Further Pure A-Level Mathematics
Compulsory Course Component
Core 1

COMPLEX NUMBERS
I

COMPLEX NUMBERS I

Lesson 1

Further A-Level Pure Mathematics : Core 1

Complex Numbers I

1.1 Introduction

Welcome to the start of the Further Mathematics Course. The first topic, Complex Numbers, is part of the Core 1 section, compulsory for all further mathematicians at both A and AS Level.

1.2 What is a complex number ?

When solving a quadratic equation such as,

$$z^2 - 4z + 5 = 0$$

one method is to utilise the method of completing the square.

Here is what a typical solution would look like,

$$z^2 - 4z + 5 = 0$$

$$\left[z^2 - 4z \right] + 5 = 0$$

$$\left[(z - 2)^2 - 4 \right] + 5 = 0$$

$$(z - 2)^2 + 1 = 0 \qquad \text{This is completed square form}$$

$$(z - 2)^2 = -1$$

$$z - 2 = \pm\sqrt{-1} \qquad \text{By square rooting both sides}$$

$$z = 2 \pm \sqrt{-1}$$

$$z = 2 \pm \mathrm{i}$$

The first four lines of this solution are familiar territory. Depending upon what you already knows about complex numbers will determine how you feel about the final four lines. Historically, mathematicians would abandon the solution at the point where $\sqrt{-1}$ occurred. However, embracing the strange final result opened a doorway into a wonderful and enlarged world of mathematics; All that was already accepted sat quite comfortably within a new land in which $\mathrm{i} = \sqrt{-1}$ and $\mathrm{i}^2 = -1$. As a subject, complex numbers became mainstream in the 18th century but they had been known “as a curiosity” for around two hundred years before that.

When solving any equation, one method to check that a solution is valid is to substitute the proposed solution back into the original equation. If it is a valid solution then it will make the original equation true. For example, to check that $x = 4$ is a solution to the equation $5x + 3 = 23$, replace the x on the left hand side with 4, and show that it then turns the left hand side into the right hand side. So, by way of introducing some complex number arithmetic, it will now be shown that $2 + \mathrm{i}$ is a valid solution to the equation $z^2 - 4z + 5 = 0$

1.3 Complex Number Arithmetic

Show by means of substitution that $2 + i$ is a solution to $z^2 - 4z + 5 = 0$

Teaching Video : http://www.NumberWonder.co.uk/Video/v9085(1).mp4

[3 marks]

1.4 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable

Marks Available : 40

Question 1

Use the method of completing the square to show that the quadratic equation,

$$z^2 - 6z + 13 = 0$$

has solutions,

$$z = 3 \pm 2\mathrm{i}$$

[3 marks]

Question 2

Use the method of completing the square to show that the quadratic equation,

$$z^2 - 4z + 29 = 0$$

has solutions

$$z = 2 \pm 5\mathrm{i}$$

[3 marks]

Question 3

Use the method of completing the square to show that the quadratic equation,

$$z^2 + 8z + 21 = 0$$

has solutions

$$z = -4 \pm \sqrt{5}\,\mathrm{i}$$

[3 marks]

Question 4

The quadratic equation $az^2 + bz + c = 0$ has solutions given by the formula,

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Show how this formula could be used to solve the quadratic equation,

$$z^2 + 5z + 25 = 0$$

Present your answer in a simplified, elegant form.

[3 marks]

Question 5

Show by means of substitution that $z = 3 + \mathrm{i}$ is a solution of the equation,

$$z^2 - 6z + 10 = 0$$

[3 marks]

Question 6

$$f(z) = z^2 - 2z + 17$$

Show by means of substitution that $z = 1 - 4\mathrm{i}$ is a solution to $f(z) = 0$

[3 marks]

Question 7

(i) Show by substitution that $z = 1 + \mathrm{i}$ is a solution of the equation,

$$z^4 + 4 = 0$$

[4 marks]

(ii) Guess another solution to the equation $z^4 + 4 = 0$ and then verify that your guess is good.

[4 marks]

Question 8

Prove that $(a + b\mathrm{i})(a - b\mathrm{i})$ is a real number for any real numbers a and b

[3 marks]

Question 9

(i) Pick any two positive real integers a and b such that $a > b$.

[1 mark]

(ii) Calculate

$$(a + b\mathrm{i})^2$$

writing your answer in the form $u + v\mathrm{i}$ where u and v are real integers.

[2 marks]

(iii) Calculate $\sqrt{u^2 + v^2}$

[1 mark]

(iv) Demonstrate that, not only is $\sqrt{u^2 + v^2}$ an integer, but that u, v and $\sqrt{u^2 + v^2}$ are a Pythagorean integer triple.

[3 marks]

Isn't that amazing ?

[0 marks]

Question 10

Use the method of completing the square to show that the quadratic equation,

$$7z^2 - 3z + 3 = 0$$

has solutions

$$z = \frac{3}{14} \pm \frac{5\sqrt{3}}{14}\,\mathrm{i}$$

[4 marks]

1.5 Answers

1.5.1 Solution to 1.3 Complex Number Arithmetic

$$\begin{aligned}
\text{LHS} &= z^2 - 4z + 5 \\
&= (2 + \mathrm{i})^2 - 4(2 + \mathrm{i}) + 5 \\
&= (2 + \mathrm{i})(2 + \mathrm{i}) - 8 - 4\mathrm{i} + 5 \\
&= 4 + 2\mathrm{i} + 2\mathrm{i} + \mathrm{i}^2 - 8 - 4\mathrm{i} + 5 \\
&= 4 + 2\mathrm{i} + 2\mathrm{i} - 1 - 8 - 4\mathrm{i} + 5 \\
&= 0 \\
&= \text{RHS} \qquad \square
\end{aligned}$$

[3 marks]

1.5.2 Solutions to 1.4 Exercise

Answer 1

$$\begin{aligned}
z^2 - 6z + 13 &= 0 \\
\left[z^2 - 6z \right] + 13 &= 0 \\
\left[(z - 3)^2 - 9 \right] + 13 &= 0 \\
(z - 3)^2 + 4 &= 0 \\
(z - 3)^2 &= -4 \\
z - 3 &= \pm\sqrt{(-1)(4)} \\
z &= 3 \pm 2\mathrm{i}
\end{aligned}$$

[3 marks]

Answer 2

$$\begin{aligned}
z^2 - 4z + 29 &= 0 \\
\left[z^2 - 4z \right] + 29 &= 0 \\
\left[(z - 2)^2 - 4 \right] + 29 &= 0 \\
(z - 2)^2 + 25 &= 0 \\
(z - 2)^2 &= -25 \\
z - 2 &= \pm\sqrt{(-1)(25)} \\
z &= 2 \pm 5\mathrm{i}
\end{aligned}$$

[3 marks]

Answer 3

$$z^2 + 8z + 21 = 0$$
$$\left[z^2 + 8z \right] + 21 = 0$$
$$\left[(z + 4)^2 - 16 \right] + 21 = 0$$
$$(z + 4)^2 + 5 = 0$$
$$(z + 4)^2 = -5$$
$$z + 4 = \pm\sqrt{(-1)(5)}$$
$$z = -4 \pm \sqrt{5}\,\mathrm{i}$$

[3 marks]

Answer 4

$$z^2 + 5z + 25 = 0$$
$$\begin{aligned} z &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-5 \pm \sqrt{5^2 - 4 \times 1 \times 25}}{2 \times 1} \\ &= \frac{-5 \pm \sqrt{25 - 100}}{2} \\ &= \frac{-5}{2} \pm \frac{\sqrt{-75}}{2} \\ &= -\frac{5}{2} \pm \frac{\sqrt{-1}\sqrt{75}}{2} \\ &= -\frac{5}{2} \pm \frac{5\sqrt{3}}{2}\mathrm{i} \end{aligned}$$

[3 marks]

Answer 5

$$\begin{aligned} \text{LHS} &= z^2 - 6z + 10 \\ &= (3 + \mathrm{i})^2 - 6(3 + \mathrm{i}) + 10 \\ &= (3 + \mathrm{i})(3 + \mathrm{i}) - 18 - 6\mathrm{i} + 10 \\ &= 9 + 3\mathrm{i} + 3\mathrm{i} + \mathrm{i}^2 - 18 - 6\mathrm{i} + 10 \\ &= 9 + 3\mathrm{i} + 3\mathrm{i} - 1 - 18 - 6\mathrm{i} + 10 \\ &= 0 \\ &= \text{RHS} \qquad \square \end{aligned}$$

[3 marks]

Answer 6

$$\begin{aligned}
\text{LHS} &= f(z) \\
&= z^2 - 2z + 17 \\
&= (1 - 4\mathrm{i})^2 - 2(1 - 4\mathrm{i}) + 17 \\
&= (1 - 4\mathrm{i})(1 - 4\mathrm{i}) - 2 + 8\mathrm{i} + 17 \\
&= 1 - 4\mathrm{i} - 4\mathrm{i} + 16\,\mathrm{i}^2 - 2 + 8\mathrm{i} + 17 \\
&= 1 - 4\mathrm{i} - 4\mathrm{i} - 16 - 2 + 8\mathrm{i} + 17 \\
&= 0 \\
&= \text{RHS} \qquad \square
\end{aligned}$$

[3 marks]

Answer 7

(i)

$$\begin{aligned}
\text{LHS} &= z^4 + 4 \\
&= (1 + \mathrm{i})^4 + 4 \\
&= (1 + \mathrm{i})^2\ (1 + \mathrm{i})^2 + 4 \\
&= \left(1 + 2\mathrm{i} + \mathrm{i}^2\right)\left(1 + 2\mathrm{i} + \mathrm{i}^2\right) + 4 \\
&= \left(1 + 2\mathrm{i} - 1\right)\left(1 + 2\mathrm{i} - 1\right) + 4 \\
&= (2\mathrm{i})(2\mathrm{i}) + 4 \\
&= 4\,\mathrm{i}^2 + 4 \\
&= 4(-1) + 4 \\
&= -4 + 4 \\
&= 0 \\
&= \text{RHS} \qquad \square
\end{aligned}$$

[4 marks]

(ii) Other three roots are,

$$1 - \mathrm{i}, \qquad -1 + \mathrm{i}, \qquad -1 - \mathrm{i}$$

Whichever is guessed it then needs verification as in part (i)

[4 marks]

Answer 8

$$\begin{aligned}
(a + b\mathrm{i})\ (a - b\mathrm{i}) &= a^2 - ab\mathrm{i} + ab\mathrm{i} - b^2\,\mathrm{i}^2 \\
&= a^2 - b^2(-1) \\
&= a^2 + b^2 \text{ which is a real number} \qquad \square
\end{aligned}$$

[3 marks]

Answer 9

Individual choice

[7 marks]

Answer 10

$$7z^2 - 3z + 3 = 0$$

$$\left[\, 7z^2 - 3z \,\right] + 3 = 0$$

$$7\left[\, z^2 - \frac{3}{7}z \,\right] + 3 = 0$$

$$7\left[\, \left(z - \frac{3}{14}\right)^2 - \frac{9}{196} \,\right] + 3 = 0$$

$$7\left(z - \frac{3}{14}\right)^2 - \frac{9}{28} + 3 \times \frac{28}{28} = 0$$

$$7\left(z - \frac{3}{14}\right)^2 - \frac{9}{28} + \frac{84}{28} = 0$$

$$7\left(z - \frac{3}{14}\right)^2 = -\frac{75}{28}$$

$$\left(z - \frac{3}{14}\right)^2 = -\frac{75}{196}$$

$$\left(z - \frac{3}{14}\right) = \pm\frac{\sqrt{-75}}{14}$$

$$z = \frac{3}{14} \pm \frac{\sqrt{-1 \times 3 \times 25}}{14}$$

$$= \frac{3}{14} \pm \frac{5\sqrt{3}}{14}\,\mathrm{i}$$

[4 marks]

2.1 Complex Number Division

A complex number, z, is a number of the form,

$$z = a + b\mathrm{i}$$

where a and b are real numbers, $\mathrm{i} = \sqrt{-1}$ and $\mathrm{i}^2 = -1$

Given two complex numbers,

$$z = a + b\mathrm{i} \qquad \text{and} \qquad w = c + d\mathrm{i}$$

the arithmetic operations of addition, subtraction and multiplication can be performed with the two numbers and the result is in the form of a complex number.

Addition :

$$z + w = (a + b\mathrm{i}) + (c + d\mathrm{i})$$
$$= (a + c) + (b + d)\mathrm{i}$$

As a, b, c and d are all real numbers, $(a + c)$ and $(b + d)$ will also be real numbers which we could call u and v.

$\therefore\ z + w = u + v\mathrm{i}$ which is in the form of a complex number

Subtraction :

$$z - w = (a + b\mathrm{i}) - (c + d\mathrm{i})$$
$$= (a - c) + (b - d)\mathrm{i}$$

As a, b, c and d are all real numbers, $(a - c)$ and $(b - d)$ will also be real numbers which we could call u and v.

$\therefore\ z - w = u + v\mathrm{i}$ which is in the form of a complex number

Multiplication :

$$zw = (a + b\mathrm{i})(c + d\mathrm{i})$$
$$= ac + ad\mathrm{i} + bc\mathrm{i} + bd\,\mathrm{i}^2$$
$$= ac + ad\mathrm{i} + bc\mathrm{i} + bd\,(-1)$$
$$= (ac - bd) + (ad + bc)\mathrm{i}$$

As a, b, c and d are all real numbers, $(ac - bd)$ and $(ad + bc)$ will also be real numbers which we could call u and v.

$\therefore\ zw = u + v\mathrm{i}$ which is in the form of a complex number

This all leads to an obvious question; "What about division ?".

Can any complex number, divided by any other (non zero) complex number be manipulated into the form of a complex number ?

Teaching Video : http://www.NumberWonder.co.uk/Video/v9085(2).mp4

Division : $\dfrac{z}{w} = \dfrac{a + b\mathrm{i}}{c + d\mathrm{i}}$

[3 marks]

2.2 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 45

Question 1

Three complex numbers are,

$$u = 5 - 3\text{i} \qquad v = 4 + 2\text{i} \quad \text{and} \quad w = \frac{u}{v}$$

(i) Write down v^* the complex conjugate of v.

[1 mark]

(ii) Show how to use v^* to write w in the form $a + b\text{i}$ where $a, b \in \mathbb{Q}$

[3 marks]

Question 2

Given that $u = 3 + \text{i}$ write $\dfrac{u}{u^*}$ in the form $a + b\text{i}$ where $a, b \in \mathbb{Q}$

[3 marks]

Question 3

Given that,

$$z_1 = 1 + \mathrm{i} \qquad z_2 = 2 + \mathrm{i} \quad \text{and} \quad z_3 = 3 + \mathrm{i}$$

write each of the following in the form $a + b\mathrm{i}$ where $a, b \in \mathbb{Q}$

(i) $\dfrac{z_1 z_2}{z_3}$

[3 marks]

(ii) $\dfrac{(z_2)^2}{z_1}$

[3 marks]

(iii) $\dfrac{2z_1 + 5z_3}{z_2}$

[3 marks]

Question 4

Simplify, $\dfrac{2 + \mathrm{i}}{2 - \mathrm{i}} - \dfrac{2 - \mathrm{i}}{2 + \mathrm{i}}$

[4 marks]

Question 5

Given that

$$w = \sqrt{3} + \sqrt{2}\,\mathrm{i}$$

write w^{-1} in the form $a + b\mathrm{i}$ where $a, b \in \mathbb{R}$

[4 marks]

Question 6

Simplify, $\dfrac{1}{i} + \dfrac{1}{1 + i} + \dfrac{1}{1 - i}$

[4 marks]

Question 7

By using the binomial expansion, or otherwise, show that,

$$(1 + 2i)^5 = 41 - 38i$$

[4 marks]

Question 8

The complex number z is defined by,

$$z = \frac{3 + q\mathrm{i}}{q - 5\mathrm{i}} \qquad \text{where } q \in \mathbb{R}$$

Given that the real part of z is $\frac{1}{13}$,

(i) find the possible values of q

[5 marks]

(ii) write the possible values of z in the form $a + b\mathrm{i}$ where a and b are real constants.

[3 marks]

Question 9

$$z = 4 - \mathrm{i}\sqrt{2}$$

Use algebra to express $\dfrac{z + 4}{z - 3}$ in the form $p + q\mathrm{i}\sqrt{2}$

where p and q are rational numbers.

[5 marks]

2.3 Answers

2.3.1 Solutions to 2.1 Complex Number Division

Division :

$$\frac{z}{w} = \frac{a + b\mathrm{i}}{c + d\mathrm{i}} \times \frac{c - d\mathrm{i}}{c - d\mathrm{i}}$$

$$= \frac{ac - ad\mathrm{i} + bc\mathrm{i} - bd\,\mathrm{i}^2}{c^2 - cd\mathrm{i} + cd\mathrm{i} - d\,\mathrm{i}^2}$$

$$= \frac{(\,ac + bd\,) + (bc - ad\,)\mathrm{i}}{c^2 + d^2}$$

$$= \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}\,\mathrm{i}$$

As a, b, c and d are all real numbers, each of $\dfrac{ac + bd}{c^2 + d^2}$ and $\dfrac{bc - ad}{c^2 + d^2}$ will will also real numbers which we could call u and v.

$$\therefore \quad \frac{z}{w} = u + v\mathrm{i}$$ which is in the form of a complex number $\square$

[3 marks]

2.3.2 Solution to 2.2 Exercise

Answer 1

(i)

$$v^* = 4 - 2\mathrm{i}$$

[1 mark]

(ii)

$$\begin{aligned} w &= \frac{u}{v} \\ &= \frac{5 - 3\mathrm{i}}{4 + 2\mathrm{i}} \times \frac{4 - 2\mathrm{i}}{4 - 2\mathrm{i}} \\ &= \frac{20 - 10\mathrm{i} - 12\mathrm{i} + 6\,\mathrm{i}^2}{16 - 8\mathrm{i} + 8\mathrm{i} - 4\,\mathrm{i}^2} \\ &= \frac{14 - 22\mathrm{i}}{20} \\ &= \frac{7}{10} - \frac{11}{10}\,\mathrm{i} \end{aligned}$$

[3 marks]

Answer 2

$$\begin{aligned} \frac{u}{u^*} &= \frac{3 + \mathrm{i}}{3 - \mathrm{i}} \times \frac{3 + \mathrm{i}}{3 + \mathrm{i}} \\ &= \frac{9 + 6\mathrm{i} + \mathrm{i}^2}{9 - 3\mathrm{i} + 3\mathrm{i} - \mathrm{i}^2} \\ &= \frac{8 + 6\mathrm{i}}{10} \\ &= \frac{4}{5} + \frac{3}{5}\,\mathrm{i} \end{aligned}$$

[4 marks]

Answer 3

(i)

$$\frac{z_1 z_2}{z_3} = \frac{(1 + i)(2 + i)}{(3 + i)}$$

$$= \frac{2 + 3i + i^2}{(3 + i)}$$

$$= \frac{(1 + 3i)}{(3 + i)} \times \frac{(3 - i)}{(3 - i)}$$

$$= \frac{3 - i + 9i - 3i^2}{9 - 3i + 3i - i^2}$$

$$= \frac{6 + 8i}{10} = \frac{3}{5} + \frac{4}{5}i$$

[3 marks]

(ii)

$$\frac{(z_2)^2}{z_1} = \frac{(2 + i)(2 + i)}{(1 + i)}$$

$$= \frac{4 + 4i + i^2}{1 + i}$$

$$= \frac{3 + 4i}{1 + i} \times \frac{1 - i}{1 - i}$$

$$= \frac{3 - 3i + 4i - 4i^2}{1 - i + i - i^2}$$

$$= \frac{7}{2} + \frac{1}{2}i$$

[3 marks]

(iii)

$$\frac{2z_1 + 5z_3}{z_2} = \frac{2(1 + i) + 5(3 + i)}{(2 + i)}$$

$$= \frac{2 + 2i + 15 + 5i}{2 + i}$$

$$= \frac{17 + 7i}{2 + i} \times \frac{2 - i}{2 - i}$$

$$= \frac{34 - 17i + 14i - 7i^2}{4 - 2i + 2i - i^2}$$

$$= \frac{41 - 3i}{5} = \frac{41}{5} - \frac{3}{5}i$$

[3 marks]

Answer 4

$$\frac{2+i}{2-i}-\frac{2-i}{2+i}=\frac{2+i}{2-i}\times\frac{2+i}{2+i}-\frac{2-i}{2+i}\times\frac{2-i}{2-i}$$

$$=\frac{4+2i+2i+i^2}{4+2i-2i-i^2}-\frac{4-2i-2i+i^2}{4+2i-2i-i^2}$$

$$=\frac{3+4i}{5}-\frac{3-4i}{5}$$

$$=\frac{8}{5}i$$

[4 marks]

Answer 5

$$w^{-1}=\frac{1}{w}$$

$$=\frac{1}{\sqrt{3}+\sqrt{2}\,i}\times\frac{\sqrt{3}-\sqrt{2}\,i}{\sqrt{3}-\sqrt{2}\,i}$$

$$=\frac{\sqrt{3}-\sqrt{2}\,i}{3-\sqrt{6}\,i+\sqrt{6}\,i-2i^2}$$

$$=\frac{\sqrt{3}-\sqrt{2}\,i}{5}$$

$$=\frac{\sqrt{3}}{5}-\frac{\sqrt{2}}{5}i$$

[4 marks]

