit"

$$\begin{aligned} arg(z_4) &= \pi - arctan\left(\frac{4}{3}\right) \\ &= \pi - 0.927 \\ &= 2.21^c \end{aligned}$$

[4 marks]

5.6.2 Solutions to 5.5 Exercise

Answer 1

For $z_1 = -3 - 4\text{i}$

Think of this as “a clockwise half a turn then anticlockwise a bit”

$$\begin{aligned} arg(z_1) &= -\pi + arctan\left(\frac{4}{3}\right) \\ &= -\pi + 0.927 \\ &= -2.21^{c} \end{aligned}$$

For $z_2 = 2 - 2\text{i}$

Think of this as “clockwise a bit” or even “clockwise an eighth of a turn”

$$\begin{aligned} arg(z_2) &= -arctan\left(\frac{2}{2}\right) \\ &= -\frac{\pi}{4} \quad \text{or} \quad -0.785^{c} \end{aligned}$$

For $z_3 = 6 + \text{i}$

Think of this as “clockwise a bit”

$$\begin{aligned} arg(z_2) &= arctan\left(\frac{1}{6}\right) \\ &= 0.165^{c} \end{aligned}$$

For $z_4 = 5\text{i}$

Think of this as “anticlockwise a quarter of a turn”

$$\begin{aligned} arg(z_4) &= \frac{\pi}{2} \\ &= 1.57^{c} \end{aligned}$$

For $z_5 = -4 + 3\text{i}$

Think of this as “anticlockwise half a turn then clockwise a bit”

$$\begin{aligned} arg(z_5) &= \pi - arctan\left(\frac{3}{4}\right) \\ &= \pi - 0.644 \\ &= 2.50^{c} \end{aligned}$$

[5 marks]

Answer 2

(a) $arg(z_1) = -\frac{\pi}{4}$ or -0.785

[2 marks]

(b)
$$z_1 z_2 = (1 - i)(3 + 4i)$$
$$= 3 + 4i - 3i - 4i^2$$
$$= 7 + i$$

[2 marks]

(c)
$$\frac{z_2}{z_1} = \frac{3 + 4i}{1 - i}$$
$$= \frac{3 + 4i}{1 - i} \times \frac{1 + i}{1 + i}$$
$$= \frac{3 + 3i + 4i + 4i^2}{1 - i^2}$$
$$= \frac{-1 + 7i}{2}$$
$$= -\frac{1}{2} + \frac{7}{2}i$$

[3 marks]

Answer 3

(a)
$$z = \frac{26}{2 - 3i}$$
$$= \frac{26}{2 - 3i} \times \frac{2 + 3i}{2 + 3i}$$
$$= \frac{26(2 + 3i)}{4 - 9i^2}$$
$$= \frac{26(2 + 3i)}{13}$$
$$= 2(2 + 3i)$$
$$= 4 + 6i$$

[2 marks]

(b)
$$z^2 = (4 + 6i)^2$$
$$= (4 + 6i)(4 + 6i)$$
$$= 16 + 24i + 24i + 36i^2$$
$$= -20 + 48i$$

[2 marks]

(c)

$$\begin{aligned}|z| &= \sqrt{4^2 + 6^2}\\ &= \sqrt{52}\\ &= 2\sqrt{13} \quad \text{or} \quad 7.21\end{aligned}$$

[2 marks]

(d) On the Argand diagram, $z^2 = -20 + 48\text{i}$
This is located in the second quadrant.

$$\begin{aligned}arg\left(z^2\right) &= \pi - arctan\left(\frac{48}{20}\right)\\ &= \pi - 1.18\\ &= 1.97^c\end{aligned}$$

[2 marks]

Answer 4

(a)

$$\begin{aligned}\frac{z_2}{z_1} &= \frac{1 + p\text{i}}{5 + 3\text{i}}\\ &= \frac{1 + p\text{i}}{5 + 3\text{i}} \times \frac{5 - 3\text{i}}{5 - 3\text{i}}\\ &= \frac{5 - 3\text{i} + 5p\text{i} - 3p\,\text{i}^2}{25 - 9\,\text{i}^2}\\ &= \frac{5 + 3p - 3\text{i} + 5p\text{i}}{34}\\ &= \frac{5 + 3p}{34} + \frac{5p - 3}{34}\,\text{i}\end{aligned}$$

[3 marks]

(b)

$$\begin{aligned}arg\left(\frac{z_2}{z_1}\right) &= \frac{\pi}{4}\\ tan\left(arg\left(\frac{z_2}{z_1}\right)\right) &= tan\left(\frac{\pi}{4}\right)\\ \frac{5p - 3}{34} \div \frac{5 + 3p}{34} &= 1 \quad \text{from part (a)}\\ \frac{5p - 3}{34} \times \frac{34}{5 + 3p} &= 1\\ \frac{5p - 3}{5 + 3p} &= 1\\ 5p - 3 &= 5 + 3p\\ 2p &= 8\\ p &= 4\end{aligned}$$

