9.4 Answers

9.4.1 Solution to 9.2 Teaching Video Example

Keep in mind that, $arg\,(z) = 0$ is a "half-line" starting at (but excluding) the point at the origin (which is undefined) and heading off in the positive x-axis direction. Questions involving arguments can often be visualised immediately by utilising a knowledge of graph transformations applied to this half-line.

It helps to have a sense of how the question should progress !

In this case, it's apparent from the following rewrite;

$$arg(z - 3 + 2i) = \frac{2\pi}{3}$$

$$arg(z - (3 - 2i)) = \frac{2\pi}{3}$$

that the half-line starts at the point $(3, -2)$ and heads off at 120°

To obtain the Cartesian equation,

$$arg(z - 3 + 2i) = \frac{2\pi}{3}$$

$$arg(x + yi - 3 + 2i) = \frac{2\pi}{3}$$

$$arg((x - 3) + (y + 2)i) = \frac{2\pi}{3}$$

(gathering reals together and, separately, imaginaries together)

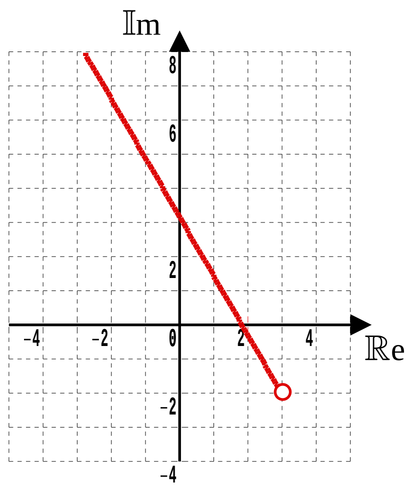
$$\frac{y+2}{x-3} = tan\left(\frac{2\pi}{3}\right)$$

$$\frac{y+2}{x-3} = -\sqrt{3}$$

$$y + 2 = -\sqrt{3}\,(x - 3)$$

$$y = -\sqrt{3}x + 3\sqrt{3} - 2 \qquad x < 3$$

(For plotting : $y = -1.73x + 3.20$)



[4 marks]

9.4.2 Solutions to 9.3 Exercise

Answer 1

To obtain the Cartesian equation,

$$arg(z - 2 + 3i) = \frac{3\pi}{4}$$

$$arg(x + yi - 2 + 3i) = \frac{3\pi}{4}$$

$$arg((x - 2) + (y + 3)i) = \frac{3\pi}{4}$$

$$\frac{y + 3}{x - 2} = tan\left(\frac{3\pi}{4}\right)$$

$$\frac{y + 3}{x - 2} = -1$$

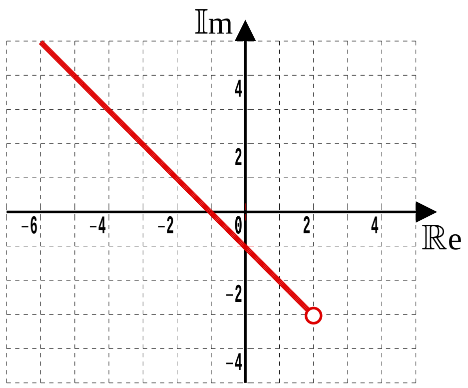
$$y + 3 = -(x - 2)$$

$$y = -x - 1, \qquad x < 2$$

With regards to the sketch,

$$arg(z - 2 + 3i) = \frac{3\pi}{4} \Rightarrow arg(z - (2 - 3i)) = \frac{3\pi}{4}$$

so the start of the half-line is at (2, – 3) and it heads off at 135°



[4 marks]

