6.4 Answers

6.4.1 Solution to 6.2 Example (Teaching Video)

The video was originally focussed on $z = \pm\sqrt{5 + 12\text{i}}$. However, the few Further A-Level examination questions on this topic are more about solving the related quadratic equation. So I've asked the question in this marginally adjusted form too.

$$z^2 - 5 - 12\text{i} = 0$$
$$z^2 = 5 + 12\text{i}$$
$$(a + b\text{i})^2 = 5 + 12\text{i}$$
$$a^2 + 2ab\text{i} + b^2\text{i}^2 = 5 + 12\text{i}$$
$$a^2 - b^2 + 2ab\text{i} = 5 + 12\text{i}$$

Two complex numbers (the one on the left hand side and the one on the right hand side) are equal if and only if the real parts are equal and the imaginary parts are equal. Therefore we can deduce that,

$$\therefore\ a^2 - b^2 = 5 \quad \text{and} \quad 2ab = 12$$
$$ab = 6$$

As a, b are integers the only possibility with $|a| > |b|$ (to have $a^2 - b^2$ positive) are, $(a, b) = (6, 1), (3, 2), (-6, -1)$ or $(-3, -2)$

Of these only $(3, 2)$ and $(-3, -2)$ also satisfy $a^2 - b^2 = 5$.

Thus, the solutions to

$$z^2 - 5 - 12\text{i} = 0$$

are $3 + 2\text{i}$ or $-3 + 2\text{i}$

These could be presented as $\pm(3 + 2\text{i})$ which has an obvious similarity with previous work with the real numbers. The solutions can, of course, be substituted back into the original equation to verify that they make it true. For example,

$$\begin{aligned} \text{LHS} &= z^2 - 5 - 12\text{i} \\ &= (\pm 1)^2 (3 + 2\text{i})^2 - 5 - 12\text{i} \\ &= 9 + 12\text{i} + 4\text{i}^2 - 5 - 12\text{i} \\ &= 9 - 5 - 4 + 12\text{i} - 12\text{i} \\ &= 0 \\ &= \text{RHS} \qquad \square \end{aligned}$$

[6 marks]

6.4.2 Solutions to 6.3 Exercise

Answer 1

(i)

$$z^2 - 7 - 24\text{i} = 0 \qquad z \in \mathbb{Z}[\text{i}]$$

$$z^2 = 7 + 24\text{i}$$

$$(a + b\text{i})^2 = 7 + 24\text{i}$$

$$a^2 + 2ab\text{i} + b^2\text{i}^2 = 7 + 24\text{i}$$

$$a^2 - b^2 + 2ab\text{i} = 7 + 24\text{i}$$

Two complex numbers (the one on the left hand side and the one on the right hand side) are equal if and only if the real parts and the imaginary parts are equal. Therefore we can deduce that,

Equating imaginary parts : $2ab = 24$

$$ab = 12$$

[2 marks]

(ii)

Equating real parts : $a^2 - b^2 = 7$

[1 mark]

(iii) As $a, b \in \mathbb{Z}$ the only possibility with $|a| > |b|$ (to have $a^2 - b^2 > 0$)

are, $(a, b) = (12, 1), \quad (6, 2), \quad (4, 3)$

$(-12, -1), (-6, -2), (-4, -3)$

Of these only $(4, 3)$ and $(-4, -3)$ also satisfy $a^2 - b^2 = 7$

Thus, the solutions to

$$z^2 - 7 - 24\text{i} = 0$$

are $4 + 3\text{i}$ or $-4 - 3\text{i}$

These could be presented as $\pm(4 + 3\text{i})$ which has an obvious similarity with previous work with the real numbers.

