## 7.4 Answers



### 7.4.1 Solution to 7.2 The Circle (Example)

**( i )**

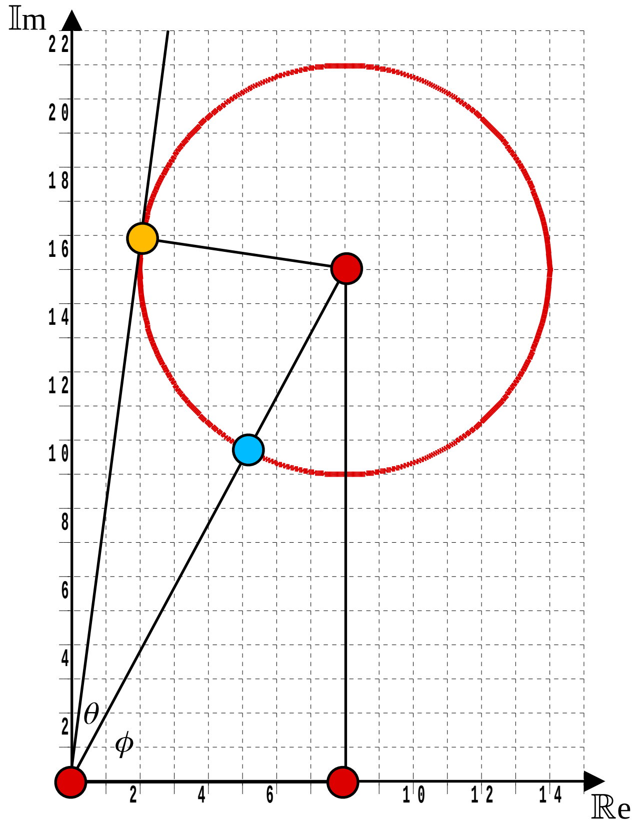
$$|z - 8 - 15\mathrm{i}| = 6$$

$$|x + y\mathrm{i} - 8 - 15\mathrm{i}| = 6$$

$$|(x - 8) + (y - 15)\mathrm{i}| = 6$$

$$(x - 8)^2 + (y - 15)^2 = 36$$

**( ii )** This is in the form of a circle with centre (8, 15) and radius 6




**( iii )** $|z|$ is the least distance from the origin to the circle.
There is an integer Pythagorean triple triangle, 8, 15, 17
Thus, as the circle has radius 6,

$$|z|_{\text{MIN}} = 17 - 6 = 11$$

**( iv )** We're after the maximum anticlockwise rotation of the positive ℝe-axis that still cuts (is a tangent to) the circle, shown as $\theta + \phi$ on the diagram.

$$\sin\theta = \frac{6}{17} \Rightarrow \theta = 0.36^c$$

$$\tan\phi = \frac{15}{8} \Rightarrow \phi = 1.08^c$$

$$\therefore [\arg z]_{\text{MAX}\,=} = 1.44^c$$

**[ 2, 2, 2, 2 marks ]**

### 7.4.2 Solutions to 7.3 Exercise

**Answer 1**

For $z_1 = -5 - 2i$

Think of this as “a clockwise half a turn then anticlockwise a bit”

$$arg(z_1) = -\pi + arctan\left(\frac{2}{5}\right)$$
$$= -\pi + 0.381$$
$$= -2.76^c$$

**[ 2 marks ]**

For $z_2 = 3 - 5i$

Think of this as “clockwise a bit”

$$arg(z_2) = -arctan\left(\frac{5}{3}\right)$$
$$= -1.03^c$$

**[ 2 marks ]**

For $z_3 = 5 + 5i$

Think of this as “anticlockwise a bit, where the bit is an eighth of a turn”

$$arg(z_3) = arctan\left(\frac{5}{5}\right)$$
$$= \frac{\pi}{4}^c \qquad \text{or} \qquad 0.785^c$$