Answer 6

$$\frac{1}{i}+\frac{1}{1+i}+\frac{1}{1-i}$$

$$=\frac{1}{i}\times\frac{i}{i}+\frac{1}{1+i}\times\frac{1-i}{1-i}+\frac{1}{1-i}\times\frac{1+i}{1+i}$$

$$=\frac{i}{i^2}+\frac{1-i}{1-i+i-i^2}+\frac{1+i}{1+i-i-i^2}$$

$$=-i+\frac{1-i}{2}+\frac{1+i}{2}$$

$$=\frac{-2i+1-i+1+i}{2}$$

$$=\frac{2-2i}{2}$$

$$=1-i$$

[4 marks]

Answer 7

$$\text{LHS} = (1 + 2i)^5$$

$$\begin{aligned}
&= \ 1 \times (1)^5 \times (2i)^0 \\
&+ \ 5 \times (1)^4 \times (2i)^1 \\
&+ 10 \times (1)^3 \times (2i)^2 \\
&+ 10 \times (1)^2 \times (2i)^3 \\
&+ \ 5 \times (1)^1 \times (2i)^4 \\
&+ \ 1 \times (1)^0 \times (2i)^5
\end{aligned}$$

$$= 1 + 10i + 40\,i^2 + 80\,i^3 + 80\,i^4 + 32\,i^5$$

$$= 1 + 10i - 40 - 80i + 80 + 32i$$

$$= 41 - 38i$$

$$= \text{RHS} \qquad \square$$

[4 marks]

Answer 8

(i)

$$z = \frac{3 + qi}{q - 5i} \times \frac{q + 5i}{q + 5i}$$

$$= \frac{3q + 15i + q^2 i + 5q\, i^2}{q^2 + 5q\, i - 5q\, i - 25\, i^2}$$

$$= \frac{-2q + \left(15 + q^2\right) i}{25 + q^2}$$

$$= -\frac{2q}{25 + q^2} + \frac{15 + q^2}{25 + q^2}\, i$$

It was stated that the real part of this is $\frac{1}{13}$

$$\therefore \ -\frac{2q}{25 + q^2} = \frac{1}{13}$$

$$-26q = 25 + q^2$$

$$q^2 + 26q + 25 = 0$$

$$(q + 1)(q + 25) = 0$$

Either $(q + 1) = 0$ or $(q + 25) = 0$

Either $q = -1$ or $q = -25$

[5 marks]

(ii)

$$z = -\frac{2q}{25 + q^2} + \frac{15 + q^2}{25 + q^2}\, i$$

Either $z = \frac{1}{13} + \frac{15 + (-1)^2}{25 + (-1)^2}\, i$ or $z = \frac{1}{13} + \frac{15 + (-25)^2}{25 + (-25)^2}\, i$

Either $z = \frac{1}{13} + \frac{16}{26}\, i$ or $z = \frac{1}{13} + \frac{640}{650}\, i$

Either $z = \frac{1}{13} + \frac{8}{13}\, i$ or $z = \frac{1}{13} + \frac{64}{65}\, i$

[3 marks]

Answer 9

$$\frac{z+4}{z-3} = \frac{4 - i\sqrt{2} + 4}{4 - i\sqrt{2} - 3}$$

$$= \frac{8 - i\sqrt{2}}{1 - i\sqrt{2}} \times \frac{1 + i\sqrt{2}}{1 + i\sqrt{2}}$$

$$= \frac{8 + 8\sqrt{2}\,i - \sqrt{2}\,i - 2i^2}{1 - i\sqrt{2} + i\sqrt{2} - 2i^2}$$

$$= \frac{10 + 7\sqrt{2}\,i}{3}$$

$$= \frac{10}{3} + \frac{7\sqrt{2}}{3}\,i$$

[5 marks]

3.1 Roots and the Complex Plane

Lesson 1, Question 7 asked for a proof that $z = 1 + i$ is a solution to

$$z^4 + 4 = 0$$

It then asked that another solution to this quartic equation be guessed.

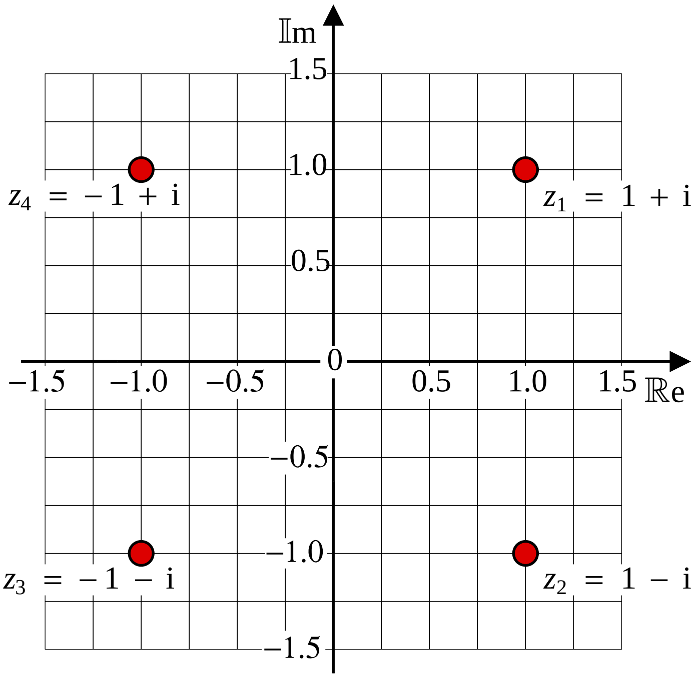
In fact this quartic equation (a polynomial of degree 4) has four complex roots,

$z_1 = 1 + i \qquad z_2 = 1 - i \qquad z_3 = -1 - i \qquad z_4 = -1 + i$

A two dimensional picture, called an Argand Diagram, can be drawn to show where the roots of this quartic equation lie on the complex plane.
Given the root of a polynomial, $z_{ROOT} = a + bi$ the real part, a, is plotted on the x-axis and the imaginary part, b, is plotted on the y-axis.
In fact, we stop using the terms x-axis and y-axis and talk instead of the *Real*-axis, $\mathbb{R}$e, and the *Imaginary*-axis, $\mathbb{I}$m. The two dimensional surface in which the real and imaginary axes reside is termed the complex plane, $\mathbb{C}$.

An Argand Diagram of the Complex Plane
showing the four complex roots of the equation

$$z^4 + 4 = 0$$



One method of verifying that each of these is a roots is valid would be to substitute them, one at a time into the quartic and show that they make it true, as in Lesson 1.

3.2 The Fundamental Theorem of Algebra

The Fundamental Theorem of Algebra states that any polynomial in z of degree n has exactly n roots, where some or all of the roots may be complex numbers. The polynomial can even have complex coefficients.

This theorem tells us that, for example, $z^4 + 4 = 0$, which is a polynomial in z of degree 4, has four roots. So once four roots have been found, there is no need to look for any more.

3.3 Unique Factorisation

The Fundamental Theorem of Algebra ties in with fact that for any polynomial in z of degree one or more unique factorisation holds.

When working with the integers, $\mathbb{Z}$, The Fundamental Theorem of Arithmetic tells us that every number that in not prime can be written as a product of primes in, essentially, only one way.
For example,

$$120 = 2^3 \times 3 \times 5$$

The “only one way” is formally referred to as “unique factorisation”.

Previously, polynomials in x with integer coefficients have been studied and these also have unique factorisation.
For example,

$$x^3 + 4x^2 + 4x + 3 = \left(x^2 + x + 1\right)\left(x + 3\right)$$

The factor theorem is of use in trying to find the unique factorisation of such polynomials. In fact, the unique factorisation holds, not just for integer coefficients, but also for real number coefficients and, indeed for complex coefficients with one restriction. The restriction is that the degree of the polynomial must be one or more. To understand the restriction consider the polynomial in z of degree 0,

$$p(z) = 6z^0$$

This does not have a unique factorisation.

For example, here is one factorisation of 6,

$$6 = (1 + \mathrm{i})(1 - \mathrm{i}) \times 3$$

and another,

$$6 = \left(1 + \sqrt{5}\,\mathrm{i}\right)\left(1 - \sqrt{5}\,\mathrm{i}\right)$$

and another,

$$6 = 2\left(\sqrt{2} + \mathrm{i}\right)\left(\sqrt{2} - \mathrm{i}\right)$$

This single counter example, is enough to show that unique factorisation does not hold for complex polynomials in z of degree 0.

3.4 Solving Polynomial Equations

Complex Conjugate Pair
If $p(z)$ is a polynomial with real coefficients and z_1 is a complex root of $p(z) = 0$ then z_1^* is also a root of $p(z) = 0$.
Such roots are termed a complex conjugate pair.

Further A-Level Examination Question from January 2009, FP1, Q2

$$f(z) = 2z^4 - 14z^3 + 33z^2 - 26z + 10$$

Given $z = 3 + \mathrm{i}$ is a solution of the equation $f(z) = 0$, solve $f(z) = 0$ completely.

Teaching Video : http://www.NumberWonder.co.uk/Video/v9085(3).mp4

[5 marks]

3.5 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 57

Question 1

Given that $z = 3$ is the real solution of the equation,

$$z^3 - 5z^2 + 11z - 15 = 0$$

find the two complex solutions of this equation.

[5 marks]

Question 2

$$f(z) = z^3 - 6z^2 + 21z - 26$$

(i) Show that $f(2) = 0$

[1 mark]

(ii) Hence solve $f(z) = 0$ completely.

[4 marks]

Question 3

Further A-Level Examination Question from June 2014, FR1 (R), Q1

The roots of the equation,

$$2z^3 - 3z^2 + 8z + 5 = 0$$

are z_1, z_2 and z_3.

Given that $z_1 = 1 + 2i$, find z_2 and z_3

[5 marks]

Question 4

Further A-Level Examination Question from June 2009, FP1, Q1

Given that $x = \mathrm{i}$ is a complex solution of the equation,

$$x^4 + 6x^3 + 26x^2 + 6x + 25 = 0$$

(a) express $x^4 + 6x^3 + 26x^2 + 6x + 25$ as a product of two quadratic factors

[4 marks]

(b) Hence solve completely the equation $x^4 + 6x^3 + 26x^2 + 6x + 25 = 0$

[2 marks]

Question 5

Further A-Level Examination Question from June 2018, F1, Q5

Given that

$$z^4 - 6z^3 + 34z^2 - 54z + 225 = \left(z^2 + 9\right)\left(z^2 + az + b\right)$$

where a and b are real numbers,

(a) find the value of a and the value of b

[2 marks]

(b) Hence find the exact roots of the equation

$$z^4 - 6z^3 + 34z^2 - 54z + 225 = 0$$

[4 marks]

(c) Show your roots on a single Argand diagram

[2 marks]

Question 6

Further A-Level Examination Question from June 2013, FP1, Q3

Given that $x = \frac{1}{2}$ is a root of the equation,

$$2x^3 - 9x^2 + kx - 13 = 0 \qquad k \in \mathbb{Z}$$

find

(a) the value of k

[3 marks]

(b) the other two roots of the equation

[4 marks]

Question 7

Further A-Level Examination Question from January 2010, FP1, Q6

Given that 2 and 5 + 2i are roots of the equation

$$z^3 - 12z^2 + cz + d = 0, \qquad c, d \in \mathbb{R}$$

(a) write down the other complex root of the equation

[1 mark]

(b) Find the value of c and the value of d

[5 marks]

(c) Show the three roots of this equation on a single Argand diagram.

[2 marks]

Question 8

Further A-Level Examination Question from June 2006, FP1, Q4

Given that $3 - 2\mathrm{i}$ is a solution of the equation

$$x^4 - 6x^3 + 19x^2 - 36x + 78 = 0$$

(a) solve the equation completely

[7 marks]

(b) Show on a single Argand diagram the four points that represent the roots of the equation.

[2 marks]

Question 9
Find the four roots of the equation

$$z^4 - 16 = 0$$

[4 marks]

3.6 Answers

3.6.1 Solution to 3.4 Example

As $f(z)$ has real coefficients, two solutions are $z = 3 \pm \mathrm{i}$

$\therefore$ Two factors are,

$$\begin{aligned}
&(z - 3 - \mathrm{i})(z - 3 + \mathrm{i}) \\
&= ((z - 3) - \mathrm{i})((z - 3) + \mathrm{i}) \\
&= (z - 3)^2 - \mathrm{i}^2 \\
&= z^2 - 6z + 9 + 1 \\
&= z^2 - 6z + 10
\end{aligned}$$

$$\begin{array}{r|l}
 & \phantom{2z^4 - 14z^3 + {}} 2z^2 - 2z + 1 \\
\hline
z^2 - 6z + 10 & 2z^4 - 14z^3 + 33z^2 - 26z + 10 \\
 & 2z^4 - 12z^3 + 20z^2 \\
\cline{2-2}
 & - 2z^3 + 13z^2 - 26z \\
 & - 2z^3 + 12z^2 - 20z \\
\cline{2-2}
 & \phantom{2z^4 - 2z^3 + {}} z^2 - 6z + 10 \\
 & \phantom{2z^4 - 2z^3 + {}} z^2 - 6z + 10 \\
\cline{2-2}
 & 0
\end{array}$$

$$\begin{gathered}
2z^4 - 14z^3 + 33z^2 - 26z + 10 = 0 \\
\left(z^2 - 6z + 10\right)\left(2z^2 - 2z + 1\right) = 0 \\
\text{Either } z^2 - 6z + 10 = 0 \text{ or } 2z^2 - 2z + 1 = 0 \\
z = 3 \pm \mathrm{i} \text{ or } z = \frac{2 \pm \sqrt{(-2)^2 - 4 \times 2 \times 1}}{2 \times 2} \\
z = \frac{2 \pm \sqrt{-4}}{4} \\
z = \frac{1}{2} \pm \frac{1}{2}\mathrm{i}
\end{gathered}$$

[5 marks]

3.6.2 Solutions to 3.5 Exercise

Answer 1

As $z = 3$ is a root, $(z - 3)$ must be a factor.
Using the technique of Polynomial Division from the Year 1 course,

$$\begin{array}{r|rrrrrrr}
\multicolumn{1}{r}{} & & & z^2 & - & 2z & + & 5 \\
\cline{2-8}
z - 3 & z^3 & - & 5z^2 & + & 11z & - & 15 \\
 & z^3 & - & 3z^2 & & & & \\
\cline{2-4}
 & & - & 2z^2 & + & 11z & & \\
 & & - & 2z^2 & + & 6z & & \\
\cline{3-5}
 & & & & & 5z & - & 15 \\
 & & & & & 5z & - & 15 \\
\cline{5-8}
 & & & & & & & 0 \leftarrow \text{ as expected}
\end{array}$$

So, we can now factorise the cubic into one linear and one quadratic factor,

$$z^3 - 5z^2 + 11z - 15 = 0$$

$$(z - 3)(z^2 - 2z + 5) = 0$$

$$\text{Either} \quad z - 3 = 0 \quad \text{or} \quad z^2 - 2z + 5 = 0$$

We already know about the root from the linear factor so the task now is to get the two complex roots from the quadratic factor. Your calculator will let you check the answers but, of course, you need to show how to get them.

I'm going to use the well known formula for solving quadratic equations.

$$z^2 - 2z + 5 = 0$$

$$\begin{aligned}
z &= \frac{2 \pm \sqrt{(-2)^2 - 4 \times 1 \times 5}}{2 \times 1} \\
&= \frac{2 \pm \sqrt{-16}}{2} \\
&= \frac{2 \pm \sqrt{16 \times (-1)}}{2} \\
&= \frac{2 \pm 4\text{i}}{2} \\
&= 1 \pm 2\text{i} \qquad \text{are the two complex roots of the equation}
\end{aligned}$$

[5 marks]

Answer 2

(i)

$$
\begin{aligned}
f(z) &= z^3 - 6z^2 + 21z - 26 \\
f(2) &= 2^3 - 6 \times 2^2 + 21 \times 2 - 26 \\
&= 8 - 24 + 42 - 26 \\
&= 0 \qquad \square
\end{aligned}
$$

[1 mark]

(ii) As $z = 2$ is a root, $(z - 2)$ must be a factor.

$$
\begin{array}{r|lllll}
 & & 4z^2 & -\ 4z & +\ 13 & \\
\cline{2-5}
z - 2 & z^3 & -\ 6z^2 & +\ 21z & -\ 26 & \\
 & z^3 & -\ 2z^2 & & & \\
\cline{2-3}
 & & -\ 4z^2 & +\ 21z & & \\
 & & -\ 4z^2 & -\ 8z & & \\
\cline{3-4}
 & & & 13z & -\ 26 & \\
 & & & 13z & -\ 26 & \\
\cline{4-5}
 & & & & 0 & \leftarrow \text{ as expected}
\end{array}
$$

So, we can now factorise the cubic into one linear and one quadratic factor,

$$z^3 - 5z^2 + 11z - 15 = 0$$

$$(z - 2)(z^2 - 4z + 13) = 0$$

Either $\quad z - 2 = 0 \quad$ or $\quad z^2 - 4z + 13 = 0$

We already know about the root from the linear factor so the task now is to get the two complex roots from the quadratic factor.

Using the formula for solving quadratic equations.

$$
\begin{aligned}
z^2 - 4z + 13 &= 0 \\
z &= \frac{4 \pm \sqrt{(-4)^2 - 4 \times 1 \times 13}}{2 \times 1} \\
&= \frac{4 \pm \sqrt{-36}}{2} \\
&= \frac{4 \pm \sqrt{36 \times (-1)}}{2} \\
&= \frac{4 \pm 6\mathrm{i}}{2} \\
&= 2 \pm 3\mathrm{i} \qquad \text{are the two complex roots of the equation}
\end{aligned}
$$

[4 marks]

Answer 3

As $z_1 = 1 + 2\mathrm{i}$ is a root, and the polynomial has integer (and $\therefore$ real) coefficients, the roots must occur in conjugate pairs and so $z_2 = 1 - 2\mathrm{i}$ must also be a root.

Finding the product of the two associated factors...

$$\begin{aligned}(z - 1 - 2\mathrm{i})\,(z - 1 + 2\mathrm{i}) &= (z - 1)^2 - 4\,\mathrm{i}^2 \\ &= z^2 - 2z + 1 + 4 \\ &= z^2 - 2z + 5\end{aligned}$$

So we now know that,

$$\left(z^2 - 2z + 5\right)\left(Az + B\right) = 2z^3 - 3z^2 + 8z + 5$$

$$\therefore \quad Az + B = 2z + 1$$

$$2z^3 - 3z^2 + 8z + 5 = 0$$

$$\left(z^2 - 2z + 5\right)\left(2z + 1\right) = 0$$

Thus : $\quad z_1 = 1 + 2\mathrm{i} \qquad z_2 = 1 - 2\mathrm{i} \qquad$ and $\qquad z_3 = -\dfrac{1}{2}$

[5 marks]

Answer 4

(a) As $x_1 = \mathrm{i}$ is a root, and the polynomial has integer (and $\therefore$ real) coefficients, the roots must occur in conjugate pairs and so $x_2 = -\,\mathrm{i}$ must also be a root.