Answer 2

To obtain the Cartesian equation,

$$arg\ (z - 3 - 4\mathrm{i}) = -\frac{2\pi}{3}$$

$$arg(x + y\mathrm{i} - 3 - 4\mathrm{i}) = -\frac{2\pi}{3}$$

$$arg((x - 3) + (y - 4)\mathrm{i}) = -\frac{2\pi}{3}$$

$$\frac{y - 4}{x - 3} = tan\left(-\frac{2\pi}{3}\right)$$

$$\frac{y - 4}{x - 3} = \sqrt{3}$$

$$y - 4 = \sqrt{3}\,(x - 3)$$

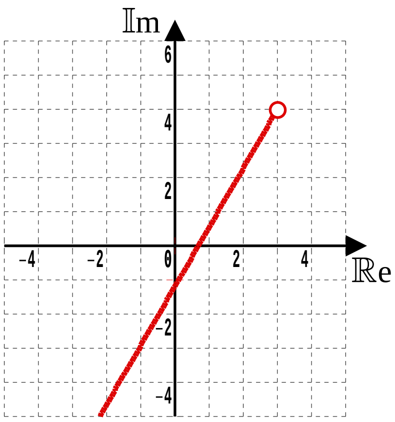
$$y = \sqrt{3}\,x + 4 - 3\sqrt{3} \qquad x < 3$$

(To plot this line use $y = 1.73x - 1.20$)

With regards to the sketch,

$$arg(z - 3 - 4\mathrm{i}) = -\frac{2\pi}{3} \Rightarrow arg(z - (3 + 4\mathrm{i})) = -\frac{2\pi}{3}$$

so the start of the half-line is at (3, 4) and it heads off at – 120°



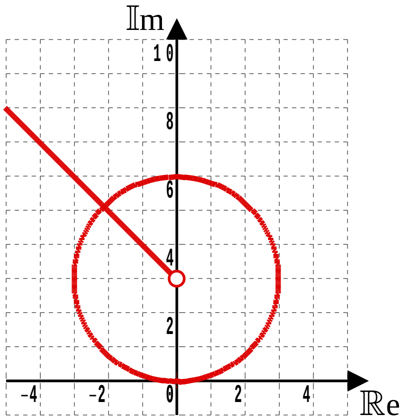
[4 marks]

Answer 3

(a) $|z - 3i| = 3$ is a circle with centre (0, 3) and radius 3

[2 marks]

(b) $arg(z - 3i) = \frac{3\pi}{4}$ is a half-line, starting at (0, 3) and heading off at 135°



$$\text{Circle}: \; x^2 + (y - 3)^2 = 9$$

$$\text{Line}: \; y = -x + 3$$

$$x^2 + ((-x + 3) - 3)^2 = 9$$

$$x^2 + (-x)^2 = 9$$

$$2x^2 = 9$$

$$x^2 = \frac{9}{2}$$

$$x = -\frac{3}{\sqrt{2}} \qquad \text{(Reject positive solution)}$$

$$x = -\frac{3\sqrt{2}}{2}$$

$$\therefore \; y = 3 + \frac{3\sqrt{2}}{2}$$

$$z = -\frac{3\sqrt{2}}{2} + \left(3 + \frac{3\sqrt{2}}{2}\right)i$$

[4 marks]

Answer 4

(a) $\left| z + \sqrt{3}\,\mathrm{i} \right| = 3$ is a circle with centre (0, $-\sqrt{3}$) and radius 3

[2 marks]

(b) $arg\,(z) = \dfrac{\pi}{6}$ is a half-line, starting at (0, 0) and heading off at 30°

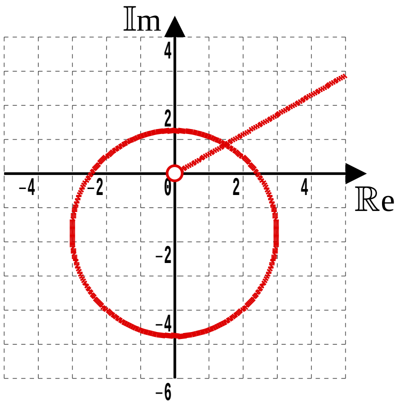
To obtain the Cartesian equation, $arg\,(z) = \dfrac{\pi}{6}$

$$arg\,(x + y\mathrm{i}) = \frac{\pi}{6}$$

$$\frac{y}{x} = tan\left(\frac{\pi}{6}\right)$$

$$\frac{y}{x} = \frac{1}{\sqrt{3}}$$

$$y = \frac{1}{\sqrt{3}}\,x \qquad x > 0$$



Circle : $x^2 + \left(y + \sqrt{3}\right)^2 = 16$ Line : $y = \dfrac{1}{\sqrt{3}}\,x$ or $y = \dfrac{\sqrt{3}}{3}\,x$