**[ 2 marks ]**

For $z_4 = -1 + 6i$

Think of this as “anticlockwise half a turn then clockwise a bit”

$$arg(z_4) = \pi - arctan\left(\frac{6}{1}\right)$$
$$= \pi - 1.41^c$$
$$= 1.74^c$$

**[ 2 marks ]**

**Answer 2**

**( a )**

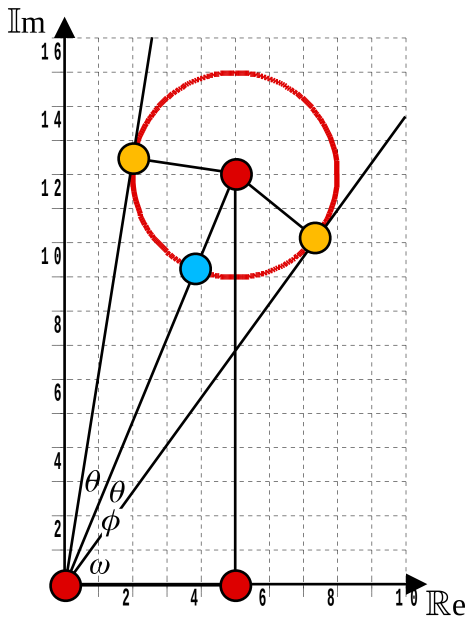
$$|z - 5 - 12i| = 3$$

$$|x + yi - 5 - 12i| = 3$$

$$|(x - 5) + (y - 12)i| = 3$$

$$(x - 5)^2 + (y - 12)^2 = 9$$

This is in the form of a circle with centre (5, 12) and radius 3




**[ 3 marks ]**

**( iii )** There is an integer Pythagorean triple triangle, 5, 12, 13

Thus, as the circle has radius 3.

$$|z|_{MIN} = 13 - 3 = 10$$

$$|z|_{MAX} = 13 + 3 = 16$$

**[ 4 marks ]**

**( iv )**

$$\sin\theta = \frac{3}{13} \Rightarrow \theta = 0.233^c$$

$$\tan\phi = \frac{12}{5} \Rightarrow \phi = 1.176^c$$

$$\therefore [\arg z]_{MAX =} = \phi + \theta$$

$$= 1.409^c$$

$$[\arg z]_{MIN} = \omega$$

$$= \phi - \theta$$

$$= 0.943^c$$

**[ 4 marks ]**

**Answer 3**

**( i )**

$$\left| z - 24 - 7\mathrm{i} \right| = 5$$

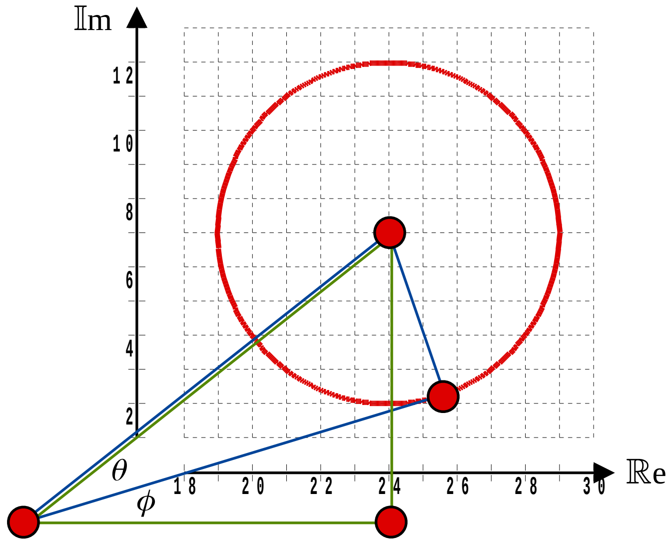
$$\left| x + y\mathrm{i} - 24 - 7\mathrm{i} \right| = 5$$

$$\left| (x - 24) + (y - 7)\mathrm{i} \right| = 5$$

$$(x - 24)^2 + (y - 7)^2 = 25$$

**[ 2 marks ]**

**( ii )** This is in the form of a circle with centre ( 24, 7 ) and radius 5




The circle is plotted accurately but the green and blue triangles have the origin as a common point which is far further to the left than shown.