Finding the product of the two associated factors...

$$\begin{aligned}(x - \mathrm{i})\,(x + \mathrm{i}) &= x^2 - \mathrm{i}^2 \\ &= x^2 + 1\end{aligned}$$

So we now know that,

$$\begin{aligned}x^4 + 6x^3 + 26x^2 + 6x + 25 &= \left(x^2 + 1\right)\left(Ax^2 + Bx + C\right) \\ &= \left(x^2 + 1\right)\left(x^2 + 6x + 25\right)\end{aligned}$$

~ Alternatively, a polynomial long division could be used ~

[4 marks]

(b)

$$\left(x^2 + 6x + 25\right) = 0$$

$$x = \frac{-6 \pm \sqrt{6^2 - 4 \times 1 \times 25}}{2 \times 1}$$

$$x = \frac{-6 \pm \sqrt{-1 \times 64}}{2}$$

$$x = -3 \pm 4\mathrm{i}$$

The four roots of the quartic are thus

$$x_1 = \mathrm{i} \qquad x_2 = -\,\mathrm{i} \qquad x_3 = -3 + 4\mathrm{i} \qquad \text{and} \qquad x_4 = -3 - 4\mathrm{i}$$

[2 marks]

Answer 5

(a) By inspection,

$$z^4 - 6z^3 + 34z^2 - 54z + 225 = \left(z^2 + 9\right)\left(z^2 + az + b\right)$$
$$= \left(z^2 + 9\right)\left(z^2 - 6z + 25\right)$$

~ Alternatively, a polynomial long division could be used ~

[2 marks]

(b)

$$z^4 - 6z^3 + 34z^2 - 54z + 225 = 0$$
$$\left(z^2 + 9\right)\left(z^2 - 6z + 25\right) = 0$$

Either $z^2 + 9 = 0$ or $z^2 - 6z + 25 = 0$

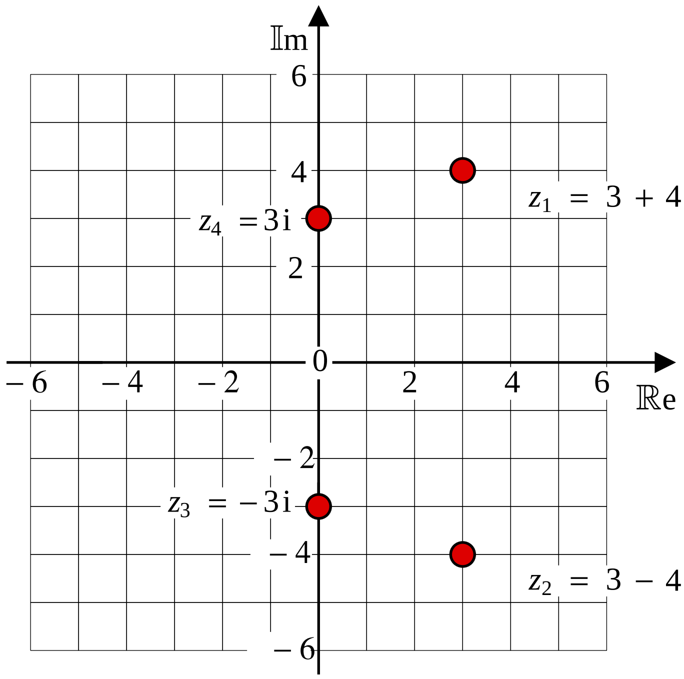
$$z^2 = -9 \qquad z = \frac{6 \pm \sqrt{(-6)^2 - 4 \times 1 \times 25}}{2 \times 1}$$

$$z = \pm\sqrt{-1 \times 9} \qquad = \frac{6 \pm \sqrt{-1 \times 64}}{2}$$

$$z = \pm 3\text{i} \qquad = 3 \pm 4\text{i}$$

[4 marks]

(c)



[2 marks]

Answer 6

(a)

$$2x^3 - 9x^2 + kx - 13 = 0 \qquad k \in \mathbb{Z}$$

As $x = \dfrac{1}{2}$ is a root,

$$2 \times \left(\frac{1}{2}\right)^3 - 9 \times \left(\frac{1}{2}\right)^2 + k \times \left(\frac{1}{2}\right) - 13 = 0$$

$$\frac{1}{4} - \frac{9}{4} + \frac{k}{2} - 13 = 0$$

$$1 - 9 + 2k - 52 = 0$$

$$2k = 60$$

$$k = 30$$

[3 marks]

(b) As $x = \dfrac{1}{2}$ is a root, $(2x - 1)$ must be a factor

$$\begin{array}{r|l}
 & \quad\;\; x^2 - 4x + 13 \\
\cline{2-2}
2x - 1 & 2x^3 - 9x^2 + 30x - 13 \\
 & \underline{x^3 - x^2} \\
 & \quad - 8x^2 + 30x \\
 & \quad \underline{- 8x^2 + 4x} \\
 & \qquad\quad 26x - 13 \\
 & \qquad\quad \underline{26z - 13} \\
 & \qquad\qquad\quad 0 \leftarrow \text{as expected}
\end{array}$$

Thus,

$$2x^3 - 9x^2 + 30x - 13 = 0$$

$$(2x - 1)(x^2 - 4x + 13) = 0$$

Either $2x - 1 = 0$ or $x^2 - 4x + 13 = 0$

$$2x = 1 \qquad\qquad x = \frac{4 \pm \sqrt{(-4)^2 - 4 \times 1 \times 13}}{2 \times 1}$$

$$x = \frac{1}{2} \qquad\qquad = \frac{4 \pm \sqrt{-1 \times 36}}{2}$$

$$= 2 \pm 3i$$

[4 marks]

Answer 7

(a) As the cubic has real number coefficients, and complex roots will occur as a conjugate pair.

So the "third root" must be $5 - 2\text{i}$

[1 mark]

(b) Written as factors, because we know the roots, the cubic is,

$$(z - 2)\,(z - 5 - 2\text{i})\,(z - 5 + 2\text{i})$$

$$= \left(z - 2\right)\left(\left(z - 5\right)^2 - (2\text{i})^2\right)$$

$$= \left(z - 2\right)\left(z^2 - 10z + 25 - 4\text{i}^2\right)$$

$$= \left(z - 2\right)\left(z^2 - 10z + 29\right)$$

Multiplication grid :

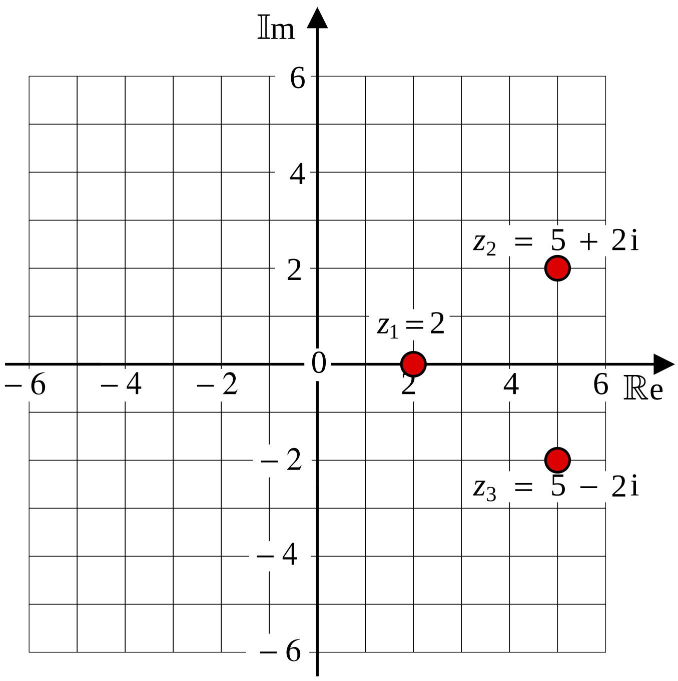
	z^2	$-10z$	29
z	z^3	$-10z^2$	$29z$
-2	$-2z^2$	$20z$	-58

$$= z^3 - 12z + 49z - 58$$

$$\therefore\ c = 49 \quad \text{and} \quad d = -58$$

[5 marks]

(c)



[2 marks]

Answer 8

(a) As $x_1 = 3 - 2i$ is a root, and the polynomial has integer (and $\therefore$ real) coefficients, the roots must occur in conjugate pairs.
So $x_2 = 3 + 2i$ must also be a root.

Finding the product of the two associated factors...

$$\begin{aligned}(x - 3 - 2i)(x - 3 + 2i) &= (x - 3)^2 - 4i^2 \\ &= x^2 - 6x + 9 + 4 \\ &= x^2 - 6x + 13\end{aligned}$$

Now to use this quadratic factor to extract the other quadratic factor from the quartic equation,

$$\begin{array}{r|l} & x^2 + 0x + 6 \\ \hline x^2 - 6x + 13 & x^4 - 6x^3 + 19x^2 - 36x + 78 \\ & \underline{x^4 - 6x^3 + 13x^2} \\ & \qquad\qquad 6x^2 - 36x + 78 \\ & \qquad\qquad \underline{6x^2 - 36x + 78} \\ & \qquad\qquad\qquad\qquad 0 \leftarrow \text{As expected}\end{array}$$

Thus,

$$x^4 - 6x^3 + 19x^2 - 36x + 78 = 0$$

$$\left(x^2 - 6x + 13\right)\left(x^2 + 6\right) = 0$$

Either $x^2 - 6x + 13 = 0$ which yields x_1 and x_2

Or $x^2 + 6 = 0$

$$x^2 = -6$$

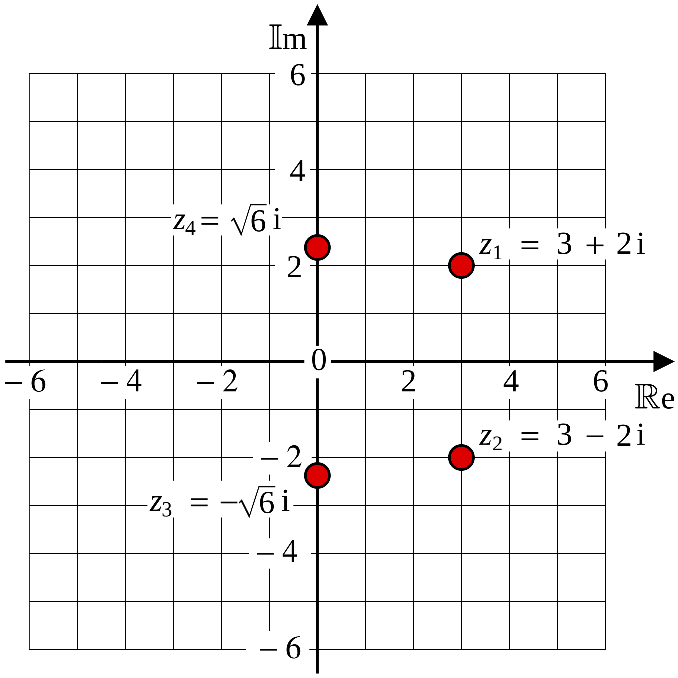
$$x = \pm\sqrt{6}\, i$$

So the solutions to the quartic are,

$x_1 = 3 - 2i \qquad x_2 = 3 + 2i \qquad x_3 = \sqrt{6}\, i \qquad$ and $\qquad x_4 = -\sqrt{6}\, i$

[7 marks]

(b)



[2 marks]

Answer 9

$$z^4 - 16 = 0$$

$$\left(z^2\right)^2 - 4^2 = 0$$

$$\left(z^2 - 4\right)\left(z^2 + 4\right) = 0$$

$$\left(z^2 - (2)^2\right)\left(z^2 - (2i)^2\right) = 0$$

$$(z - 2)(z + 2)(z - 2i)(z + 2i) = 0$$

Thus the roots are,

$z_1 = 2 \qquad z_2 = -2 \qquad z_3 = 2i \qquad$ and $\qquad z_4 = -2i$

[4 marks]

4.1 To Simplify, Complexify

Mathematicians' ongoing fascination with Polynomials stems from the fact that they are one of the most simple class of function. Many other more complicated functions can be approximated with a polynomial.

For example, using the Binomial Theorem,

$$\sqrt{4 + x} = 2 + \frac{x}{4} - \frac{x^2}{64} + \frac{x^3}{512} + \ldots \qquad x \in \mathbb{R}, \qquad |x| < 4$$

Polynomials over the reals are, of course, easy to differentiate and integrate. When over the real numbers they have a structure that is very like that of the prime numbers and, in particular, they have the property of unique factorisation. However, one unsatisfactory consequence of restricting polynomials to the real numbers is that the number of roots for a polynomial of any given degree varies. For example, consider the generalised polynomial of degree two, a quadratic;

$$a x^2 + bx + c = 0$$

There are,

- Two roots if the discriminant is positive
- One root (sometimes called a repeated root) is the discriminant is zero
- No roots at all if the discriminant is negative

In moving to complex numbers, the desirable property of unique factorisation is retained except for the trivial case of a polynomial of degree zero. One of the many benefits of working with polynomials over the complex numbers is that the number of roots for a polynomial of any given degree no longer varies. This is of such importance that it has the grand title, “The fundamental theorem of algebra”.

The Fundamental Theorem of Algebra

Any polynomial in z of degree n has exactly n roots, where some or all of the roots may be complex numbers.

The polynomial can even have complex coefficients.

An oft repeated mathematical quote is,

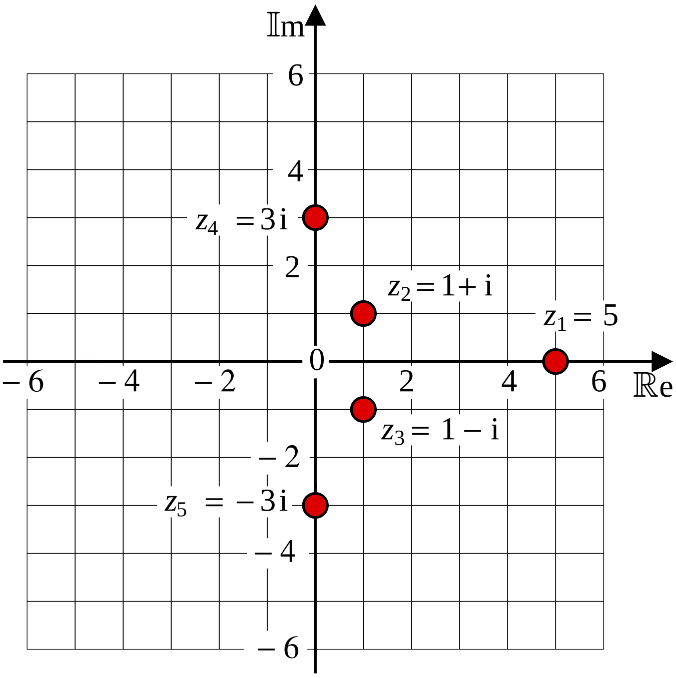
“to simplify, complexify”

which is saying that for many mathematical problems the easy way to tackle them is to consider them, not just over the real numbers, but in the world of the complex numbers. The “problems” referred to are not just in mathematics. Complex numbers have traditionally been used in physics and engineering. In electrical engineering, for example, circuits involving resistors, capacitors and inductors are far easier and more elegantly modelled using complex numbers. More recently, complex numbers have become a part of Biology with the realisation that nature can be effectively modelled using Fractal Geometry which has, at its core (you guessed it) Complex Numbers !

4.2 A Polynomial From Its Roots

In Lesson 3 the problem of finding the roots of a polynomial was tackled, often using a few clues, such as being given one root and asked to find the others. The factor theorem was useful along with the knowledge that, if the polynomial had only real coefficients, any complex roots came as a conjugate pair. How would this work in reverse ? In other words, given the roots of a polynomial, how might the polynomial they came from be derived ?

The roots of a mystery polynomial are shown below on an Argand diagram.
Find a polynomial that they could be the root of.



Teaching Video : http://www.NumberWonder.co.uk/Video/v9085(4).mp4

	z^3	$-5z^2$	$+9z$	-45
z^2				
$-2z$				
$+2$				

Multiplication Grid the for Teaching Video

[5 marks]

The final statement in the video is not precise : It should be that "This polynomial is the only polynomial with highest order term z^5 (note the coefficient of 1) that will have z_1, z_2, z_3, z_4 and z_5 as its only roots".

Question 1 (iii) in Exercise 4.3, will emphasise this subtlety.

4.3 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 40

Question 1

A quadratic has roots,

$$z_1 = 7 + 5\mathrm{i} \qquad \text{and} \qquad z_2 = 7 - 5\mathrm{i}$$

(i) Write down the two factors of the quadratic.

[1 mark]

(ii) Show how the factors may be multiplied together to give the quadratic,

$$z^2 - 14z + 74$$

[2 marks]

(iii) Explain why

$$2z^2 - 28z + 148 = 0$$

would have the same roots.

[2 marks]

Question 2

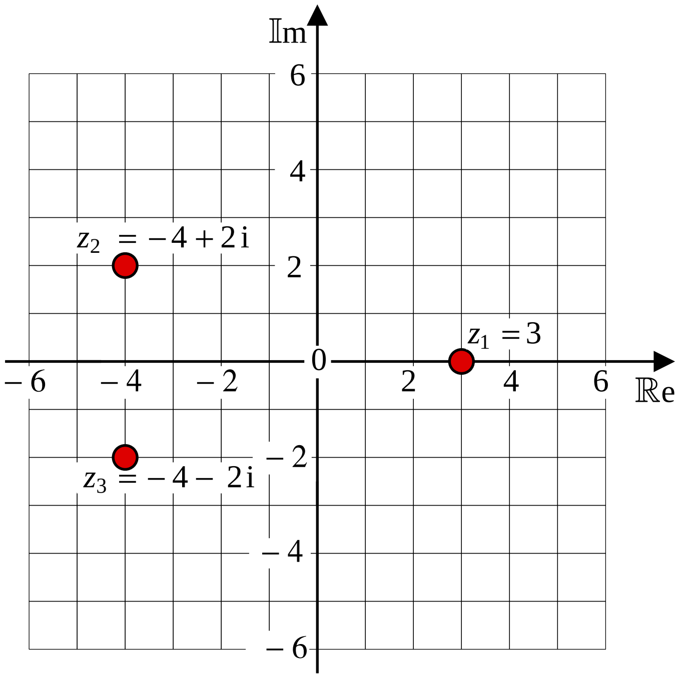
A polynomial, $p(z)$ has the four roots,

$$z_1 = 1 \qquad z_2 = \mathrm{i} \qquad z_3 = -1 \qquad \text{and} \qquad z_4 = -\mathrm{i}$$

Find a quartic polynomial that will have z_1, z_2, z_3 and z_4 as its only roots.

[4 marks]

Question 3



(i) Which two roots form a conjugate pair ?

[1 mark]

(ii) What is the mathematical theorem called that tells you that $p(z)$ will be a cubic ?

[1 mark]

(iii) Explain why $p(z)$ will have real coefficients.

[1 mark]

(iv) Find a cubic that will have z_1, z_2 and z_3 as its only roots.

[4 marks]

Question 4

Further A-Level Examination Question from June 2014, FP1, Q3

Given that 2 and $1 - 5\text{i}$ are roots of the equation

$$x^3 + px^2 + 30x + q = 0 \qquad p, q \in \mathbb{R}$$

(a) write down the third root of the equation

[1 mark]

(b) Find the value of p and the value of q

[5 marks]

(c) Show the three roots of this equation on a single Argand diagram

[2 marks]

Question 5

Further A-Level Examination Question from May 2016, FP1, Q4

$$z = \frac{4}{1 + \mathrm{i}}$$

Find in the form $a + b\mathrm{i}$, where $a, b \in \mathbb{R}$

(a) z

[2 marks]

(b) z^2

[2 marks]

Given that z is a complex root of the quadratic equation,

$$x^2 + px + q = 0$$

where p and q are real integers,

(c) find the value of p and the value of q

[3 marks]

Question 6

Further A-Level Examination Question from June 2019, Core 1, Q1

$$f(z) = z^4 + az^3 + bz^2 + cz + d$$

where a, b, c and d are real constants.

Given that $-1 + 2\mathrm{i}$ and $3 - \mathrm{i}$ are two roots of the equation $f(z) = 0$

(a) show all the roots of $f(z) = 0$ on a single Argand diagram

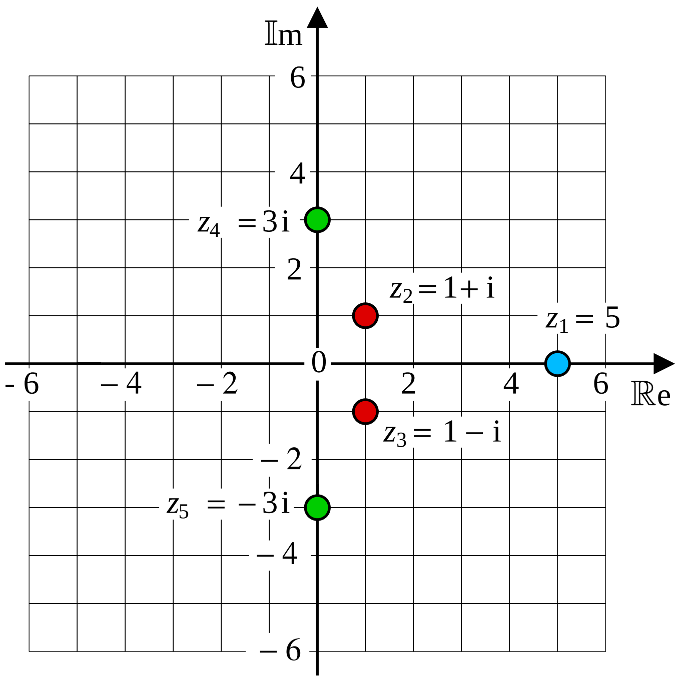
[4 marks]

(b) find the values of a, b, c, and d

[5 marks]

4.4 Answers

4.4.1 Solution to 4.2 A Polynomial From Its Roots



The roots on the Argand diagram have been coloured to show that z_1 is a real root (blue), z_2 and z_3 are a conjugate pair (red), and z_4 and z_5 are a second conjugate pair (green).

Writing out the factors in order of difficulty, and grouped in conjugate pairs where possible,

$$(z - 5)(z - 3\mathrm{i})(z + 3\mathrm{i})(z - 1 - \mathrm{i})(z - 1 + \mathrm{i})$$

$$= \left(z - 5\right)\left(z^2 + 9\right)\left((z - 1)^2 - \mathrm{i}^2\right)$$

$$= \left(z^3 - 5z^2 + 9z - 45\right)\left(z^2 - 2z + 2\right)$$

	z^3	$-5z^2$	$+9z$	-45
z^2	z^5	$-5z^4$	$9z^3$	$-45z^2$
$-2z$	$-2z^4$	$10z^3$	$-18z^2$	$90z$
$+2$	$2z^3$	$-10z^2$	$18z$	-90

$$= z^5 - 7z^4 + 21z^3 - 73z^2 + 108z - 90$$

Such answers can be checked online in Wolfram Alpha where typing this quintic in will result in the five roots being displayed.

[5 marks]

4.4.2 Solutions to 4.3 Exercise

Answer 1

(i)

$$(z - 7 - 5\mathrm{i})\ (z - 7 + 5\mathrm{i})$$

[1 mark]

(ii)

$$\begin{aligned} & (z - 7 - 5\mathrm{i})\ (z - 7 + 5\mathrm{i}) \\ &= \left((z - 7)^2 - (5\mathrm{i})^2 \right) \\ &= \left(z^2 - 14z + 49 - 25\,\mathrm{i}^2 \right) \\ &= z^2 - 14z + 74 \qquad \square \end{aligned}$$

[2 marks]

(iii)

$$2\,z^2 - 28z + 148 = 0$$

$$2\left(z^2 - 14z + 74 \right) = 0$$

Either 2 = 0 which it clearly does not,

or $z^2 - 14z + 74 = 0$ which we already know has roots $7 \pm 5\mathrm{i}$

$\therefore\ \ 2\,z^2 - 28z + 148 = 0$ has the same roots as $z^2 - 14z + 74 = 0$ $\quad\square$

[2 marks]

Answer 2

From the roots,

$$z_1 = 1 \qquad z_2 = \mathrm{i} \qquad z_3 = -1 \qquad \text{and} \qquad z_4 = -\mathrm{i}$$

the factors can be written out directly, taking care to put the conjugate pair next to each other (not critical in this case),

$$\begin{aligned} & (z - 1)\ (z + 1)\ (z - \mathrm{i})\ (z + \mathrm{i}) \\ &= \left(z^2 - 1 \right) \left(z^2 - \mathrm{i}^2 \right) \\ &= \left(z^2 - 1 \right) \left(z^2 + 1 \right) \\ &= z^4 - 1 \end{aligned}$$

[4 marks]

Answer 3

(i) The conjugate pair are z_2 and z_3

(ii) The Fundamental Theorem of Algebra.