[3 marks]

(iv)

$$(\pm 1)^2 (4 + 3\text{i})^2 = 16 + 24\text{i} + 9\text{i}^2$$

$$= 16 + 24\text{i} - 9$$

$$= 7 + 24\text{i} \text{ as expected}$$

[2 marks]

Answer 2

(i)

$$z^2 + 9 - 40\mathrm{i} = 0 \qquad z \in \mathbb{Z}[\mathrm{i}]$$

$$z^2 = -9 + 40\mathrm{i}$$

$$(a + b\mathrm{i})^2 = -9 + 40\mathrm{i}$$

$$a^2 + 2ab\mathrm{i} + b^2\mathrm{i}^2 = -9 + 40\mathrm{i}$$

$$a^2 - b^2 + 2ab\mathrm{i} = -9 + 40\mathrm{i}$$

Two complex numbers (the one on the left hand side and the one on the right hand side) are equal if and only if the real parts and the imaginary parts are equal. Therefore we can deduce that,

Equating imaginary parts : $2ab = 40$

$$ab = 20$$

[2 marks]

(ii)

Equating real parts : $a^2 - b^2 = -9$

[1 mark]

(iii) As $a, b \in \mathbb{Z}$ the only possibility with $|a| < |b|$ (to have $a^2 - b^2 < 0$)

are, $(a, b) = (1, 20), \quad (2, 10), \quad (4, 5)$

$(-1, -20), (-2, -10), (-4, -5)$

Of these only $(4, 5)$ and $(-4, -5)$ also satisfy $a^2 - b^2 = -9$

Thus, the solutions to

$$z^2 + 9 - 40\mathrm{i} = 0$$

are $4 + 5\mathrm{i}$ or $-4 - 5\mathrm{i}$

These could be presented as $\pm(4 + 5\mathrm{i})$ which has an obvious similarity with previous work with the real numbers

[3 marks]

(iv)

$$(\pm 1)^2 (4 + 5\mathrm{i})^2 = 16 + 40\mathrm{i} + 25\mathrm{i}^2$$

$$= 16 + 40\mathrm{i} - 25$$

$$= -9 + 40\mathrm{i} \quad \text{as expected}$$

[2 marks]

Answer 3

(a)

$$z^2 - 40 + 42\mathrm{i} = 0 \qquad z \in \mathbb{Z}[\mathrm{i}]$$

$$z^2 = 40 - 42\mathrm{i}$$

$$(a + b\mathrm{i})^2 = 40 - 42\mathrm{i}$$

$$a^2 + 2ab\mathrm{i} + b^2\mathrm{i}^2 = 40 - 42\mathrm{i}$$

$$a^2 - b^2 + 2ab\mathrm{i} = 40 - 42\mathrm{i}$$

Equating imaginary parts : $2ab = -42$

$$ab = -21$$

Equating real parts : $a^2 - b^2 = 40$

As $a, b \in \mathbb{Z}$ the only possibility with $|a| > |b|$ (to have $a^2 - b^2 > 0$)

are, $(a, b) = (21, -1)$, $(7, -3)$

$(-21, 1)$, $(-7, 3)$

Of these only $(7, -3)$ and $(-7, 3)$ also satisfy $a^2 - b^2 = -9$

Thus, the solutions to

$$z^2 - 40 + 42\mathrm{i} = 0$$

are $7 - 3\mathrm{i}$ or $-7 + 3\mathrm{i}$

These could be presented as $\pm(7 - 3\mathrm{i})$ which has an obvious similarity with previous work with the real numbers

[5 marks]

(b) **(i)** $a = 1$, $b = 0$ and $c = -40 + 42\mathrm{i}$

$$z = \frac{0 \pm \sqrt{0^2 - 4 \times 1 \times (-40 + 42\mathrm{i})}}{2 \times 1}$$

$$= \frac{\pm\sqrt{4} \times \sqrt{40 - 42\mathrm{i}}}{2}$$

$$= \pm\sqrt{40 - 42\mathrm{i}}$$

[2 marks]

(ii) The formula hasn't told us anything that wasn't obvious anyway !
We could have got this “result” much more easily like this;

$$z^2 - 40 + 42\mathrm{i} = 0$$

$$z^2 = 40 - 42\mathrm{i}$$

$$z = \pm\sqrt{40 - 42\mathrm{i}}$$

[1 mark]