$$x^2 + \left(\frac{1}{\sqrt{3}}\,x + \sqrt{3}\right)^2 = 9$$

$$x^2 + \frac{1}{3}x^2 + 2\,x + 3 = 9$$

$$3\,x^2 + x^2 + 6\,x + 9 = 27$$

$$4\,x^2 + 6\,x - 18 = 0$$

$$2\,x^2 + 3x - 9 = 0$$

$$x = \frac{-3 \pm \sqrt{3^2 - 4 \times 2 \times (-9)}}{2 \times 2}$$

$$= \frac{-3 \pm 9}{4} = \frac{3}{2} \qquad \text{(Reject solution of } -3\text{)}$$

$$\therefore\ z = \frac{3}{2} + \frac{\sqrt{3}}{2}\,\mathrm{i}$$

[4 marks]

Answer 5

$arg(z + 4) = \frac{\pi}{3}$ is a half-line, starting at $(-4,\ 0)$ and heading off at 60°

[2 marks]

To obtain the Cartesian equation, $arg(z + 4) = \frac{\pi}{3}$

$$arg(x + y\text{i} + 4) = \frac{\pi}{3}$$

$$arg(x + 4 + y\text{i}) = \frac{\pi}{3}$$

$$\frac{y}{x + 4} = tan\left(\frac{\pi}{3}\right)$$

$$\frac{y}{x + 4} = \sqrt{3}$$

$$y = \sqrt{3}\,x + 4\sqrt{3} \qquad x > -4$$

Thus, the perpendicular line through the origin is, $y = -\frac{1}{\sqrt{3}}\,x$

These two lines will meet when, $\sqrt{3}\,x + 4\sqrt{3} = -\frac{1}{\sqrt{3}}\,x$

Multiply through by $\sqrt{3}$

$$3x + 12 = -x$$

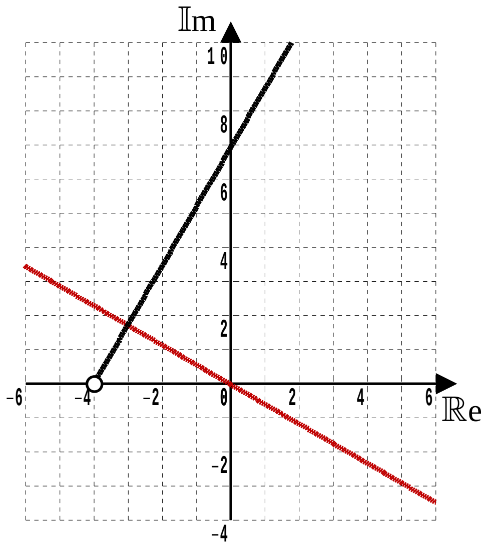
$$4x = -12$$

$$x = -3 \qquad \text{and so} \qquad y = \sqrt{3}$$

To get the distance from the origin,

$$|z|_{\text{MIN}} = \sqrt{(-3)^2 + \left(\sqrt{3}\right)^2}$$

$$= 2\sqrt{3}$$



[4 marks]

Answer 6

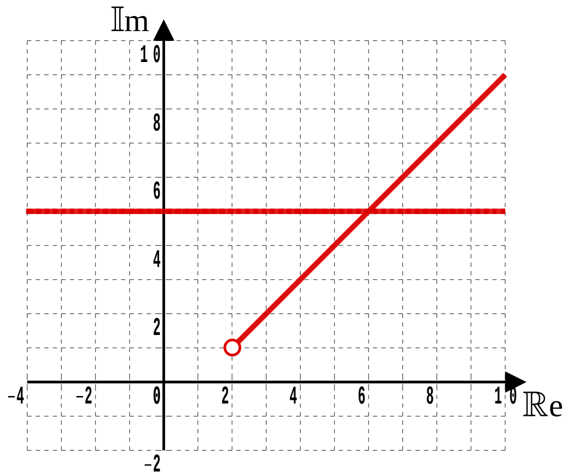
(a) $|z - 2i| = |z - 8i|$ is the perpendicular bisector of the line segment between (0, 2) and (0, 8) which is the line $y = 5$

[2 marks]

(b) $arg(z - 2 - i) = \dfrac{\pi}{4}$ is a half-line, starting at (2, 1), heading off at 45°

The equation of this line will be $y = x - 1$

[3 marks]



(c)

$$z = 6 + 5i$$

[2 marks]