(iii) The two complex roots are a conjugate pair.

(iv) From the roots,

$$z_1 = 3 \qquad z_2 = -4 + 2\text{i} \qquad \text{and} \qquad z_3 = -4 - 2\text{i}$$

the factors can be written out directly, taking care to put the conjugate pair next to each other,

$$(z - 3)\,(z + 4 - 2\text{i})\,(z + 4 + 2\text{i})$$
$$= (z - 3)\left((z + 4)^2 - 4\,\text{i}^2\right)$$
$$= (z - 3)\left(z^2 + 8z + 20\right)$$

Multiplication grid :

	z^2	$8z$	20
z	z^3	$8z^2$	$20z$
-3	$-3z^2$	$-24z$	-60

$$= z^3 + 5z^2 - 4z - 60$$

[1, 1, 1, 4 marks]

Answer 4

(a)

$$z_3 = 1 + 5\text{i}$$

[1 mark]

(b) From the roots,

$$z_1 = 2 \qquad z_2 = 1 - 5\text{i} \qquad \text{and} \qquad z_3 = 1 + 5\text{i}$$

the factors can be written out directly, taking care to put the conjugate pair next to each other,

$$(z - 2)\,(z - 1 + 5\text{i})\,(z - 1 - 5\text{i})$$
$$= (z - 2)\left((z - 1)^2 - 25\,\text{i}^2\right)$$
$$= (z - 2)\left(z^2 - 2z + 26\right)$$

Multiplication grid :

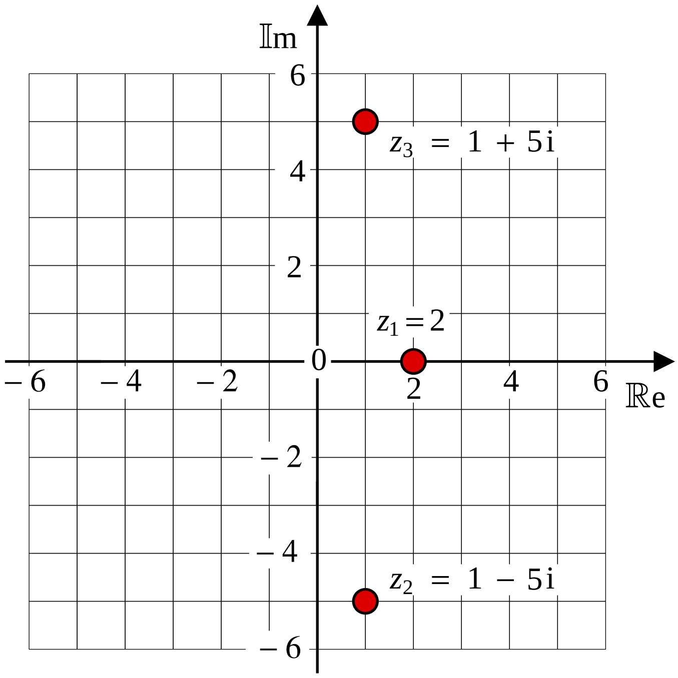
	z^2	$-2z$	26
z	z^3	$-2z^2$	$26z$
-2	$-2z^2$	$4z$	-52

$$= z^3 - 4z^2 + 30z - 52$$

$$\therefore\ p = -4 \qquad \text{and} \qquad q = -52$$

[5 marks]

(c)



[2 marks]

Answer 5

(a)

$$z = \frac{4}{1 + i} \times \frac{1 - i}{1 - i}$$

$$= \frac{4\,(1 - i)}{2}$$

$$= 2 - 2i$$

[2 marks]

(b)

$$z^2 = (2 - 2i)^2$$

$$= 4 - 8i + 4i^2$$

$$= -8i$$

[2 marks]

(c) From the roots,

$$z_1 = 2 - 2i \qquad \text{and} \qquad z_2 = 2 + 2i$$

the factors can be written out directly,

$$(z - 2 + 2i)(z - 2 - 2i)$$

$$= (z - 2)^2 - 4i^2$$

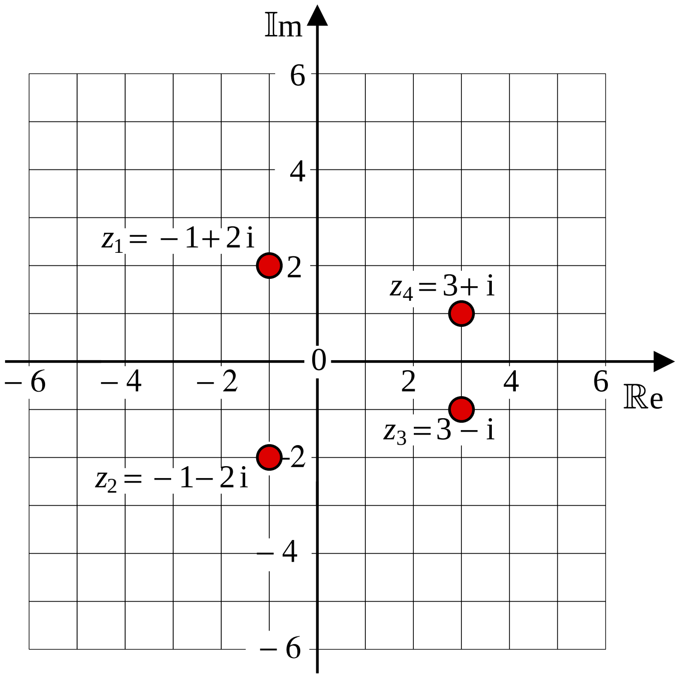
$$= z^2 - 4z + 8$$

$$\therefore\ p = -4 \qquad \text{and} \qquad q = 8$$

[3 marks]

Answer 6

(a)



[4 marks]

(b) From the roots,

$z_1 = -1 + 2i \qquad z_2 = -1 - 2i \qquad z_3 = 3 - i \qquad \text{and} \qquad z_4 = 3 + i$

the factors can be written out directly, taking care to group each conjugate pair,

$$(z + 1 - 2i)(z + 1 + 2i)(z - 3 + i)(z - 3 - i)$$

$$= \left((z + 1)^2 - (2i)^2 \right) \left((z - 3)^2 - i^2 \right)$$

$$= \left(z^2 + 2z + 1 - 4\,i^2 \right) \left(z^2 - 6z + 9 - i^2 \right)$$

$$= \left(z^2 + 2z + 5 \right) \left(z^2 - 6z + 10 \right)$$

Multiplication grid :

	z^2	$2z$	5
z^2	z^4	$2\,z^3$	$5\,z^2$
$-6z$	$-6\,z^3$	$-12\,z^2$	$-30z$
10	$10\,z^2$	$20z$	50

$$= z^4 - 4\,z^3 + 3\,z^2 - 10z + 50$$

Matching this up with, $f(z) = z^4 + a\,z^3 + b\,z^2 + cz + d$ gives us that

$$a = -4, \quad b = 3. \quad c = -10 \quad \text{and} \quad d = 50$$

[5 marks]

Lesson 5

5.1 Modulus

For a given a complex number, $z = a + b\text{i}$, the modulus is denoted $|z|$.
It is the distance from the origin to that number on an Argand diagram.
In other words,

$$|z| = \sqrt{a^2 + b^2}$$

5.2 The Principal Argument

For a given complex number, $z = a + b\text{i}$, the principal argument is denoted $Arg(z)$. It is the angle between the positive real axis and the line joining the point of interest to the origin on an Argand diagram. The calculation varies depending upon which of the four quadrants on the Argand diagram the point resides. This will be looked at shortly in this lesson's teaching video.

First, however, some scene setting.

As always in mathematics an anticlockwise rotation is considered to be positive. and a clockwise rotation negative. By convention, arguments are given in radians and, to make the function one-to-one, in the range,

$$-\pi < arg(z) \leqslant \pi$$

Thus, to rotate from the positive x-axis direction onto points in either the 1st or the 2nd quadrants the rotation is given as a positive (anticlockwise) rotation.

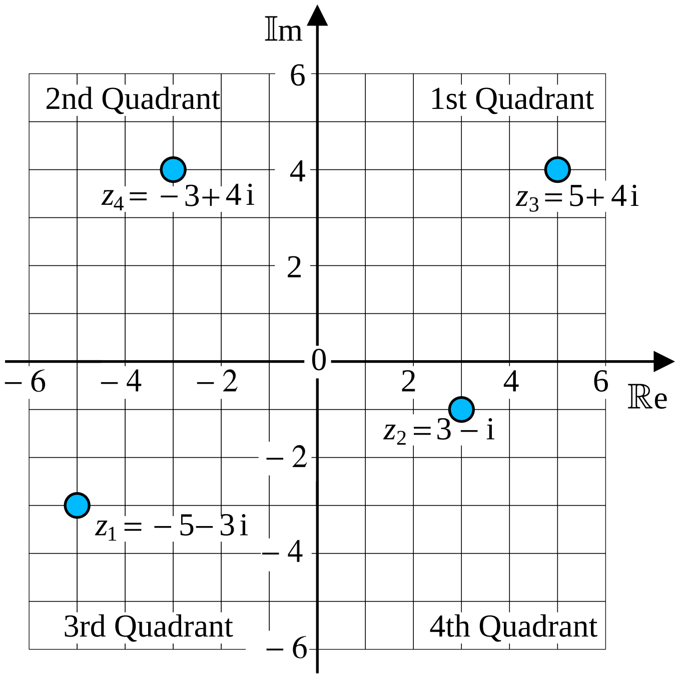
To rotate from the positive x-axis direction onto points in either the 3rd or the 4th quadrants the rotation is given as a negative (clockwise) rotation.

An argument given in this range is formally termed a “Principal Argument” but often the word “principal” is dropped. In examination questions on arguments, answers should always be given in the range $-\pi < arg(z) < \pi$ unless otherwise instructed.

An argument outside the range $-\pi < arg(z) < \pi$ can be brought back into range by simply adding or subtracting a multiple of 2π. However, it is better to get into the habit of generating answers within the required range directly rather than having to “rescue” out of range answers.

In University courses an argument outside of the range $-\pi < arg(z) < \pi$ is written $arg\ (z)$ whereas a principle argument is denoted $Arg\ (z)$, with the capital letter A. In the Further Mathematics A-level course this distinction is not made, although teachers often, without thinking, (correctly) put the capital letter in !

5.3 Example on Finding Principal Arguments



For each of the complex numbers on the Argand diagram find the principal argument.

Teaching Video : http://www.NumberWonder.co.uk/Video/v9085(5).mp4

[4 marks]

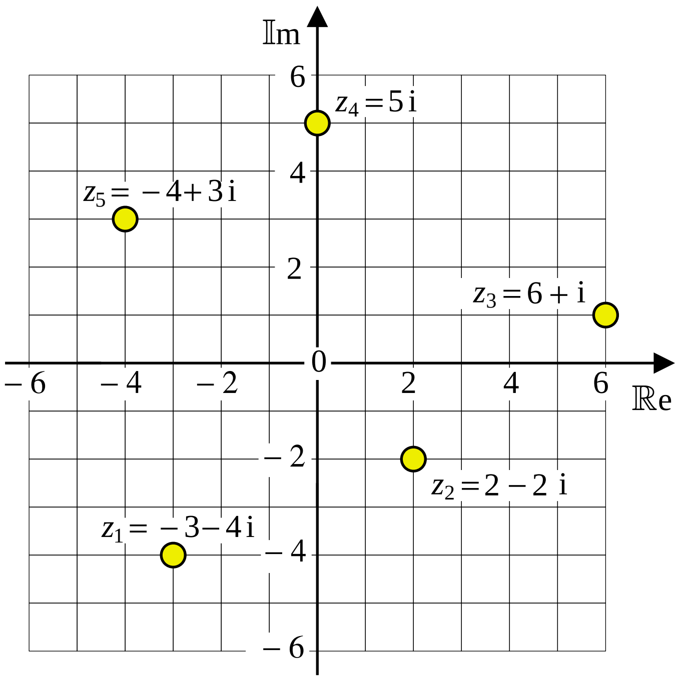
5.5 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 40

Question 1

For each of the complex numbers on the Argand diagram find the principal argument.



[5 marks]

Question 2

Further A-Level Examination Question from January 2012, FP1, Q1

Given that

$$z_1 = 1 - \mathrm{i}$$

(a) find $arg(z_1)$

[2 marks]

Given also that $z_2 = 3 + 4\mathrm{i}$, find, in the form $a + \mathrm{i}b$, $a, b \in \mathbb{R}$

(b) $z_1 z_2$

[2 marks]

(c) $\dfrac{z_2}{z_1}$

[3 marks]

In part (b) and part (c) you must show all your working clearly.

Question 3

$$z = \frac{26}{2 - 3\mathrm{i}}$$

(a) Find z in the form $a + b\mathrm{i}$ where $a, b \in \mathbb{R}$

[2 marks]

(b) Find z^2 in the form $a + b\mathrm{i}$ where $a, b \in \mathbb{R}$

[2 marks]

(c) Determine $|z|$

[2 marks]

(d) Calculate $arg\left(z^2\right)$ giving your answer in radians to 2 decimal places.

[2 marks]

Question 4

Further A-Level Examination Question from January 2007, FP1, Q3

The complex numbers z_1 and z_2 are given by

$$z_1 = 5 + 3\mathrm{i}$$

$$z_2 = 1 + p\mathrm{i}$$

where p is an integer.

(a) Find $\dfrac{z_2}{z_1}$ in the form $a + \mathrm{i}b$, where a and b are expressed in terms of p

[3 marks]

Given that,

$$arg\left(\frac{z_2}{z_1}\right) = \frac{\pi}{4}$$

(b) find the value of p

[2 marks]

Question 5

Further A-Level Examination Question from January 2014, IAL, FP1, Q5

$$z = 5 + \mathrm{i}\sqrt{3} \qquad\qquad w = \sqrt{3} - \mathrm{i}$$

(a) Find the value of $|w|$

[1 mark]

Find in the form $a + \mathrm{i}b$, where a and b are real numbers,

(b) zw, showing clearly how you obtained your answer.

[2 marks]

(c) $\dfrac{z}{w}$, showing clearly how you obtained your answer.

[3 marks]

Given that

$$arg(z + \lambda) = \frac{\pi}{3}, \qquad \text{where } \lambda \text{ is a real constant}$$

(d) find the value of λ

[2 marks]

Question 6

Further A-Level Examination Question from January 2019, IAL, F1, Q9

The complex numbers z_1 and z_2 are given by

$$z_1 = -1 - \mathrm{i} \qquad \text{and} \qquad z_2 = 3 - 4\mathrm{i}$$

(a) Find the argument of the complex number $z_1 - z_2$
Give your answer in radians to 3 decimal places.

[3 marks]

(b) Find the complex number $\frac{z_1}{z_2}$ in the form $a + b\mathrm{i}$
where a and b are rational numbers.

[3 marks]

(c) Find the modulus of $\frac{z_1}{z_2}$ giving your answer as a simplified surd.

[2 marks]

(d) Find the values of the real constants p and q such that

$$\frac{p + \mathrm{i}q - 8z_1}{p - \mathrm{i}q - 8z_2} = 3\mathrm{i}$$

[5 marks]

5.6 Answers

5.6.1 Solution to 5.3 Example on Finding Principal Arguments

For $z_1 = -5 - 3i$

Think of this as “a clockwise half a turn then anticlockwise a bit”

$$\begin{aligned} arg(z_1) &= -\pi + arctan\left(\frac{3}{5}\right) \\ &= -\pi + 0.540 \\ &= -2.60^c \end{aligned}$$

For $z_2 = 3 - i$

Think of this as “clockwise a bit”

$$\begin{aligned} arg(z_2) &= 0 - arctan\left(\frac{1}{3}\right) \\ &= 0 - 0.322 \\ &= -0.322^c \end{aligned}$$

For $z_3 = 5 + 4i$

Think of this as “anticlockwise a bit”

$$\begin{aligned} arg(z_3) &= arctan\left(\frac{4}{5}\right) \\ &= 0.675^c \end{aligned}$$

For $z_4 = -3 + 4i$

Think of this as “anticlockwise half a turn then clockwise a bit”

$$\begin{aligned} arg(z_4) &= \pi - arctan\left(\frac{4}{3}\right) \\ &= \pi - 0.927 \\ &= 2.21^c \end{aligned}$$

[4 marks]

5.6.2 Solutions to 5.5 Exercise

Answer 1

For $z_1 = -3 - 4\text{i}$

Think of this as "a clockwise half a turn then anticlockwise a bit"

$$arg(z_1) = -\pi + arctan\left(\frac{4}{3}\right)$$
$$= -\pi + 0.927$$
$$= -2.21^c$$

For $z_2 = 2 - 2\text{i}$

Think of this as "clockwise a bit" or even "clockwise an eighth of a turn"

$$arg(z_2) = -arctan\left(\frac{2}{2}\right)$$
$$= -\frac{\pi}{4} \quad \text{or} \quad -0.785^c$$

For $z_3 = 6 + \text{i}$

Think of this as "clockwise a bit"

$$arg(z_2) = arctan\left(\frac{1}{6}\right)$$
$$= 0.165^c$$

For $z_4 = 5\text{i}$

Think of this as "anticlockwise a quarter of a turn"

$$arg(z_4) = \frac{\pi}{2}$$
$$= 1.57^c$$

For $z_5 = -4 + 3\text{i}$

Think of this as "anticlockwise half a turn then clockwise a bit"

$$arg(z_5) = \pi - arctan\left(\frac{3}{4}\right)$$
$$= \pi - 0.644$$
$$= 2.50^c$$

[5 marks]

Answer 2

(a) $arg(z_1) = -\dfrac{\pi}{4}$ or -0.785

[2 marks]

(b)

$$\begin{aligned} z_1 z_2 &= (1 - i)(3 + 4i) \\ &= 3 + 4i - 3i - 4i^2 \\ &= 7 + i \end{aligned}$$

[2 marks]

(c)

$$\begin{aligned} \frac{z_2}{z_1} &= \frac{3 + 4i}{1 - i} \\ &= \frac{3 + 4i}{1 - i} \times \frac{1 + i}{1 + i} \\ &= \frac{3 + 3i + 4i + 4i^2}{1 - i^2} \\ &= \frac{-1 + 7i}{2} \\ &= -\frac{1}{2} + \frac{7}{2} i \end{aligned}$$

[3 marks]

Answer 3

(a)

$$\begin{aligned} z &= \frac{26}{2 - 3i} \\ &= \frac{26}{2 - 3i} \times \frac{2 + 3i}{2 + 3i} \\ &= \frac{26(2 + 3i)}{4 - 9i^2} \\ &= \frac{26(2 + 3i)}{13} \\ &= 2(2 + 3i) \\ &= 4 + 6i \end{aligned}$$

[2 marks]

(b)

$$\begin{aligned} z^2 &= (4 + 6i)^2 \\ &= (4 + 6i)(4 + 6i) \\ &= 16 + 24i + 24i + 36i^2 \\ &= -20 + 48i \end{aligned}$$

[2 marks]

(c)

$$\begin{aligned} |z| &= \sqrt{4^2 + 6^2} \\ &= \sqrt{52} \\ &= 2\sqrt{13} \quad \text{or} \quad 7.21 \end{aligned}$$

[2 marks]

(d) On the Argand diagram, $z^2 = -20 + 48i$
This is located in the second quadrant.

$$\begin{aligned} arg\left(z^2\right) &= \pi - arctan\left(\frac{48}{20}\right) \\ &= \pi - 1.18 \\ &= 1.97^c \end{aligned}$$

[2 marks]

Answer 4

(a)

$$\begin{aligned} \frac{z_2}{z_1} &= \frac{1 + pi}{5 + 3i} \\ &= \frac{1 + pi}{5 + 3i} \times \frac{5 - 3i}{5 - 3i} \\ &= \frac{5 - 3i + 5pi - 3p\,i^2}{25 - 9\,i^2} \\ &= \frac{5 + 3p - 3i + 5pi}{34} \\ &= \frac{5 + 3p}{34} + \frac{5p - 3}{34}\,i \end{aligned}$$

[3 marks]

(b)

$$\begin{aligned} arg\left(\frac{z_2}{z_1}\right) &= \frac{\pi}{4} \\ tan\left(arg\left(\frac{z_2}{z_1}\right)\right) &= tan\left(\frac{\pi}{4}\right) \\ \frac{5p - 3}{34} \div \frac{5 + 3p}{34} &= 1 \qquad \text{from part (a)} \\ \frac{5p - 3}{34} \times \frac{34}{5 + 3p} &= 1 \\ \frac{5p - 3}{5 + 3p} &= 1 \\ 5p - 3 &= 5 + 3p \\ 2p &= 8 \\ p &= 4 \end{aligned}$$

[2 marks]

Answer 5

(a)

$$|w| = |\sqrt{3} - \mathrm{i}|$$
$$= \sqrt{(\sqrt{3})^2 + (-1)^2}$$
$$= 2$$

[1 mark]

(b)

$$zw = \left(5 + \sqrt{3}\,\mathrm{i}\right)\left(\sqrt{3} - \mathrm{i}\right)$$
$$= 5\sqrt{3} - 5\mathrm{i} + 3\mathrm{i} - \sqrt{3}\,\mathrm{i}^2$$
$$= 6\sqrt{3} - 2\mathrm{i}$$