Answer 4

(i) $\left(1+\sqrt{7}\,\mathrm{i}\right)\left(1-\sqrt{7}\,\mathrm{i}\right) = 1 - 7\,\mathrm{i}^2$

$$= 1 + 7$$

$$= 8$$

[1 mark]

(ii) $(1+\mathrm{i})(1-\mathrm{i})(1+\mathrm{i})(1-\mathrm{i})(1+\mathrm{i})(1-\mathrm{i}) = 2\times 2\times 2$

$$= 8$$

[2 marks]

(iii) The part (i) and part (ii) answers show that the number 8 can be factorised in at least two different ways and so the factorisation is not unique.
Other factorisations are possible such as,

$$\left(\sqrt{3}+\sqrt{5}\,\mathrm{i}\right)\left(\sqrt{3}-\sqrt{5}\,\mathrm{i}\right) = 3 - 5\,\mathrm{i}^2$$

$$= 3 + 5$$

$$= 8$$

[1 mark]

Answer 5

$$p = (3+2\mathrm{i})(3-2\mathrm{i})$$

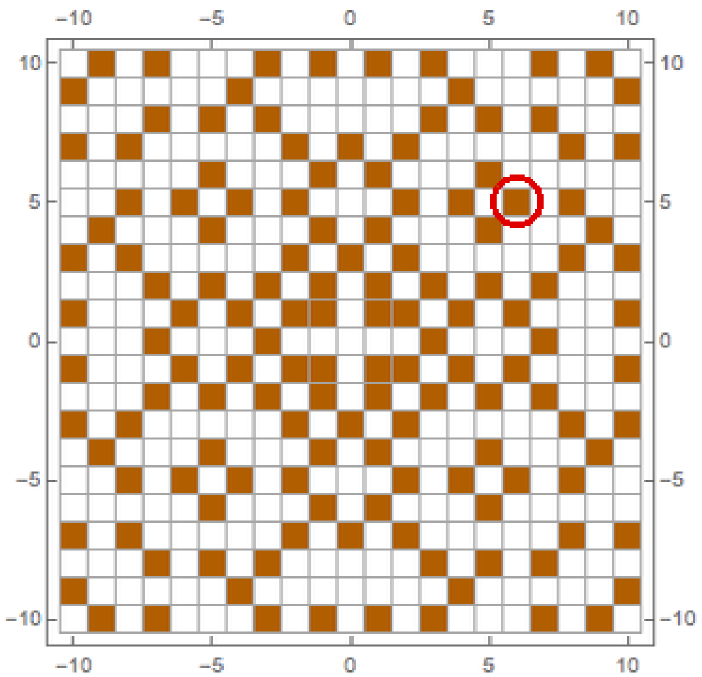
$$= 9 - 6\mathrm{i} + 6\mathrm{i} - 4\,\mathrm{i}^2$$

$$= 13$$

[1 mark]

Answer 6

(i)



[1 mark]

(ii) Yes, $6 + 5\mathrm{i}$ is a Gaussian prime.
It has exactly two factorised in $\mathbb{Z}[\mathrm{i}]$, itself and the number 1.

[1 mark]

Answer 7

$$z^3 + \mathrm{i}\,z^2 + 6z = 0 \qquad z \in \mathbb{C}$$

$$z\left(z^2 + \mathrm{i}\,z + 6\right) = 0$$

$$\text{Either } z = 0 \quad \text{or} \quad z^2 + \mathrm{i}\,z + 6 = 0$$

With $a = 1$, $b = \mathrm{i}$ and $c = 6$

$$\begin{aligned} z &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-\mathrm{i} \pm \sqrt{\mathrm{i}^2 - 4 \times 1 \times 6}}{2 \times 1} \\ &= \frac{-\mathrm{i} \pm \sqrt{-1 - 24}}{2} \\ &= \frac{-\mathrm{i} \pm \sqrt{-25}}{2} \\ &= \frac{-\mathrm{i} \pm 5\mathrm{i}}{2} \\ &= -3\mathrm{i} \text{ or } 2\mathrm{i} \end{aligned}$$

So, in total there are the three roots, $-3\mathrm{i}$, 0, $2\mathrm{i}$

[5 marks]