**[ 2 marks ]**

**( iii )** $|z|$ is the greatest distance from the origin to the circle.
There is an integer Pythagorean triple triangle, 24, 7, 25
Thus, as the circle has radius 5,

$$|z|_{\text{MIN}} = 25 + 5$$

$$= 30$$

**[ 2 marks ]**

**( iv )** We're after the minimum anticlockwise rotation of the positive $x$-axis that is cutting the circle. Let $(\phi + \theta)$ be the angle in the Pythagorean integer triangle, shown green (but not to scale) above. Let $\theta$ be the angle in the blue triangle with a right angle where the circle's radius touches the tangent from the origin to the circle.

$$tan\ (\phi + \theta) = \frac{7}{24} \Rightarrow (\phi + \theta) = 0.2838^{c}$$

$$sin\ \theta = \frac{5}{25} \Rightarrow \theta = 0.2014^{c}$$

$$\therefore \phi = [\ arg\ z\ ]_{\text{MIN}} = 0.0824^{c}$$

**[ 3 marks ]**

**Answer 4**

$$z = \frac{3 + 5i}{2 - i} \times \frac{2 + i}{2 + i}$$

$$= \frac{6 + 3i + 10i + 5i^2}{4 + 2i - 2i - i^2}$$

$$= \frac{1 + 13i}{5}$$

$$= 0.2 + 2.6i$$

**( i )**

$$|z| = \sqrt{0.2^2 + 2.6^2}$$

$$= 2.61$$

**[ 4 marks ]**

**( ii )**

$$arg\, z = arctan\left(\frac{2.6}{0.2}\right)$$

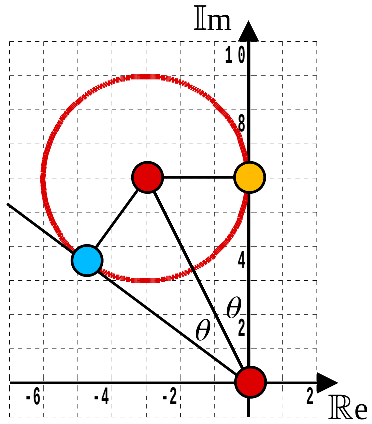
$$= 1.49^c$$

**[ 2 marks ]**

**Answer 5**

$$|z + 3 - 6\mathrm{i}| = 3$$
$$|x + y\mathrm{i} + 3 - 6\mathrm{i}| = 3$$
$$|(x + 3) + (y - 6)\mathrm{i}| = 3$$
$$(x + 3)^2 + (y - 6)^2 = 9$$

This is in the form of a circle with centre (– 3, 6) and radius 3




We're after the maximum anticlockwise rotation of a line that started on the positive $\mathbb{R}$e-axis that is cutting the circle. This is the line, tangential to the circle on the diagram and meeting the circle at the point coloured blue.

Use can be made of the fact that the Imaginary axis is tangential to the circle and meeting the circle at the point coloured yellow.

A property of tangents to a circle from a common point is that the two angles marked $\theta$ are equal. So the required angle is a quarter turn anticlockwise plus two further turns anticlockwise of $\theta$.