[2 marks]

(c)

$$\frac{z}{w} = \frac{5 + \sqrt{3}\,\mathrm{i}}{\sqrt{3} - \mathrm{i}}$$
$$= \frac{5 + \sqrt{3}\,\mathrm{i}}{\sqrt{3} - \mathrm{i}} \times \frac{\sqrt{3} + \mathrm{i}}{\sqrt{3} + \mathrm{i}}$$
$$= \frac{5\sqrt{3} + 5\mathrm{i} + 3\mathrm{i} + \sqrt{3}\,\mathrm{i}^2}{4}$$
$$= \frac{4\sqrt{3} + 8\mathrm{i}}{4}$$
$$= \sqrt{3} + 2\mathrm{i}$$

[3 marks]

(d)

$$arg(z + \lambda) = \frac{\pi}{3}$$
$$tan\left(arg\left(5 + \sqrt{3}\,\mathrm{i} + \lambda\right)\right) = tan\left(\frac{\pi}{3}\right)$$
$$tan\left(arg\left((5 + \lambda) + \sqrt{3}\,\mathrm{i}\right)\right) = \sqrt{3}$$
$$\frac{\sqrt{3}}{5 + \lambda} = \sqrt{3}$$
$$1 = 5 + \lambda$$
$$\lambda = -4$$

[2 marks]

Answer 6

(a)

$$\begin{aligned} arg(z_1 - z_2) &= arg(\ -1 - i - 3 + 4i\ ) \\ &= arg\ (-4 + 3i\ ) \\ &= \pi - arctan\left(\frac{3}{4}\right) \\ &= \pi - 0.6435 \\ &= 2.498^c \end{aligned}$$

[3 marks]

(b)

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{-1 - i}{3 - 4i} \\ &= \frac{-1 - i}{3 - 4i} \times \frac{3 + 4i}{3 + 4i} \\ &= \frac{-3 - 4i - 3i - 4\,i^2}{9 - 16\,i^2} \\ &= \frac{1 - 7i}{25} \\ &= \frac{1}{25} - \frac{7}{25}\,i \end{aligned}$$

[3 marks]

(c)

$$\begin{aligned} \left|\frac{z_1}{z_2}\right| &= \left|\frac{1}{25} - \frac{7}{25}\,i\right| \\ &= \sqrt{\left(\frac{1}{25}\right)^2 + \left(\frac{7}{25}\right)^2} \\ &= \sqrt{\frac{1 + 49}{625}} \\ &= \frac{\sqrt{50}}{25} \\ &= \frac{5\sqrt{2}}{25} \\ &= \frac{\sqrt{2}}{5} \end{aligned}$$

[2 marks]

(d)

$$\frac{p + \mathrm{i}q - 8z_1}{p - \mathrm{i}q - 8z_2} = 3\mathrm{i}$$

$$p + \mathrm{i}q - 8z_1 = 3\mathrm{i}\,(p - \mathrm{i}q - 8z_2)$$

$$p + \mathrm{i}q - 8(-1 - \mathrm{i}) = 3\mathrm{i}p - 3\,\mathrm{i}^2 q - 24\mathrm{i}\,(3 - 4\mathrm{i})$$

$$p + \mathrm{i}q + 8 + 8\mathrm{i} = 3\mathrm{i}p + 3q - 72\mathrm{i} + 96\,\mathrm{i}^2$$

$$p + \mathrm{i}q + 8 + 8\mathrm{i} = 3\mathrm{i}p + 3q - 72\mathrm{i} - 96$$

$$p + \mathrm{i}q = 3\mathrm{i}p + 3q - 80\mathrm{i} - 104$$

$$p + \mathrm{i}q = 3q - 104 + (3p - 80)\,\mathrm{i}$$

$$\mathrm{Re}: \; p = 3q - 104 \;\Rightarrow\; p - 3q = -104$$

$$\mathrm{Im}: \; q = 3p - 80 \;\Rightarrow\; -3p + q = -80$$

$$3p - 9q = -312$$

$$3p - q = 80 \qquad \text{Subtract}$$

$$-8q = -392$$

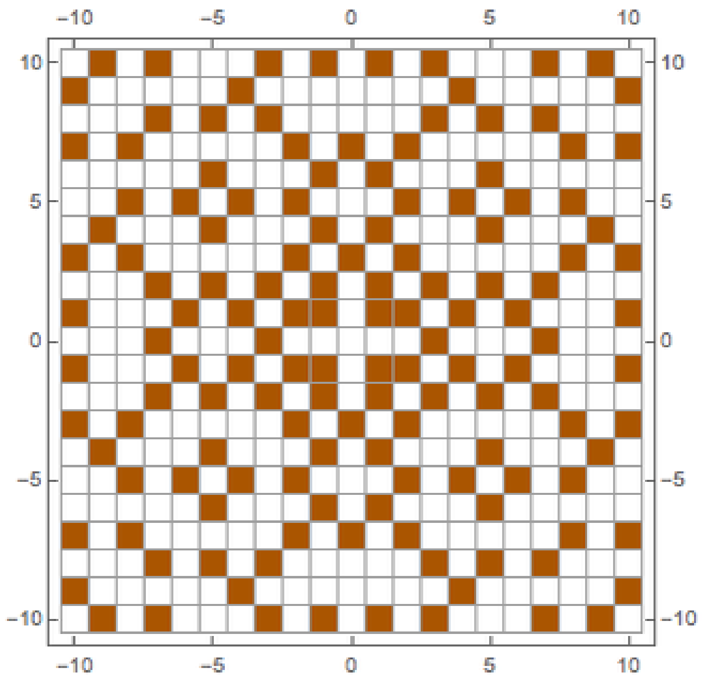
$$q = 49$$

$$\therefore\; p = 43$$

[5 marks]

6.1 The Gaussian Integers

The Gaussian integers are complex numbers of the form $a + b\mathrm{i}$ where both a and b are integers. For this subset of complex numbers unique factorisation holds. They are attractive to work with because each Gaussian integer is either a prime Gaussian integer or a composite Gaussian integer. A composite Gaussian integer can be decomposed into a unique product of prime Gaussian integers. So, some concepts, familiar from earlier work with the real integers, get a new lease of life with the Gaussian integers. The statement, $z \in \mathbb{Z}[\mathrm{i}]$ says "z is a Gaussian integer".



The integer grid shows Gaussian primes for values around zero. On the real axis observe that 3 and 7 are Gaussian prime but 2 and 5 are not. What are the factors of these composite Gaussian integers 2 and 5 ? With a little thought,

- 2 factorises into $(1 + \mathrm{i})\ (1 - \mathrm{i})$
- 5 factorises into $(2 + \mathrm{i})\ (2 - \mathrm{i})$

From a careful look at the diagram, $(1 \pm \mathrm{i})$ and $(2 \pm \mathrm{i})$ are all Gaussian prime. There is no further factorising to do and the unique prime factorisation of 2 and 5 in Gaussian integers has been found.

In lesson 3, the number 6 was used to show that, in the general, complex numbers don't have unique factorisation because 6 could be factorized in more than one way,

- $6 = 3(1 + \mathrm{i})\ (1 - \mathrm{i})$
- $6 = \left(1 + \sqrt{5}\,\mathrm{i}\right)\left(1 - \sqrt{5}\,\mathrm{i}\right)$
- $6 = 2\left(\sqrt{2} + \mathrm{i}\right)\left(\sqrt{2} - \mathrm{i}\right)$

The second and third of these is not valid if only Gaussian integers are being considered and so this is an illustration that in $\mathbb{Z}[i]$ the composite number 6 does indeed have the unique factorisation,

- $6 = 3(1 + i)(1 - i)$

where each of these three factors is a Gaussian prime (easily checked on the grid).

6.2 The Square Root of a Complex Number

In the teaching video I will show you a very clever way to square root a Gaussian integer if it has an answer that is also a Gaussian integer.

Find z such that $z^2 - 5 - 12i = 0$, $\quad z \in \mathbb{Z}[i]$

Teaching Video : http://www.NumberWonder.co.uk/Video/v9085(6).mp4

[6 marks]

I am indebted to Dr Neha Agrawal of Murray Edwards College,
Cambridge University, for teaching me this method
of finding complex square roots in $\mathbb{Z}[i]$

6.3 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 50

Question 1

A complex number $z = a + b\text{i}$, where a and b are real numbers, is sought that satisfies the equation,

$$z^2 - 7 - 24\text{i} = 0 \qquad z \in \mathbb{Z}[\text{i}]$$

(i) Show that $ab = 12$

[2 marks]

(ii) Write down the value of $a^2 - b^2$

[1 mark]

(iii) Hence find the two values of z such that

$$z^2 - 7 - 24\text{i} = 0 \qquad z \in \mathbb{Z}[\text{i}]$$

[3 marks]

(iv) Show that your part (iii) solutions, when squared are equal to $7 + 24\text{i}$ as you would expect.

[2 marks]

Question 2

A complex number $z = a + b\text{i}$, where a and b are real numbers, is sought that satisfies the equation,

$$z^2 + 9 - 40\text{i} = 0 \qquad z \in \mathbb{Z}[\text{i}]$$

(i) Show that $ab = 20$

[2 marks]

(ii) Write down the value of $a^2 - b^2$

[1 mark]

(iii) Hence find the two values of z such that

$$z^2 + 9 - 40\text{i} = 0 \qquad z \in \mathbb{Z}[\text{i}]$$

[3 marks]

(iv) Show that your part (iii) solutions, when squared, are equal to $-9 + 40\text{i}$ as you would expect.

[2 marks]

Question 3

A complex number $z = a + b\text{i}$, where a and b are real numbers, is sought that satisfies the equation,

$$z^2 - 40 + 42\text{i} = 0 \qquad z \in \mathbb{Z}[\text{i}]$$

(a) Find the two possible values of z

[5 marks]

(b) **(i)** The formula to solve quadratic equations of the form

$$a z^2 + bz + c = 0 \qquad z \in \mathbb{C}$$

where a, b and c are complex numbers is

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Use this formula to try and solve,

$$z^2 - 40 + 42\text{i} = 0 \qquad z \in \mathbb{Z}[\text{i}]$$

[2 marks]

(ii) Comment on the usefulness of this solution technique in this case.

[1 mark]

Question 4

(i) Work out $\left(1 + \sqrt{7}\,\mathrm{i}\right)\left(1 - \sqrt{7}\,\mathrm{i}\right)$

[1 mark]

(ii) Work out $(1 + \mathrm{i})(1 - \mathrm{i})(1 + \mathrm{i})(1 - \mathrm{i})(1 + \mathrm{i})(1 - \mathrm{i})$

[2 marks]

(iii) Explain how your part (i) and part (ii) answers show that complex numbers, in general, do not have unique factorisation.

[1 mark]

Question 5

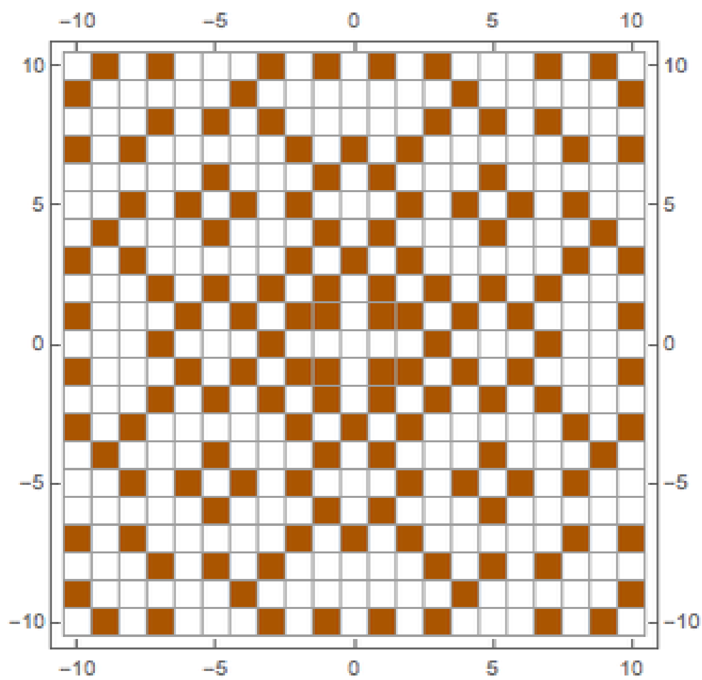
A certain number p is a prime integer in $\mathbb{R}$ but is not a Gaussian prime integer in $\mathbb{Z}[\mathrm{i}]$. When considered to be a Gaussian integer it is composite and has the unique factorization,

$$p = (3 + 2\mathrm{i})(3 - 2\mathrm{i})$$

What number is p ?

[1 mark]

Question 6



(i) On the grid of Gaussian integers, circle $6 + 5\mathrm{i}$

(ii) Hence state if $6 + 5\mathrm{i}$ is a Gaussian prime or not.

[1, 1 marks]

Question 7

A polynomial $p(z)$ has some coefficients that are complex numbers,

$$z^3 + \mathrm{i}\, z^2 + 6z = 0 \qquad z \in \mathbb{C}$$

The formula to solve quadratic equations of the form

$$a z^2 + bz + c = 0 \qquad z \in \mathbb{C}$$

where a, b and c are complex numbers is

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Show how this formula may be used to find the three roots of $p(z)$

[5 marks]

Question 8

A complex number $z = a + b\mathrm{i}$, where a and b are real numbers, is sought that satisfies the equation,

$$z^2 + 3 - 2\mathrm{i} = 0 \qquad z \in \mathbb{Z}[\mathrm{i}]$$

(i) Show that $ab = 1$

[2 marks]

(ii) Find the value of $a^2 - b^2$

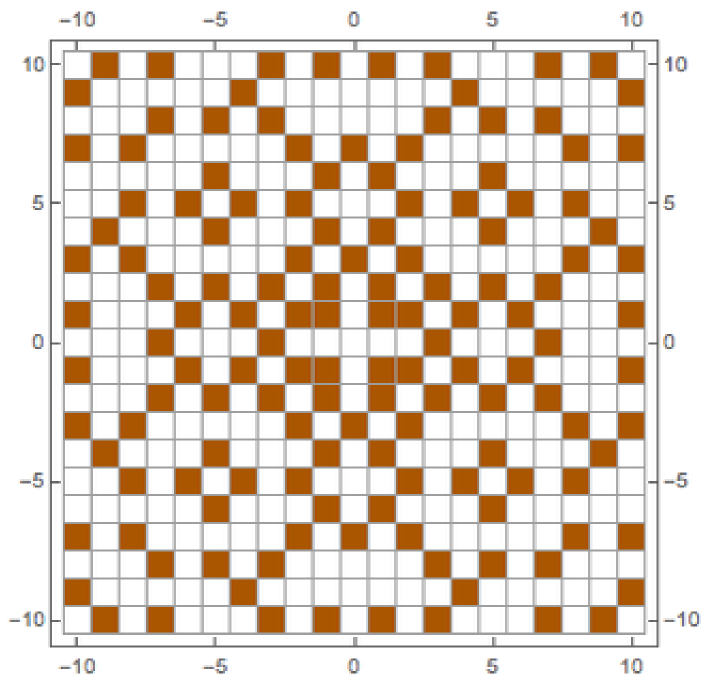
[1 mark]

(iii) Explain why there is no Gaussian integer that satisfies the equation

$$z^2 + 3 - 2\mathrm{i} = 0 \qquad z \in \mathbb{Z}[\mathrm{i}]$$

[2 marks]

(iv) On the grid of Gaussian integers, circle the complex number $-3 + 2\mathrm{i}$



[1 mark]

(v) Explain how your part (iv) answer confirms that there is no complex number that satisfies the equation $z^2 + 3 - 2\mathrm{i} = 0$, $z \in \mathbb{Z}[\mathrm{i}]$

[2 marks]

Question 9

A-Level Examination Question from June 2004, P4, Q3

The complex number $z = a + b\text{i}$, where a and b are real numbers, satisfies the equation,

$$z^2 + 16 - 30\text{i} = 0$$

(a) Show that $ab = 15$

[2 marks]

(b) Write down a second equation in a and b and hence find the roots of

$$z^2 + 16 - 30\text{i} = 0$$

[4 marks]

6.4 Answers

6.4.1 Solution to 6.2 Example (Teaching Video)

The video was originally focussed on $z = \pm\sqrt{5 + 12i}$. However, the few Further A-Level examination questions on this topic are more about solving the related quadratic equation. So I've asked the question in this marginally adjusted form too.

$$z^2 - 5 - 12i = 0$$
$$z^2 = 5 + 12i$$
$$(a + bi)^2 = 5 + 12i$$
$$a^2 + 2abi + b^2 i^2 = 5 + 12i$$
$$a^2 - b^2 + 2abi = 5 + 12i$$

Two complex numbers (the one on the left hand side and the one on the right hand side) are equal if and only if the real parts are equal and the imaginary parts are equal. Therefore we can deduce that,

$$\therefore\ a^2 - b^2 = 5 \quad \text{and} \quad 2ab = 12$$
$$ab = 6$$

As a, b are integers the only possibility with $|a| > |b|$ (to have $a^2 - b^2$ positive)

are, $(a, b) = (6, 1), (3, 2), (-6, -1)$ or $(-3, -2)$

Of these only $(3, 2)$ and $(-3, -2)$ also satisfy $a^2 - b^2 = 5$.

Thus, the solutions to

$$z^2 - 5 - 12i = 0$$

$$\text{are } 3 + 2i \text{ or } -3 + 2i$$

These could be presented as $\pm(3 + 2i)$ which has an obvious similarity with previous work with the real numbers. The solutions can, of course, be substituted back into the original equation to verify that they make it true. For example,

$$\begin{aligned} \text{LHS} &= z^2 - 5 - 12i \\ &= (\pm 1)^2 (3 + 2i)^2 - 5 - 12i \\ &= 9 + 12i + 4i^2 - 5 - 12i \\ &= 9 - 5 - 4 + 12i - 12i \\ &= 0 \\ &= \text{RHS} \qquad \square \end{aligned}$$

[6 marks]

6.4.2 Solutions to 6.3 Exercise

Answer 1

(i)

$$z^2 - 7 - 24\text{i} = 0 \qquad z \in \mathbb{Z}[\text{i}]$$

$$z^2 = 7 + 24\text{i}$$

$$(a + b\text{i})^2 = 7 + 24\text{i}$$

$$a^2 + 2ab\text{i} + b^2\text{i}^2 = 7 + 24\text{i}$$

$$a^2 - b^2 + 2ab\text{i} = 7 + 24\text{i}$$

Two complex numbers (the one on the left hand side and the one on the right hand side) are equal if and only if the real parts and the imaginary parts are equal. Therefore we can deduce that,

Equating imaginary parts : $\quad 2ab = 24$

$$ab = 12$$

[2 marks]

(ii)

Equating real parts : $\quad a^2 - b^2 = 7$

[1 mark]

(iii) As $a, b \in \mathbb{Z}$ the only possibility with $|a| > |b|$ (to have $a^2 - b^2 > 0$)

are, $(a, b) = \ (12,\ 1),\quad (6,\ 2),\quad (4,\ 3)$

$$(-12, -1),\ (-6, -2),\ (-4, -3)$$

Of these only $(4,\ 3)$ and $(-4, -3)$ also satisfy $a^2 - b^2 = 7$

Thus, the solutions to

$$z^2 - 7 - 24\text{i} = 0$$

are $4 + 3\text{i}$ or $-4 - 3\text{i}$

These could be presented as $\pm(4 + 3\text{i})$ which has an obvious similarity with previous work with the real numbers.