Answer 8

$$z^2 + 3 - 2\text{i} = 0 \qquad z \in \mathbb{Z}[\text{i}]$$

$$z^2 = -3 + 2\text{i}$$

$$(a + b\text{i})^2 = -3 + 2\text{i}$$

$$a^2 + 2ab\text{i} + b^2\text{i}^2 = -3 + 2\text{i}$$

$$a^2 - b^2 + 2ab\text{i} = -3 + 2\text{i}$$

(i) Equating imaginary parts : $2ab = 2$

$$ab = 1$$

[2 marks]

(ii) Equating real parts : $a^2 - b^2 = -3$

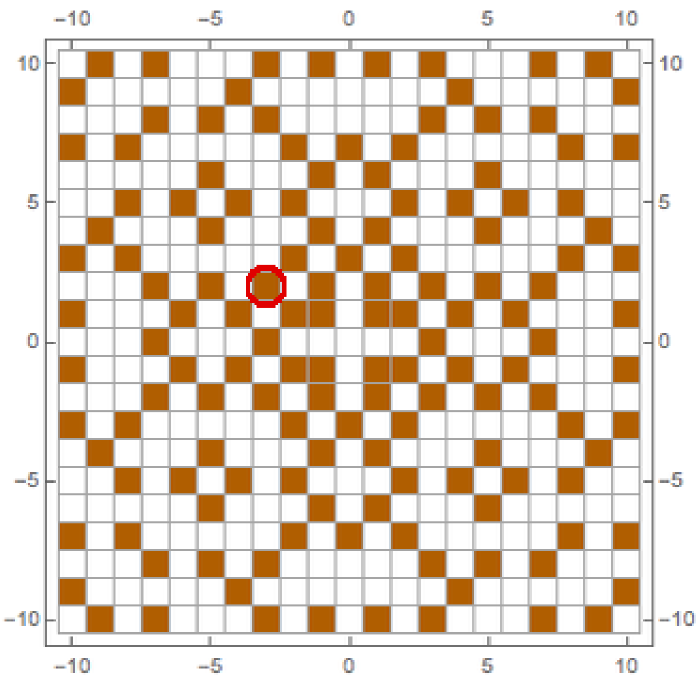
[1 mark]

(iii) As $a, b \in \mathbb{Z}$ the with $ab = 1$ only possibilities are $|a| = |b| = 1$
but this clearly cannot satisfy the requirement that $a^2 - b^2 = -3$
Thus there are no Gaussian integers that satisfy the equation,

$$z^2 + 3 - 2\text{i} = 0 \qquad z \in \mathbb{Z}[\text{i}] \qquad \square$$

[2 marks]

(iv)



[1 mark]

(v) The complex number is shown as a prime on the grid of Gaussian integers, confirming that it cannot be factorised in $\mathbb{Z}[\text{i}]$

[2 marks]

Answer 9

(a)

$$z^2 + 16 - 30\text{i} = 0$$
$$z^2 = -16 + 30\text{i}$$
$$(a + b\text{i})^2 = -16 + 30\text{i}$$
$$a^2 + 2ab\text{i} + b^2\text{i}^2 = -16 + 30\text{i}$$
$$a^2 - b^2 + 2ab\text{i} = -16 + 30\text{i}$$

Equating imaginary parts : $2ab = 30$

$ab = 15$ ☐

[2 marks]

(b) Equating real parts : $a^2 - b^2 = -16$

As $a, b \in \mathbb{Z}$ the only possibilities with $|a| < |b|$ (to get $a^2 - b^2 < 0$)

are, $(a, b) = (1, 15), (3, 5), (-1, -15)$ and $(-3, -5)$

Of these only $(3, 5)$ and $(-3, -5)$ also satisfy $a^2 - b^2 = -16$

Thus, the solutions to

$$z^2 + 16 - 30\text{i} = 0$$

are $3 + 5\text{i}$ or $-3 - 5\text{i}$

These could be presented as $\pm(3 + 5\text{i})$

which has an obvious similarity with previous work with the real numbers.

[4 marks]