Thus,

$$\begin{aligned}
\therefore\ [\, arg\, z\,]_{\mathrm{MAX}} &= \frac{\pi}{2} + 2\theta \\
&= \frac{\pi}{2} + 2\, arcsin\left( \frac{3}{\sqrt{6^2 + 3^2}} \right) \\
&= \frac{\pi}{2} + 2\, arcsin\left( \frac{3}{\sqrt{45}} \right) \\
&= \frac{\pi}{2} + 2\, arcsin\left( \frac{3}{3\sqrt{5}} \right) \\
&= \frac{\pi}{2} + 2\, arcsin\left( \frac{1}{\sqrt{5}} \right) \qquad \square
\end{aligned}$$

**[ 4 marks ]**

**Answer 6**

**( i )**

$$
\begin{aligned}
|z| &= \sqrt{(-1)^2 + \left(\sqrt{3}\,\right)^2} \\
&= \sqrt{1 + 3} \\
&= \sqrt{4} \\
&= 2
\end{aligned}
$$

**[ 1 mark ]**

**( ii )**

$$
\begin{aligned}
\left|\frac{z}{z^*}\right| &= \left|\frac{-1 - \sqrt{3}\,\mathrm{i}}{-1 + \sqrt{3}\,\mathrm{i}}\right| \\
&= \left|\frac{-1 - \sqrt{3}\,\mathrm{i}}{-1 + \sqrt{3}\,\mathrm{i}} \times \frac{-1 - \sqrt{3}\,\mathrm{i}}{-1 - \sqrt{3}\,\mathrm{i}}\right| \\
&= \left| \frac{(-1)\left(1 + \sqrt{3}\,\mathrm{i}\right)\ (-1)\ \left(1 + \sqrt{3}\,\mathrm{i}\right)}{1 + \sqrt{3}\,\mathrm{i} - \sqrt{3}\,\mathrm{i} - 3\,\mathrm{i}^2} \right| \\
&= \left| \frac{1 + 2\sqrt{3}\,\mathrm{i} + 3\,\mathrm{i}^2}{4} \right| \\
&= \left| \frac{-2 + 2\sqrt{3}\,\mathrm{i}}{4} \right| \\
&= \left| -\frac{1}{2} + \frac{\sqrt{3}}{2}\,\mathrm{i} \right| \\
&= \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \\
&= \sqrt{\left(\frac{1}{4}\right) + \left(\frac{3}{4}\right)} \\
&= \sqrt{1} \\
&= 1
\end{aligned}
$$

**[ 4 marks ]**

**( iii )**

$$arg\, z = -\pi + arctan\left( \frac{\sqrt{3}}{1} \right)$$
$$= -\pi + \frac{\pi}{3}$$
$$= -\frac{2}{3}\pi$$

$$z^* = -1 + \sqrt{3}\ \mathrm{i}$$
$$arg\ (z^*) = \pi - arctan\left( \frac{\sqrt{3}}{1} \right)$$
$$= \pi - \frac{\pi}{3}$$
$$= \frac{2}{3}\pi$$

$$arg\left(\frac{z}{z^*}\right) = \left( -\frac{1}{2} + \frac{\sqrt{3}}{2}\ \mathrm{i} \right)$$
$$= \frac{2}{3}\pi$$

**[ 3 marks ]**

**Answer 7**

**( i )**

$$\begin{aligned} |w| &= |6 + 3\mathrm{i}| \\ &= \sqrt{6^2 + 3^2} \\ &= \sqrt{45} \\ &= 3\sqrt{5} \end{aligned}$$

**[ 1 mark ]**

**( ii )**

$$\begin{aligned} arg\, z &= arg\ ( 6 + 3\mathrm{i} ) \\ &= arctan\left( \frac{3}{6} \right) \\ &= 0.464^c \end{aligned}$$

**[ 2 marks ]**

**( iii )** The angle puts the point in the first quadrant of the complex plane which makes the question easier than it might have been !

$$\begin{aligned} arg\ ( \lambda + 5\mathrm{i} + w ) &= \frac{\pi}{4} \\ arg\ ( \lambda + 5\mathrm{i} + 6 + 3\mathrm{i} ) &= \frac{\pi}{4} \\ arctan\left( \frac{8}{\lambda + 6} \right) &= \frac{\pi}{4} \\ \left( \frac{8}{\lambda + 6} \right) &= 1 \\ 8 &= \lambda + 6 \\ \lambda &= 2 \end{aligned}$$

**[ 2 marks ]**