[3 marks]

(iv)

$$(\pm 1)^2 (4 + 3\text{i})^2 = 16 + 24\text{i} + 9\text{i}^2$$

$$= 16 + 24\text{i} - 9$$

$$= 7 + 24\text{i} \quad \text{as expected}$$

[2 marks]

Answer 2

(i)

$$z^2 + 9 - 40\mathrm{i} = 0 \qquad z \in \mathbb{Z}[\mathrm{i}]$$

$$z^2 = -9 + 40\mathrm{i}$$

$$(a + b\mathrm{i})^2 = -9 + 40\mathrm{i}$$

$$a^2 + 2ab\mathrm{i} + b^2\mathrm{i}^2 = -9 + 40\mathrm{i}$$

$$a^2 - b^2 + 2ab\mathrm{i} = -9 + 40\mathrm{i}$$

Two complex numbers (the one on the left hand side and the one on the right hand side) are equal if and only if the real parts and the imaginary parts are equal. Therefore we can deduce that,

Equating imaginary parts : $\quad 2ab = 40$

$$ab = 20$$

[2 marks]

(ii)

Equating real parts : $\quad a^2 - b^2 = -9$

[1 mark]

(iii) As $a, b \in \mathbb{Z}$ the only possibility with $|a| < |b|$ (to have $a^2 - b^2 < 0$)

are, $(a, b) = \ (1, 20), \quad (2, 10), \quad (4, 5)$

$(-1, -20), (-2, -10), \ (-4, -5)$

Of these only $(4, 5)$ and $(-4, -5)$ also satisfy $a^2 - b^2 = -9$

Thus, the solutions to

$$z^2 + 9 - 40\mathrm{i} = 0$$

are $4 + 5\mathrm{i}$ or $-4 - 5\mathrm{i}$

These could be presented as $\pm(4 + 5\mathrm{i})$ which has an obvious similarity with previous work with the real numbers

[3 marks]

(iv)

$$(\pm 1)^2 (4 + 5\mathrm{i})^2 = 16 + 40\mathrm{i} + 25\mathrm{i}^2$$

$$= 16 + 40\mathrm{i} - 25$$

$$= -9 + 40\mathrm{i} \quad \text{as expected}$$

[2 marks]

Answer 3

(a)

$$z^2 - 40 + 42\text{i} = 0 \qquad z \in \mathbb{Z}[\text{i}]$$

$$z^2 = 40 - 42\text{i}$$

$$(a + b\text{i})^2 = 40 - 42\text{i}$$

$$a^2 + 2ab\text{i} + b^2\,\text{i}^2 = 40 - 42\text{i}$$

$$a^2 - b^2 + 2ab\text{i} = 40 - 42\text{i}$$

Equating imaginary parts : $2ab = -42$

$ab = -21$

Equating real parts : $a^2 - b^2 = 40$

As $a, b \in \mathbb{Z}$ the only possibility with $|a| > |b|$ (to have $a^2 - b^2 > 0$)

are, $(a, b) = (21, -1), \quad (7, -3)$

$(-21, 1), \quad (-7, 3)$

Of these only $(7, -3)$ and $(-7, 3)$ also satisfy $a^2 - b^2 = -9$

Thus, the solutions to

$$z^2 - 40 + 42\text{i} = 0$$

are $7 - 3\text{i}$ or $-7 + 3\text{i}$

These could be presented as $\pm(7 - 3\text{i})$ which has an obvious similarity with previous work with the real numbers

[5 marks]

(b) **(i)** $a = 1$, $b = 0$ and $c = -40 + 42\text{i}$

$$z = \frac{0 \pm \sqrt{0^2 - 4 \times 1 \times (-40 + 42\text{i})}}{2 \times 1}$$

$$= \frac{\pm\sqrt{4} \times \sqrt{40 - 42\text{i}}}{2}$$

$$= \pm\sqrt{40 - 42\text{i}}$$

[2 marks]

(ii) The formula hasn't told us anything that wasn't obvious anyway !
We could have got this “result” much more easily like this;

$$z^2 - 40 + 42\text{i} = 0$$

$$z^2 = 40 - 42\text{i}$$

$$z = \pm\sqrt{40 - 42\text{i}}$$

[1 mark]

Answer 4

(i) $$\left(1+\sqrt{7}\,\mathrm{i}\right)\left(1-\sqrt{7}\,\mathrm{i}\right)=1-7\,\mathrm{i}^2$$
$$=1+7$$
$$=8$$

[1 mark]

(ii) $$(1+\mathrm{i})(1-\mathrm{i})(1+\mathrm{i})(1-\mathrm{i})(1+\mathrm{i})(1-\mathrm{i})=2\times2\times2$$
$$=8$$

[2 marks]

(iii) The part (i) and part (ii) answers show that the number 8 can be factorised in at least two different ways and so the factorisation is not unique.
Other factorisations are possible such as,

$$\left(\sqrt{3}+\sqrt{5}\,\mathrm{i}\right)\left(\sqrt{3}-\sqrt{5}\,\mathrm{i}\right)=3-5\,\mathrm{i}^2$$
$$=3+5$$
$$=8$$

[1 mark]

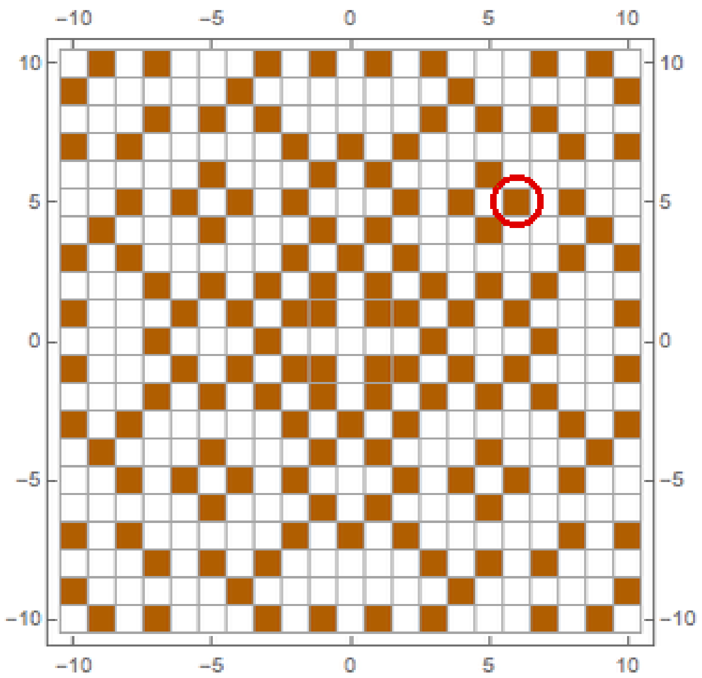
Answer 5

$$p=(3+2\mathrm{i})(3-2\mathrm{i})$$
$$=9-6\mathrm{i}+6\mathrm{i}-4\,\mathrm{i}^2$$
$$=13$$

[1 mark]

Answer 6

(i)



[1 mark]

(ii) Yes, $6 + 5i$ is a Gaussian prime.
It has exactly two factorised in $\mathbb{Z}[i]$, itself and the number 1.

[1 mark]

Answer 7

$$z^3 + i z^2 + 6z = 0 \qquad z \in \mathbb{C}$$

$$z\left(z^2 + i z + 6\right) = 0$$

Either $z = 0$ or $z^2 + i z + 6 = 0$

With $a = 1$, $b = i$ and $c = 6$

$$\begin{aligned} z &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-i \pm \sqrt{i^2 - 4 \times 1 \times 6}}{2 \times 1} \\ &= \frac{-i \pm \sqrt{-1 - 24}}{2} \\ &= \frac{-i \pm \sqrt{-25}}{2} \\ &= \frac{-i \pm 5i}{2} \\ &= -3i \text{ or } 2i \end{aligned}$$

So, in total there are the three roots, $-3i$, 0, $2i$

[5 marks]

Answer 8

$$z^2 + 3 - 2i = 0 \qquad z \in \mathbb{Z}[i]$$

$$z^2 = -3 + 2i$$

$$(a + bi)^2 = -3 + 2i$$

$$a^2 + 2abi + b^2 i^2 = -3 + 2i$$

$$a^2 - b^2 + 2abi = -3 + 2i$$

(i) Equating imaginary parts : $2ab = 2$

$$ab = 1$$

[2 marks]

(ii) Equating real parts : $a^2 - b^2 = -3$

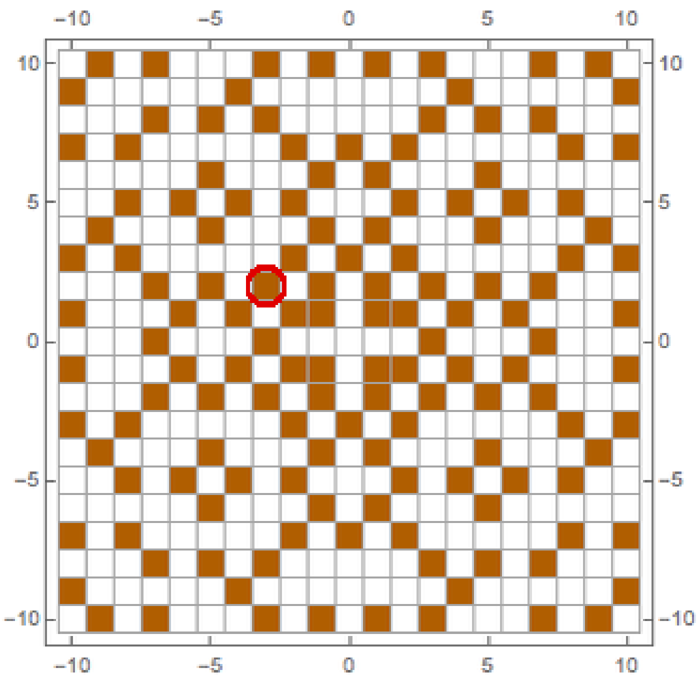
[1 mark]

(iii) As $a, b \in \mathbb{Z}$ the with $ab = 1$ only possibilities are $|a| = |b| = 1$ but this clearly cannot satisfy the requirement that $a^2 - b^2 = -3$
Thus there are no Gaussian integers that satisfy the equation,

$$z^2 + 3 - 2i = 0 \qquad z \in \mathbb{Z}[i] \qquad \square$$

[2 marks]

(iv)



[1 mark]

(v) The complex number is shown as a prime on the grid of Gaussian integers, confirming that it cannot be factorised in $\mathbb{Z}[i]$

[2 marks]

Answer 9

(a)

$$z^2 + 16 - 30i = 0$$

$$z^2 = -16 + 30i$$

$$(a + bi)^2 = -16 + 30i$$

$$a^2 + 2abi + b^2 i^2 = -16 + 30i$$

$$a^2 - b^2 + 2abi = -16 + 30i$$

Equating imaginary parts : $2ab = 30$

$ab = 15$ ☐

[2 marks]

(b) Equating real parts : $a^2 - b^2 = -16$

As $a, b \in \mathbb{Z}$ the only possibilities with $|a| < |b|$ (to get $a^2 - b^2 < 0$)

are, $(a, b) = (1, 15), (3, 5), (-1, -15)$ and $(-3, -5)$

Of these only $(3, 5)$ and $(-3, -5)$ also satisfy $a^2 - b^2 = -16$

Thus, the solutions to

$$z^2 + 16 - 30i = 0$$

are $3 + 5i$ or $-3 - 5i$

These could be presented as $\pm (3 + 5i)$

which has an obvious similarity with previous work with the real numbers.

[4 marks]

7.1 Loci in the Argand Diagram

When faced with a piece of algebra such as

$$y = x^2 - 6x + 5$$

most mathematicians would immediately visualise this as a geometric object, a quadratic curve passing through (0, 5) on the y-axis. Even just this initial vague visualisation may be enough to answer a question.

Or perhaps more detail is needed.

Doing some algebra, factorising, leads to,

$$y = (x - 1)(x - 5)$$

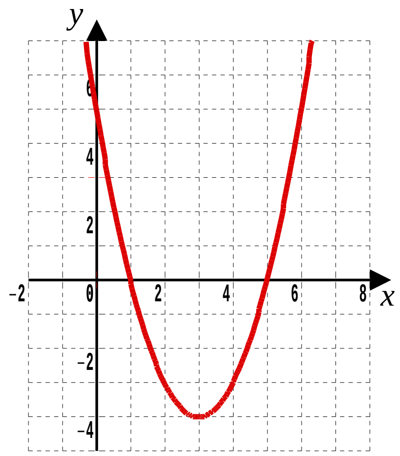
but again, faced with a piece of algebra, it is geometry that is in mind; the fact that this quadratic curve crosses the x-axis at (1, 0) and (5, 0).

Perhaps still more detail is required.

More algebra, this time completing the square, yields,

$$y = (x - 3)^2 - 4$$

and again, a visualisation that (3, – 4) is the minimum point is second nature.



$$y = x^2 - 6x + 5$$

With the foregoing in mind, it should not come as a surprise that, when faced with equations involving complex numbers, there are certain types of equation that are immediately visualised as a geometric object on an Argand diagram. In general the geometric objects are termed loci, and include familiar shapes such as straight lines, bits of straight lines, circles and ellipses.

7.2 The Circle

Given that

$$|z - 8 - 15i| = 6$$

(i) Derive the Cartesian equation of the locus of z

(ii) Sketch the locus of z on an Argand diagram.

(iii) Calculate the minimum value of $|z|$

(iv) Find the maximum value, in radians, of $\arg z$

Teaching Video : http://www.NumberWonder.co.uk/Video/v9085(7).mp4

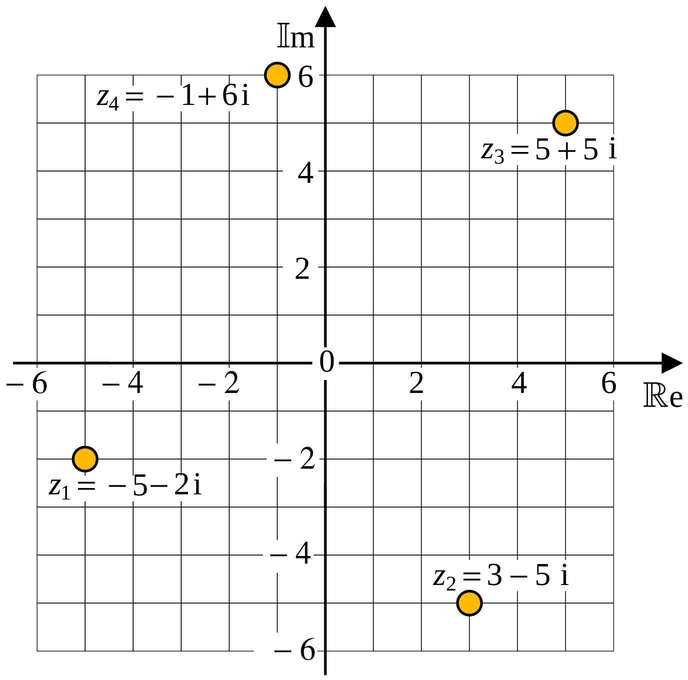
[8 marks]

7.3 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 50

Question 1

For each complex number on the Argand diagram find the principal argument.



[8 marks]

Question 2

Further A-Level Examination Question from FP2 Mock Paper, Q8

A complex number z satisfies the equation,

$$|z - 5 - 12i| = 3$$

(a) Describe in geometrical terms with the aid of a sketch, the locus of the point which represents z in the Argand diagram.

[3 marks]

For the points on this locus, find

(b) the maximum and minimum values of $|z|$

[4 marks]

(c) the maximum and minimum values for $arg(z)$, giving your answers in radians to 2 decimal places.

[4 marks]

Question 3

Given that

$$|z - 24 - 7\text{i}| = 5$$

(i) Derive the Cartesian equation of the locus of z

(ii) Sketch the locus of z on an Argand diagram

(iii) Calculate the maximum value of $|z|$

(iv) Find the minimum value, in radians, of $arg(z)$

[2, 2, 2, 3 marks]

Question 4
The complex number z is defined by

$$z = \frac{3 + 5\mathrm{i}}{2 - \mathrm{i}}$$

(i) Find $|z|$

[4 marks]

(ii) *arg z*

[2 marks]

Question 5
The complex number z satisfies

$$|z + 3 - 6\mathrm{i}| = 3$$

Show that the exact maximum value of *arg z* in the interval $[-\pi, \pi]$ is

$$\frac{\pi}{2} + 2\,arcsin\left(\frac{1}{\sqrt{5}}\right)$$

[4 marks]

Question 6

$$z = -1 - \sqrt{3}\,\mathrm{i}$$

Find **(i)** $|z|$

[1 mark]

(ii) $\left| \dfrac{z}{z^*} \right|$

[4 marks]

(iii) $arg\, z$, $arg\,(z^*)$ and $arg\left(\dfrac{z}{z^*}\right)$

giving your answers in terms of π

[3 marks]

Question 7

The complex number w is given by

$$w = 6 + 3\text{i}$$

(i) Determine the value of $|w|$

[1 mark]

(ii) Find *arg w*, giving your answer in radians to 2 decimal places

[2 marks]

Given that

$$arg\ (\lambda + 5\text{i} + w) = \frac{\pi}{4}$$

where λ is a real constant

(iii) find the value of λ

[2 marks]

7.4.1 Solution to 7.2 The Circle (Example)

(i)

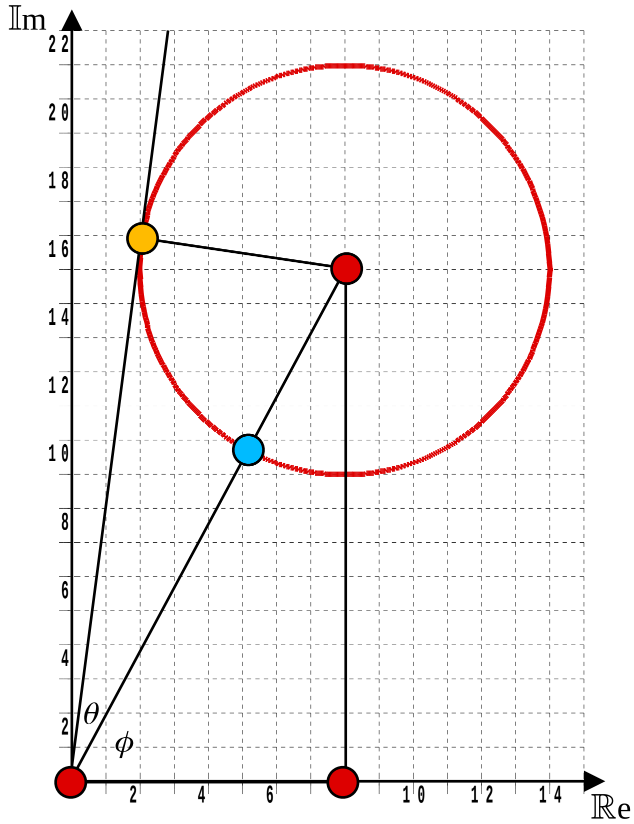
$$|z - 8 - 15\text{i}| = 6$$

$$|x + y\text{i} - 8 - 15\text{i}| = 6$$

$$|(x - 8) + (y - 15)\text{i}| = 6$$

$$(x - 8)^2 + (y - 15)^2 = 36$$

(ii) This is in the form of a circle with centre (8, 15) and radius 6



(iii) $|z|$ is the least distance from the origin to the circle.
There is an integer Pythagorean triple triangle, 8, 15, 17
Thus, as the circle has radius 6,

$$|z|_{\text{MIN}} = 17 - 6 = 11$$

(iv) We're after the maximum anticlockwise rotation of the positive ℝe-axis that still cuts (is a tangent to) the circle, shown as $\theta + \phi$ on the diagram.

$$\sin\theta = \frac{6}{17} \Rightarrow \theta = 0.36^c$$

$$\tan\phi = \frac{15}{8} \Rightarrow \phi = 1.08^c$$

$$\therefore [\arg z]_{\text{MAX}=} = 1.44^c$$

[2, 2, 2, 2 marks]

7.4.2 Solutions to 7.3 Exercise

Answer 1

For $z_1 = -5 - 2i$

Think of this as “a clockwise half a turn then anticlockwise a bit”

$$arg(z_1) = -\pi + arctan\left(\frac{2}{5}\right)$$
$$= -\pi + 0.381$$
$$= -2.76^c$$

[2 marks]

For $z_2 = 3 - 5i$

Think of this as “clockwise a bit”

$$arg(z_2) = -arctan\left(\frac{5}{3}\right)$$
$$= -1.03^c$$

[2 marks]

For $z_3 = 5 + 5i$

Think of this as “anticlockwise a bit, where the bit is an eighth of a turn”

$$arg(z_3) = arctan\left(\frac{5}{5}\right)$$
$$= \frac{\pi}{4}^c \qquad \text{or} \qquad 0.785^c$$

[2 marks]

For $z_4 = -1 + 6i$

Think of this as “anticlockwise half a turn then clockwise a bit”

$$arg(z_4) = \pi - arctan\left(\frac{6}{1}\right)$$
$$= \pi - 1.41^c$$
$$= 1.74^c$$

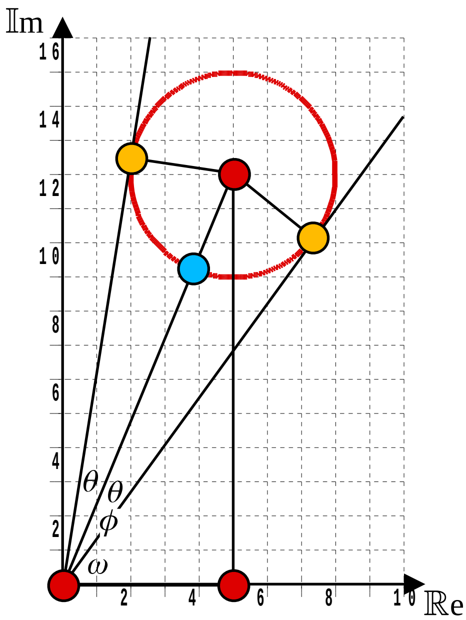
[2 marks]

Answer 2

(a)

$$|z - 5 - 12i| = 3$$
$$|x + yi - 5 - 12i| = 3$$
$$|(x - 5) + (y - 12)i| = 3$$
$$(x - 5)^2 + (y - 12)^2 = 9$$

This is in the form of a circle with centre (5, 12) and radius 3



[3 marks]

(iii) There is an integer Pythagorean triple triangle, 5, 12, 13

Thus, as the circle has radius 3.

$$|z|_{MIN} = 13 - 3 = 10$$
$$|z|_{MAX} = 13 + 3 = 16$$

[4 marks]

(iv)

$$\sin\theta = \frac{3}{13} \Rightarrow \theta = 0.233^c$$
$$\tan\phi = \frac{12}{5} \Rightarrow \phi = 1.176^c$$
$$\therefore [\arg z]_{MAX =} = \phi + \theta$$
$$= 1.409^c$$
$$[\arg z]_{MIN} = \omega$$
$$= \phi - \theta$$
$$= 0.943^c$$

[4 marks]

Answer 3

(i)

$$|z - 24 - 7\text{i}| = 5$$

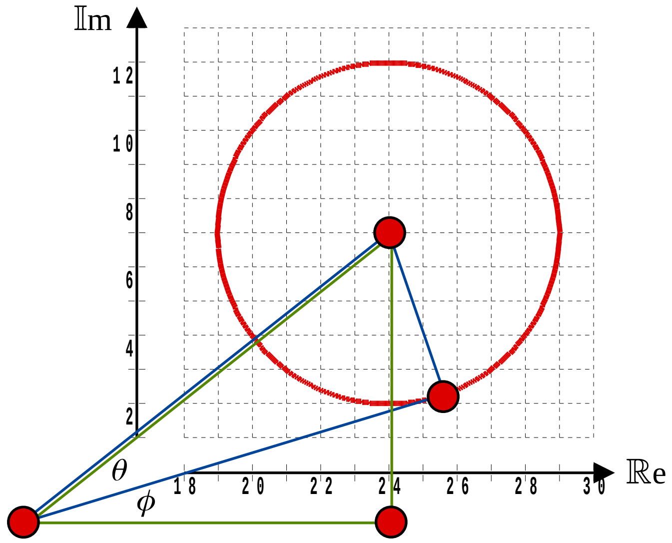
$$|x + y\text{i} - 24 - 7\text{i}| = 5$$

$$|(x - 24) + (y - 7)\text{i}| = 5$$

$$(x - 24)^2 + (y - 7)^2 = 25$$

[2 marks]

(ii) This is in the form of a circle with centre (24, 7) and radius 5



The circle is plotted accurately but the green and blue triangles have the origin as a common point which is far further to the left than shown.