[2 marks]

Answer 5

(a)

$$\begin{aligned} |w| &= \left|\sqrt{3} - \mathrm{i}\right| \\ &= \sqrt{(\sqrt{3})^2 + (-1)^2} \\ &= 2 \end{aligned}$$

[1 mark]

(b)

$$\begin{aligned} zw &= \left(5 + \sqrt{3}\,\mathrm{i}\right)\left(\sqrt{3} - \mathrm{i}\right) \\ &= 5\sqrt{3} - 5\mathrm{i} + 3\mathrm{i} - \sqrt{3}\,\mathrm{i}^2 \\ &= 6\sqrt{3} - 2\mathrm{i} \end{aligned}$$

[2 marks]

(c)

$$\begin{aligned} \frac{z}{w} &= \frac{5 + \sqrt{3}\,\mathrm{i}}{\sqrt{3} - \mathrm{i}} \\ &= \frac{5 + \sqrt{3}\,\mathrm{i}}{\sqrt{3} - \mathrm{i}} \times \frac{\sqrt{3} + \mathrm{i}}{\sqrt{3} + \mathrm{i}} \\ &= \frac{5\sqrt{3} + 5\mathrm{i} + 3\mathrm{i} + \sqrt{3}\,\mathrm{i}^2}{4} \\ &= \frac{4\sqrt{3} + 8\mathrm{i}}{4} \\ &= \sqrt{3} + 2\mathrm{i} \end{aligned}$$

[3 marks]

(d)

$$\begin{aligned} arg(z + \lambda) &= \frac{\pi}{3} \\ tan\left(arg\left(5 + \sqrt{3}\,\mathrm{i} + \lambda\right)\right) &= tan\left(\frac{\pi}{3}\right) \\ tan\left(arg\left((5 + \lambda) + \sqrt{3}\,\mathrm{i}\right)\right) &= \sqrt{3} \\ \frac{\sqrt{3}}{5 + \lambda} &= \sqrt{3} \\ 1 &= 5 + \lambda \\ \lambda &= -4 \end{aligned}$$

[2 marks]

Answer 6

(a)

$$\begin{aligned} arg(z_1 - z_2) &= arg(-1 - i - 3 + 4i) \\ &= arg(-4 + 3i) \\ &= \pi - arctan\left(\frac{3}{4}\right) \\ &= \pi - 0.6435 \\ &= 2.498^c \end{aligned}$$

[3 marks]

(b)

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{-1 - i}{3 - 4i} \\ &= \frac{-1 - i}{3 - 4i} \times \frac{3 + 4i}{3 + 4i} \\ &= \frac{-3 - 4i - 3i - 4i^2}{9 - 16i^2} \\ &= \frac{1 - 7i}{25} \\ &= \frac{1}{25} - \frac{7}{25}i \end{aligned}$$

[3 marks]

(c)

$$\begin{aligned} \left|\frac{z_1}{z_2}\right| &= \left|\frac{1}{25} - \frac{7}{25}i\right| \\ &= \sqrt{\left(\frac{1}{25}\right)^2 + \left(\frac{7}{25}\right)^2} \\ &= \sqrt{\frac{1 + 49}{625}} \\ &= \frac{\sqrt{50}}{25} \\ &= \frac{5\sqrt{2}}{25} \\ &= \frac{\sqrt{2}}{5} \end{aligned}$$

[2 marks]

(d)

$$\frac{p + \mathrm{i}q - 8z_1}{p - \mathrm{i}q - 8z_2} = 3\mathrm{i}$$

$$p + \mathrm{i}q - 8z_1 = 3\mathrm{i}\,(p - \mathrm{i}q - 8z_2)$$

$$p + \mathrm{i}q - 8(-1 - \mathrm{i}) = 3\mathrm{i}p - 3\,\mathrm{i}^2 q - 24\mathrm{i}\,(3 - 4\mathrm{i})$$

$$p + \mathrm{i}q + 8 + 8\mathrm{i} = 3\mathrm{i}p + 3q - 72\mathrm{i} + 96\,\mathrm{i}^2$$

$$p + \mathrm{i}q + 8 + 8\mathrm{i} = 3\mathrm{i}p + 3q - 72\mathrm{i} - 96$$

$$p + \mathrm{i}q = 3\mathrm{i}p + 3q - 80\mathrm{i} - 104$$

$$p + \mathrm{i}q = 3q - 104 + (3p - 80)\,\mathrm{i}$$

$$\text{Re}: \quad p = 3q - 104 \Rightarrow \quad p - 3q = -104$$

$$\text{Im}: \quad q = 3p - 80 \Rightarrow -3p + q = -80$$

$$3p - 9q = -312$$

$$3p - q = 80 \qquad \text{Subtract}$$

$$-8q = -392$$

$$q = 49$$

$$\therefore\ p = 43$$

[5 marks]