[2 marks]

(iii) $|z|$ is the greatest distance from the origin to the circle.
There is an integer Pythagorean triple triangle, 24, 7, 25
Thus, as the circle has radius 5,

$$|z|_{\text{MIN}} = 25 + 5$$

$$= 30$$

[2 marks]

(iv) We're after the minimum anticlockwise rotation of the positive x-axis that is cutting the circle. Let $(\phi + \theta)$ be the angle in the Pythagorean integer triangle, shown green (but not to scale) above. Let θ be the angle in the blue triangle with a right angle where the circle's radius touches the tangent from the origin to the circle.

$$tan\ (\phi + \theta) = \frac{7}{24} \Rightarrow (\phi + \theta) = 0.2838^c$$

$$sin\ \theta = \frac{5}{25} \Rightarrow \theta = 0.2014^c$$

$$\therefore \phi = [\, arg\, z\,]_{\text{MIN}} = 0.0824^c$$

[3 marks]

Answer 4

$$z = \frac{3 + 5i}{2 - i} \times \frac{2 + i}{2 + i}$$

$$= \frac{6 + 3i + 10i + 5i^2}{4 + 2i - 2i - i^2}$$

$$= \frac{1 + 13i}{5}$$

$$= 0.2 + 2.6i$$

(i)

$$|z| = \sqrt{0.2^2 + 2.6^2}$$

$$= 2.61$$

[4 marks]

(ii)

$$arg\, z = arctan\left(\frac{2.6}{0.2}\right)$$

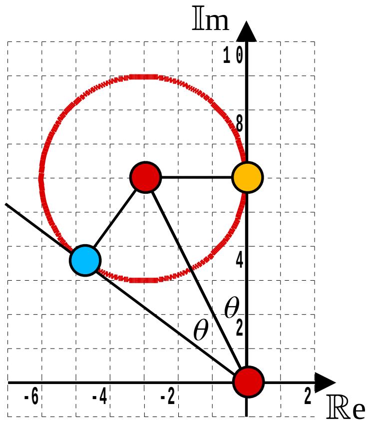
$$= 1.49^c$$

[2 marks]

Answer 5

$$\left| z + 3 - 6\mathrm{i} \right| = 3$$
$$\left| x + y\mathrm{i} + 3 - 6\mathrm{i} \right| = 3$$
$$\left| (x + 3) + (y - 6)\mathrm{i} \right| = 3$$
$$(x + 3)^2 + (y - 6)^2 = 9$$

This is in the form of a circle with centre (– 3, 6) and radius 3



We're after the maximum anticlockwise rotation of a line that started on the positive $\mathbb{R}$e-axis that is cutting the circle. This is the line, tangential to the circle on the diagram and meeting the circle at the point coloured blue.

Use can be made of the fact that the Imaginary axis is tangential to the circle and meeting the circle at the point coloured yellow.

A property of tangents to a circle from a common point is that the two angles marked θ are equal. So the required angle is a quarter turn anticlockwise plus two further turns anticlockwise of θ.

Thus,

$$\therefore\ [\, arg\, z\,]_{\text{MAX}} = \frac{\pi}{2} + 2\theta$$
$$= \frac{\pi}{2} + 2\,arcsin\left(\frac{3}{\sqrt{6^2 + 3^2}}\right)$$
$$= \frac{\pi}{2} + 2\,arcsin\left(\frac{3}{\sqrt{45}}\right)$$
$$= \frac{\pi}{2} + 2\,arcsin\left(\frac{3}{3\sqrt{5}}\right)$$
$$= \frac{\pi}{2} + 2\,arcsin\left(\frac{1}{\sqrt{5}}\right) \qquad \square$$

[4 marks]

Answer 6

(i)

$$
\begin{aligned}
|z| &= \sqrt{(-1)^2 + \left(\sqrt{3}\right)^2} \\
&= \sqrt{1 + 3} \\
&= \sqrt{4} \\
&= 2
\end{aligned}
$$

[1 mark]

(ii)

$$
\begin{aligned}
\left|\frac{z}{z^*}\right| &= \left|\frac{-1 - \sqrt{3}\,\mathrm{i}}{-1 + \sqrt{3}\,\mathrm{i}}\right| \\
&= \left|\frac{-1 - \sqrt{3}\,\mathrm{i}}{-1 + \sqrt{3}\,\mathrm{i}} \times \frac{-1 - \sqrt{3}\,\mathrm{i}}{-1 - \sqrt{3}\,\mathrm{i}}\right| \\
&= \left|\frac{(-1)\left(1 + \sqrt{3}\,\mathrm{i}\right)\ (-1)\ \left(1 + \sqrt{3}\,\mathrm{i}\right)}{1 + \sqrt{3}\,\mathrm{i} - \sqrt{3}\,\mathrm{i} - 3\,\mathrm{i}^2}\right| \\
&= \left|\frac{1 + 2\sqrt{3}\,\mathrm{i} + 3\,\mathrm{i}^2}{4}\right| \\
&= \left|\frac{-2 + 2\sqrt{3}\,\mathrm{i}}{4}\right| \\
&= \left| -\frac{1}{2} + \frac{\sqrt{3}}{2}\,\mathrm{i} \right| \\
&= \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \\
&= \sqrt{\left(\frac{1}{4}\right) + \left(\frac{3}{4}\right)} \\
&= \sqrt{1} \\
&= 1
\end{aligned}
$$

[4 marks]

(iii)

$$\begin{aligned} arg\, z &= -\pi + arctan\left(\frac{\sqrt{3}}{1} \right) \\ &= -\pi + \frac{\pi}{3} \\ &= -\frac{2}{3}\pi \end{aligned}$$

$$\begin{aligned} z^* &= -1 + \sqrt{3}\ \mathrm{i} \\ arg\left(z^*\right) &= \pi - arctan\left(\frac{\sqrt{3}}{1} \right) \\ &= \pi - \frac{\pi}{3} \\ &= \frac{2}{3}\pi \end{aligned}$$

$$\begin{aligned} arg\left(\frac{z}{z^*}\right) &= \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\ \mathrm{i} \right) \\ &= \frac{2}{3}\pi \end{aligned}$$

[3 marks]

Answer 7

(i)

$$|w| = |6 + 3\text{i}|$$
$$= \sqrt{6^2 + 3^2}$$
$$= \sqrt{45}$$
$$= 3\sqrt{5}$$

[1 mark]

(ii)

$$\arg z = \arg(6 + 3\text{i})$$
$$= \arctan\left(\frac{3}{6}\right)$$
$$= 0.464^c$$

[2 marks]

(iii) The angle puts the point in the first quadrant of the complex plane which makes the question easier than it might have been !

$$\arg(\lambda + 5\text{i} + w) = \frac{\pi}{4}$$
$$\arg(\lambda + 5\text{i} + 6 + 3\text{i}) = \frac{\pi}{4}$$
$$\arctan\left(\frac{8}{\lambda + 6}\right) = \frac{\pi}{4}$$
$$\left(\frac{8}{\lambda + 6}\right) = 1$$
$$8 = \lambda + 6$$
$$\lambda = 2$$

[2 marks]

8.1 When Circles Collide

Consider the following complex number equation with a view to plotting its locus on an Argand diagram,

$$|z + 2 - i| = |z - 4 + i|$$

From lesson 7 it is clear that,

$$|z + 2 - i| = \lambda$$

or, equivalently, $|z - (-2 + i)| = \lambda$

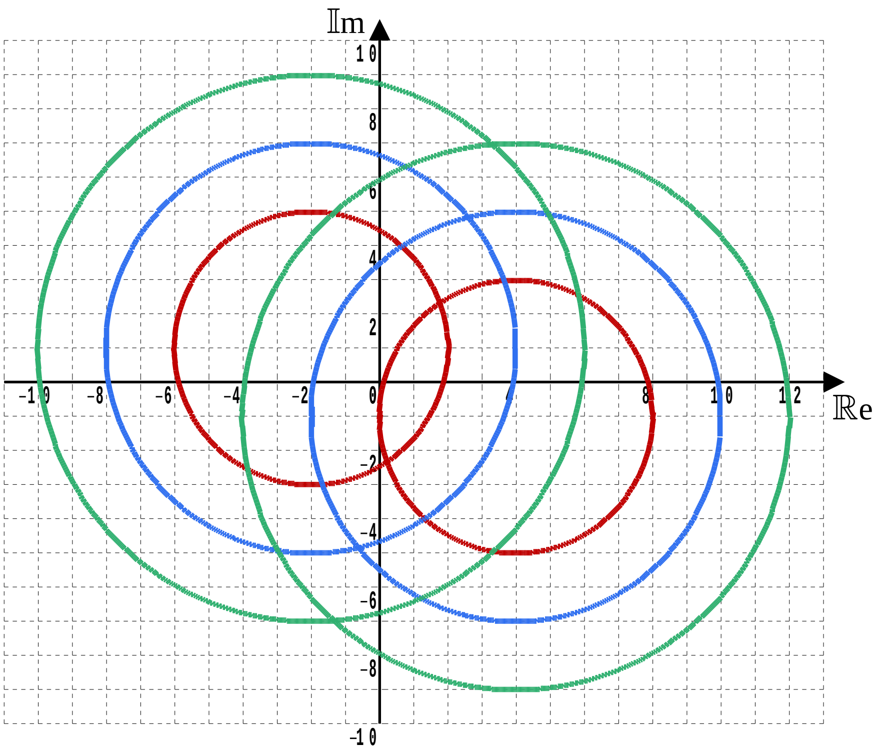
describes a circle with centre $(-2,\ 1)$ and radius λ on an Argand diagram.
Similarly,

$$|z - 4 + i| = \lambda$$

or, equivalently, $|z - (4 - i)| = \lambda$

describes a circle with centre $(4, -1)$ and radius λ on an Argand diagram.

Equating the two can be thought of as generating a sequence of circles of the same radius, but with the radius increasing in steps, say $\lambda = 4, 6, 8, \ldots$



The red circles are of radius 4, the blue of radius 6 and the green of radius 8.
The points of interest are where red meets red, blue meets blue and green meets green.

The Key Question
When those points of intersection are joined up what geometric shape is obtained ?

8.2 The Perpendicular Bisector

The quick answer to the key question is “a straight line”. However, more is true. From geometric work with a compass and straight edge, it can be seen that drawing a single pair of circles in this way is the method of obtaining a perpendicular bisector. The locus described by

$$|z + 2 - \mathrm{i}| = |z - 4 + \mathrm{i}|$$

is a line that cuts the line segment with endpoints at circle centres $(-2,\ 1)$ and $(4,\ -1)$ exactly in half and at right angles.

The Cartesian equation of this perpendicular bisector could be approximately read off from the intersecting circles diagram. Alternatively, as this very problem was tackled in the Year 1 course, we can use the mathematics already developed there.

Year 1 Revision Question

Find the Cartesian equation of the perpendicular bisector of line segment with endpoints $(-2,\ 1)$ and $(4,\ -1)$

Teaching Video : http://www.NumberWonder.co.uk/Video/v9085(8).mp4

[3 marks]

8.3 An Alternative Method

The Cartesian equation of the perpendicular bisector described by,

$$|z + 2 - \mathrm{i}| = |z - 4 + \mathrm{i}|$$

can also be found by the following alternative method which uses the technique developed in lesson 7.

$$|z + 2 - \mathrm{i}| = |z - 4 + \mathrm{i}|$$

$$|x + y\mathrm{i} + 2 - \mathrm{i}| = |x + y\mathrm{i} - 4 + \mathrm{i}|$$

$$|(x + 2) + (y - 1)\mathrm{i}| = |(x - 4) + (y + 1)\mathrm{i}|$$

Take the modulus of each side, and square to remove the square roots

$$(x + 2)^2 + (y - 1)^2 = (x - 4)^2 + (y + 1)^2$$

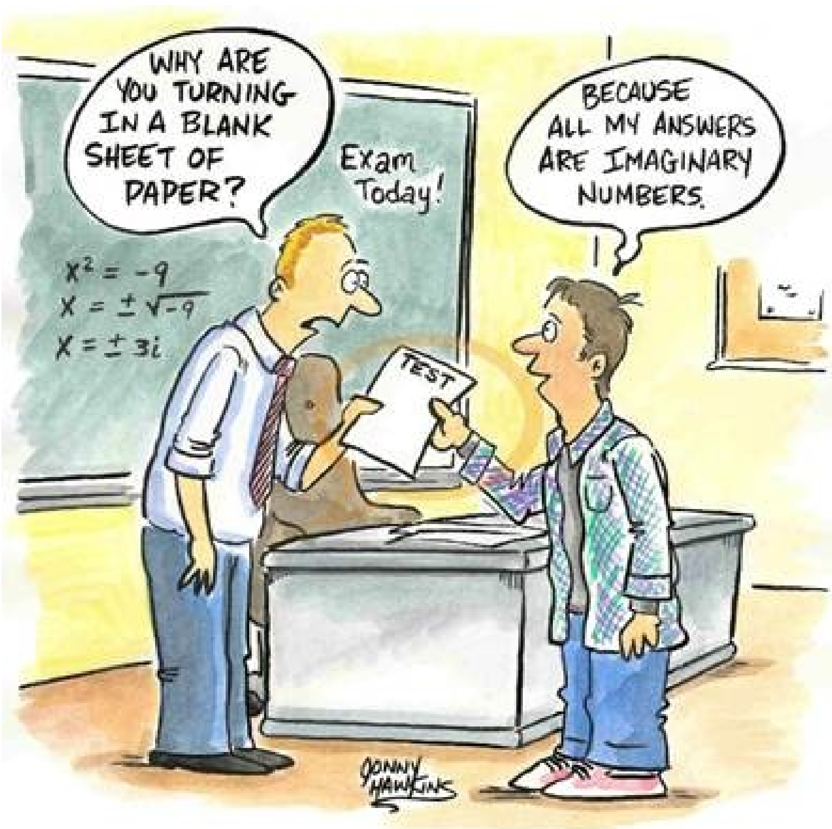
$$x^2 + 4x + 4 + y^2 - 2y + 1 = x^2 - 8x + 16 + y^2 + 2y + 1$$

$$4x - 2y + 5 = -8x + 2y + 17$$

$$12x - 12 = 4y$$

$$y = 3x - 3$$

This method will be required in the Exercise in Question 4 to get the locus of P and in Question 5 to get the requested equation of a circle.



8.4 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 50

Question 1

(a) Write down the coordinates of the centre of the circle described by

$$|z - 10 - 5i| = \lambda \qquad z \in \mathbb{C},\ \lambda \in \mathbb{R}$$

[1 mark]

(b) Write down the coordinates of the centre of the circle described by

$$|z + 6 - i| = \lambda \qquad z \in \mathbb{C},\ \lambda \in \mathbb{R}$$

[1 mark]

(c) Hence, or otherwise, determine the Cartesian equation of the locus of z when

$$|z - 10 - 5i| = |z + 6 - i|$$

[3 marks]

(d) What will be the shortest distance between the origin and the locus found in part (c) ? In other words, what is the minimum value of $|z|$?

A sketch of the locus on an Argand diagram may help !

[3 marks]

Question 2

Further A-Level Examination Question from June 2010, FP2, Q6 (edited)

A complex number z is represented by the point P in the Argand diagram.

(a) Given that,

$$|z - 6| = |z|$$

sketch the locus of P

[2 marks]

Another complex number w is represented by the point Q in the Argand diagram.

(b) Given that,

$$|w - 3 - 4\mathrm{i}| = 5$$

add a sketch of the locus of Q to your part (a) sketch

[2 marks]

(c) Find the complex numbers which satisfy both

$$|z - 6| = |z| \qquad \text{and} \qquad |w - 3 - 4\mathrm{i}| = 5$$

[2 marks]

Question 3

Sketch on the same Argand diagram the locus of points satisfying,

(a) $|z - 2\mathrm{i}| = |z - 8\mathrm{i}|$

[2 marks]

(b) $|z - 6 - 2\mathrm{i}| = 5$

[3 marks]

The complex number z satisfies both,

$$|z - 2\mathrm{i}| = |z - 8\mathrm{i}|$$

$$\text{and} \quad |z - 6 - 2\mathrm{i}| = 5$$

(c) Use your answers to parts (a) and (b) to find the two values of z

[2 marks]

Question 4

Further A-Level Examination Question from June 2014, F2, Q7

The point P represents a complex number z on an Argand diagram, where

$$|z + 1| = |2z - 1|$$

and the point Q represents a complex number w on the Argand diagram, where

$$|w| = |w - 1 + \mathrm{i}|$$

Find the exact coordinates of the points where the locus of P intersects the locus of Q

[7 marks]

Question 5

Further A-Level Exam Question from SAM, June 2017, FM Option 2, Paper 4, Q6

A curve has equation,

$$|z + 6| = 2|z - 6| \qquad z \in \mathbb{C}$$

(a) Show that the curve is a circle with equation

$$x^2 + y^2 - 20x + 36 = 0$$

[2 marks]

(b) Sketch the curve on an Argand diagram

[2 marks]

8.5.1 Solution to 8.2 The Perpendicular Bisector

To find the perpendicular bisector of the line segment with endpoints $(-2,\ 1)$ and we can first find the midpoint between those endpoints,

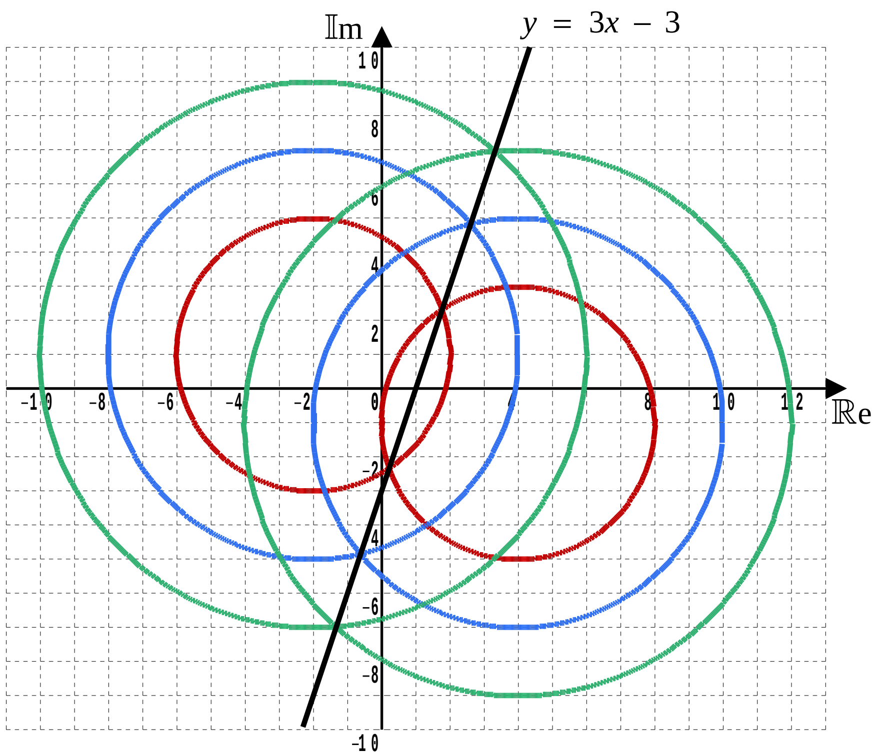
$$\begin{aligned}(\text{mean } x,\ \text{mean } y) &= \left(\frac{-2+4}{2},\ \frac{1+(-1)}{2}\right)\\ &= (1,\ 0)\end{aligned}$$

The gradient of the line between the two circle centres will be,

$$\begin{aligned}m &= \frac{\Delta Y}{\Delta X}\\ &= \frac{(-1)-(1)}{(4)-(-2)}\\ &= \frac{-2}{6}\\ &= -\frac{1}{3}\end{aligned}$$

The line perpendicular to this will have a gradient that is "the sign changed reciprocal". Thus we're after a line with gradient 3 through (1, 0),

$$\begin{aligned}y &= mx + c\\ 0 &= 3 \times 1 + c \Rightarrow c = -3\\ \therefore\ y &= 3x - 3\end{aligned}$$



[3 marks]

8.5.2 Solutions to 8.4 Exercise

Answer 1

(a) Centre is (10, 5)

[1 mark]

(b) Centre is (– 6, 1)

[1 mark]

(c)

$$\text{Midpoint}\left(\frac{10 + (-6)}{2}, \frac{5 + 1}{2}\right) = (2,\ 3)$$

$$m = \frac{\Delta Y}{\Delta X}$$

$$= \frac{5 - 1}{10 - (-6)}$$

$$= \frac{4}{16}$$

$$= \frac{1}{4} \qquad \therefore m_{PERP} = -4$$

$$y = mx + c$$

$$3 = -4 \times 2 + c \Rightarrow c = 11$$

Cartesian Equation of perpendicular bisector is $y = -4x + 11$

[3 marks]

(d) The gradient of the shortest line will be 0.25 and it has to go through (0, 0) so it will have equation,

$$y = 0.25x$$

To get the point of intersection with the perpendicular bisector,

$$0.25x = -4x + 11$$

$$x = -16x + 44$$

$$17x = 44$$

$$x = \frac{44}{17} \quad \therefore y = \frac{11}{17}$$

$$\text{Minimum Distance} = \sqrt{\left(\frac{44}{17}\right)^2 + \left(\frac{11}{17}\right)^2}$$

$$= \frac{11\sqrt{17}}{17} \qquad (2.66789...)$$

[3 marks]

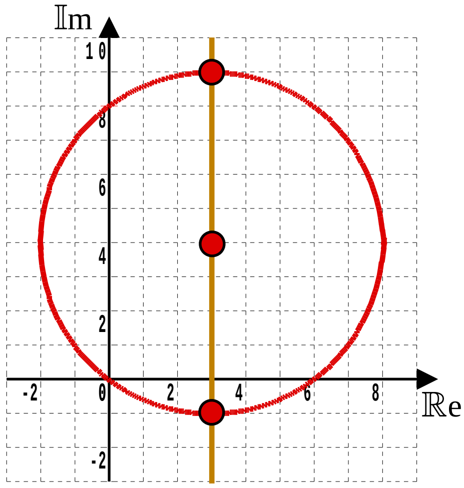
Answer 2

(a) $|z - 6|$ can be thought of as a circle of unspecified radius, but centre (6, 0)

$|z|$ can be thought of as a circle of unspecified radius, but centre (0, 0)

The locus will be the perpendicular bisector of the line segment between these two point, which will be $x = 3$

(b) $|w - 3 - 4\mathrm{i}| = 5$ can be thought of an a circle of radius 5, centre (3, 4)



(c) The complex numbers on both loci are,

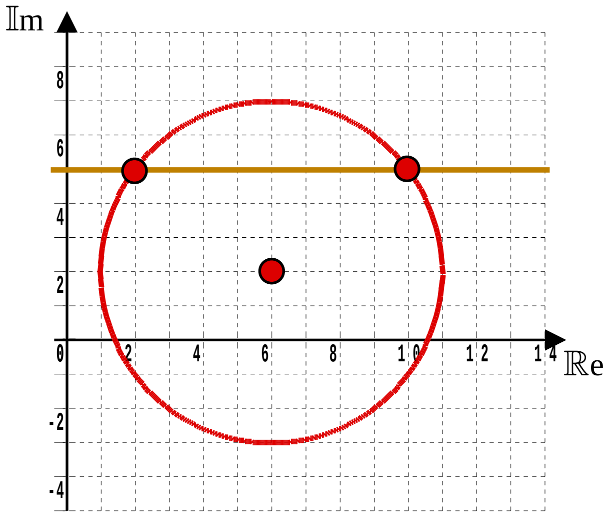
$$z_1 = w_1 = 3 - \mathrm{i} \qquad z_2 = w_2 = 3 + 9\mathrm{i}$$

[2, 2, 2 marks]

Answer 3

(a) $|z - 2\mathrm{i}| = |z - 8\mathrm{i}|$ will have locus that is the perpendicular bisector of the line segment between (0, 2) and (0, 8) which are the centres of the circles described by $|z - 2\mathrm{i}| = \lambda$ and $|z - 8\mathrm{i}| = \lambda$ where λ is a variable radius. Without calculation, this will be obviously be the line $y = 5$

(b) $|z - 6 - 2\mathrm{i}| = 5$ is a circle radius 5, centre (6, 2) on the Argand diagram.



(c) The complex numbers on both loci are

$$z_1 = 2 + 5\mathrm{i} \qquad \text{and} \qquad z_2 = 10 + 5\mathrm{i}$$

[2, 3, 2 marks]

Answer 4

For P a cautious approach is required,

$$|z + 1| = |2z - 1|$$

$$|x + y\text{i} + 1| = |2(x + y\text{i}) - 1|$$

$$|(x + 1) + y\text{i}| = |(2x - 1) + 2y\text{i}|$$

Take the modulus of each side, and square to remove the square roots

$$(x + 1)^2 + y^2 = (2x - 1)^2 + (2y)^2$$

$$x^2 + 2x + 1 + y^2 = 4x^2 - 4x + 1 + 4y^2$$

$$3x^2 - 6x + 3y^2 = 0$$

Divide through by 3

$$\left[x^2 - 2x\right] + y^2 = 0$$

$$\left[(x - 1)^2 - 1\right] + y^2 = 0$$

$$(x - 1)^2 + y^2 = 1$$

This is the equation of a circle, centre (1, 0) and radius 1

For Q it's the perpendicular bisector of the line segment between (0, 0) and (1, − 1) that is required which will be the line $y = x - 1$

Solving by substitution,

$$(x - 1)^2 + y^2 = 1$$

$$(x - 1)^2 + (x - 1)^2 = 1$$

$$2(x - 1)^2 = 1$$

$$(x - 1)^2 = \frac{1}{2}$$

$$x - 1 = \pm\frac{1}{\sqrt{2}}$$

$$x - 1 = \pm\frac{\sqrt{2}}{2}$$

$$x = 1 \pm \frac{\sqrt{2}}{2}$$

$$x = 1 - \frac{\sqrt{2}}{2} \quad \text{or} \quad x = 1 + \frac{\sqrt{2}}{2}$$

Points of intersection are

$$\left(1 - \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) \quad \text{or} \quad \left(1 + \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$

[7 marks]

Answer 5

(a)

$$|z + 6| = 2|z - 6|$$

$$|x + yi + 6| = 2|x + yi - 6|$$

$$|(x + 6) + yi| = 2|(x - 6) + yi|$$

Take the modulus of each side, and square to remove the square roots

$$(x + 6)^2 + y^2 = 4\left[(x - 6)^2 + y^2\right]$$

$$x^2 + 12x + 36 + y^2 = 4\left[x^2 - 12x + 36 + y^2\right]$$

$$x^2 + 12x + 36 + y^2 = 4x^2 - 48x + 144 + 4y^2$$

$$3x^2 - 60x + 3y^2 + 108 = 0$$

Divide through by 3

$$x^2 + y^2 - 20x + 36 = 0 \qquad \square$$

[2 marks]

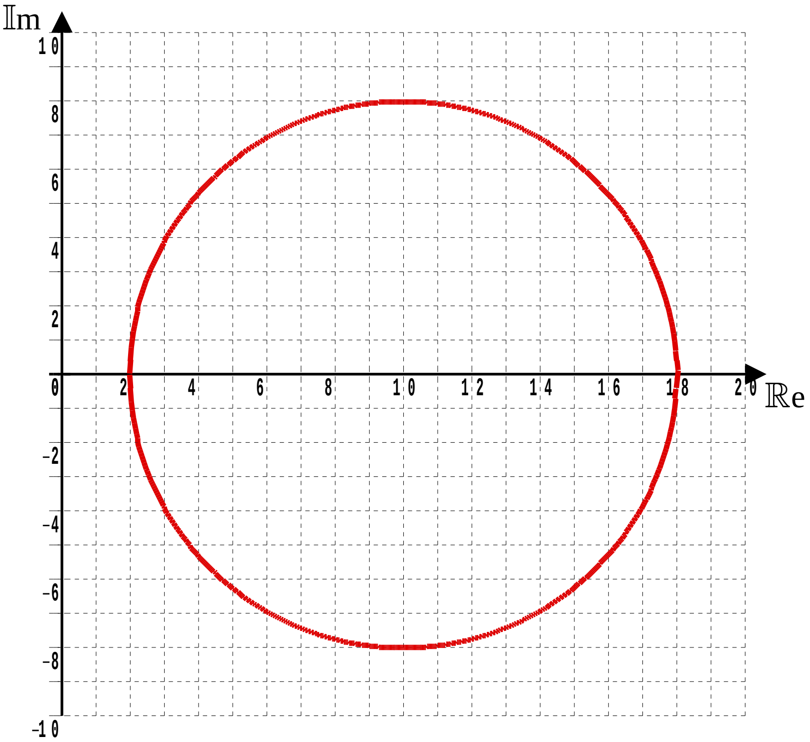
(b)

$$\left[x^2 - 20x\right] + y^2 + 36 = 0$$

$$\left[(x - 10)^2 - 100\right] + y^2 + 36 = 0$$

$$(x - 10)^2 + y^2 = 64$$

Which is recognizable as a circle, centre (10, 0) and radius 8



[2 marks]

Lesson 9

9.1 A Catalogue of Loci

Over the last few lessons, a first steps towards developing a catalogue of loci on Argand diagrams have been taken. It has thus far only involved two shapes, two objects familiar from previous study.

Generalising previous specific examples gives a catalogue with two entries;

- $|z - (a + b\text{i})| = r$ where $a, b, r \in \mathbb{R}$, and $r > 0$

  A circle with centre (a, b) and radius r

- $|z - (a + b\text{i})| = |z - (c + d\text{i})|$ where $a, b, c, d \in \mathbb{R}$

  A line that is the perpendicular bisector of the line segment between the points (a, b) and (c, d)

9.2 The Argument as a Locus

This lesson a third entry to the catalogue will be developed.

Given that $arg(z - 3 + 2\text{i}) = \dfrac{2\pi}{3}$ give the Cartesian equation of the locus and sketch the locus of z on an Argand diagram.

Teaching Video : http://www.NumberWonder.co.uk/Video/v9085(9).mp4

[4 marks]

9.3 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 33

Question 1

Given that $arg(z - 2 + 3\text{i}) = \dfrac{3\pi}{4}$ give the Cartesian equation of the locus and sketch the locus of z on an Argand diagram.

[4 marks]

Question 2

Given that $arg(z - 3 - 4\text{i}) = -\dfrac{2\pi}{3}$ give the Cartesian equation of the locus and sketch the locus of z on an Argand diagram.

[4 marks]

Question 3

Further A-Level Examination Question from June 2005, FP2, Q9

A complex number z is represented by the point P in the Argand diagram.
Given that,

$$|z - 3\text{i}| = 3$$

(a) sketch the locus of P

[2 marks]

(b) Find the complex number z which satisfies both

$$|z - 3\text{i}| = 3 \quad \text{and} \quad arg(z - 3\text{i}) = \frac{3\pi}{4}$$

[4 marks]

Question 4

Given that z satisfies,

$$\left| z + \sqrt{3}\,\mathrm{i} \right| = 3$$

(a) sketch the locus of z on an Argand diagram

[2 marks]

(b) find z that satisfies both $\left| z + \sqrt{3}\,\mathrm{i} \right| = 3$ and $arg\,(z) = \dfrac{\pi}{6}$

[4 marks]

Question 5

Given that

$$arg(z + 4) = \frac{\pi}{3}$$

(a) sketch the locus of z on an Argand diagram

[2 marks]

(b) find the minimum value of $|z|$ for points on this locus

[4 marks]

Question 6

Sketch on the same Argand diagram the locus of points satisfying,

(a) $|z - 2\mathrm{i}| = |z - 8\mathrm{i}|$

[2 marks]

(b) $arg(z - 2 - \mathrm{i}) = \frac{\pi}{4}$

[3 marks]

The complex number z satisfies both

$$|z - 2\mathrm{i}| = |z - 8\mathrm{i}| \quad \text{and} \quad arg(z - 2 - \mathrm{i}) = \frac{\pi}{4}$$

(c) Use your answers to parts (a) and (b) to find the value of z

[2 marks]

9.4 Answers

9.4.1 Solution to 9.2 Teaching Video Example

Keep in mind that, $arg\,(z) = 0$ is a “half-line” starting at (but excluding) the point at the origin (which is undefined) and heading off in the positive x-axis direction. Questions involving arguments can often be visualised immediately by utilising a knowledge of graph transformations applied to this half-line.

It helps to have a sense of how the question should progress !

In this case, it's apparent from the following rewrite;

$$arg(z - 3 + 2\mathrm{i}) = \frac{2\pi}{3}$$

$$arg(z - (3 - 2\mathrm{i})) = \frac{2\pi}{3}$$

that the half-line starts at the point $(3, -2)$ and heads off at 120°

To obtain the Cartesian equation,

$$arg(z - 3 + 2\mathrm{i}) = \frac{2\pi}{3}$$

$$arg(x + y\mathrm{i} - 3 + 2\mathrm{i}) = \frac{2\pi}{3}$$

$$arg((x - 3) + (y + 2)\mathrm{i}) = \frac{2\pi}{3}$$

(gathering reals together and, separately, imaginaries together)

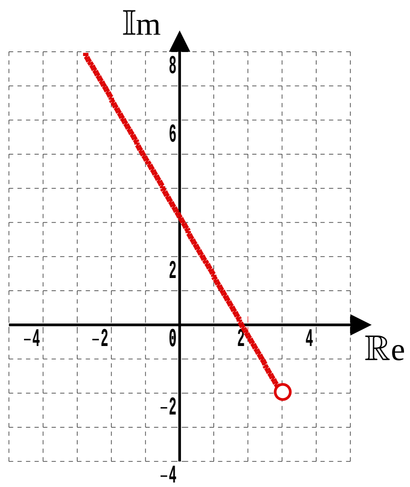
$$\frac{y + 2}{x - 3} = tan\left(\frac{2\pi}{3}\right)$$

$$\frac{y + 2}{x - 3} = -\sqrt{3}$$

$$y + 2 = -\sqrt{3}\,(x - 3)$$

$$y = -\sqrt{3}x + 3\sqrt{3} - 2 \qquad x < 3$$

(For plotting : $y = -1.73x + 3.20$)



[4 marks]

9.4.2 Solutions to 9.3 Exercise

Answer 1

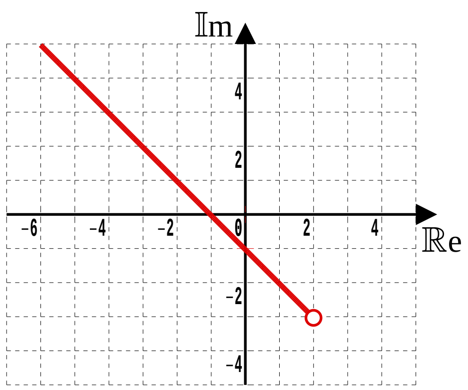
To obtain the Cartesian equation,

$$\begin{aligned}
arg(z - 2 + 3i) &= \frac{3\pi}{4} \\
arg(x + yi - 2 + 3i) &= \frac{3\pi}{4} \\
arg((x - 2) + (y + 3)i) &= \frac{3\pi}{4} \\
\frac{y + 3}{x - 2} &= tan\left(\frac{3\pi}{4}\right) \\
\frac{y + 3}{x - 2} &= -1 \\
y + 3 &= -(x - 2) \\
y &= -x - 1, \qquad x < 2
\end{aligned}$$

With regards to the sketch,

$$arg(z - 2 + 3i) = \frac{3\pi}{4} \Rightarrow arg(z - (2 - 3i)) = \frac{3\pi}{4}$$

so the start of the half-line is at $(2, -3)$ and it heads off at 135°



[4 marks]

Answer 2

To obtain the Cartesian equation,

$$arg\,(z - 3 - 4\mathrm{i}) = -\frac{2\pi}{3}$$

$$arg\,(x + y\mathrm{i} - 3 - 4\mathrm{i}) = -\frac{2\pi}{3}$$

$$arg\,((x - 3) + (y - 4)\mathrm{i}) = -\frac{2\pi}{3}$$

$$\frac{y - 4}{x - 3} = tan\left(-\frac{2\pi}{3}\right)$$

$$\frac{y - 4}{x - 3} = \sqrt{3}$$

$$y - 4 = \sqrt{3}\,(x - 3)$$

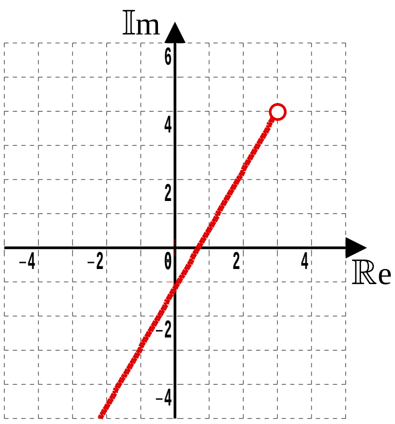
$$y = \sqrt{3}\,x + 4 - 3\sqrt{3} \qquad x < 3$$

(To plot this line use $y = 1.73x - 1.20$)

With regards to the sketch,

$$arg\,(z - 3 - 4\mathrm{i}) = -\frac{2\pi}{3} \Rightarrow arg\,(z - (3 + 4\mathrm{i})) = -\frac{2\pi}{3}$$

so the start of the half-line is at (3, 4) and it heads off at – 120°



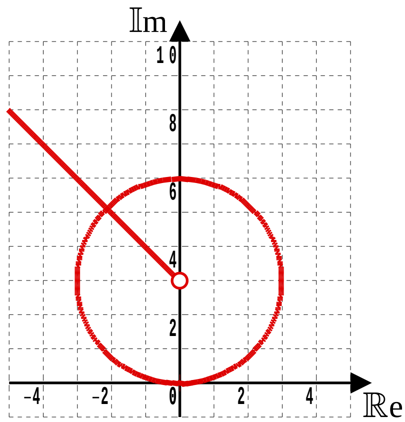
[4 marks]

Answer 3

(a) $|z - 3i| = 3$ is a circle with centre (0, 3) and radius 3

[2 marks]

(b) $arg(z - 3i) = \frac{3\pi}{4}$ is a half-line, starting at (0, 3) and heading off at 135°



Circle : $x^2 + (y - 3)^2 = 9$

Line : $y = -x + 3$

$$x^2 + ((-x + 3) - 3)^2 = 9$$

$$x^2 + (-x)^2 = 9$$

$$2x^2 = 9$$

$$x^2 = \frac{9}{2}$$

$$x = -\frac{3}{\sqrt{2}}$$ (Reject positive solution)

$$x = -\frac{3\sqrt{2}}{2}$$

$$\therefore y = 3 + \frac{3\sqrt{2}}{2}$$

$$z = -\frac{3\sqrt{2}}{2} + \left(3 + \frac{3\sqrt{2}}{2}\right)i$$

[4 marks]

Answer 4

(a) $\left| z + \sqrt{3}\,\mathrm{i} \right| = 3$ is a circle with centre (0, $-\sqrt{3}$) and radius 3

[2 marks]

(b) $arg\,(z) = \dfrac{\pi}{6}$ is a half-line, starting at (0, 0) and heading off at 30°

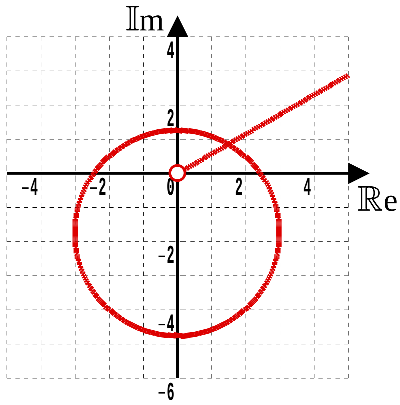
To obtain the Cartesian equation, $arg\,(z) = \dfrac{\pi}{6}$

$$arg\,(x + y\mathrm{i}) = \frac{\pi}{6}$$

$$\frac{y}{x} = tan\left(\frac{\pi}{6} \right)$$

$$\frac{y}{x} = \frac{1}{\sqrt{3}}$$

$$y = \frac{1}{\sqrt{3}}\,x \qquad x > 0$$



Circle : $x^2 + \left(y + \sqrt{3} \right)^2 = 16$ Line : $y = \dfrac{1}{\sqrt{3}}\,x$ or $y = \dfrac{\sqrt{3}}{3}\,x$

$$x^2 + \left(\frac{1}{\sqrt{3}}\,x + \sqrt{3} \right)^2 = 9$$

$$x^2 + \frac{1}{3}x^2 + 2\,x + 3 = 9$$

$$3\,x^2 + x^2 + 6\,x + 9 = 27$$

$$4\,x^2 + 6\,x - 18 = 0$$

$$2\,x^2 + 3x - 9 = 0$$

$$x = \frac{-3 \pm \sqrt{3^2 - 4 \times 2 \times (-9)}}{2 \times 2}$$

$$= \frac{-3 \pm 9}{4} = \frac{3}{2} \qquad \text{(Reject solution of } -3 \text{)}$$

$$\therefore\; z = \frac{3}{2} + \frac{\sqrt{3}}{2}\,\mathrm{i}$$

[4 marks]

Answer 5

$arg\,(\,z\,+\,4\,)\;=\;\dfrac{\pi}{3}$ is a half-line, starting at $(-4,\;0\,)$ and heading off at 60°

[2 marks]

To obtain the Cartesian equation, $arg\,(\,z\,+\,4\,)\;=\;\dfrac{\pi}{3}$

$$arg\,(\,x\,+\,y\mathrm{i}\,+\,4\,)\;=\;\frac{\pi}{3}$$

$$arg\,(\,x\,+\,4\,+\,y\mathrm{i}\,)\;=\;\frac{\pi}{3}$$

$$\frac{y}{x\,+\,4}\;=\;tan\left(\frac{\pi}{3}\right)$$

$$\frac{y}{x\,+\,4}\;=\;\sqrt{3}$$

$$y\;=\;\sqrt{3}\,x\,+\,4\sqrt{3}\qquad x\;>\;-4$$

Thus, the perpendicular line through the origin is, $y\;=\;-\dfrac{1}{\sqrt{3}}\,x$

These two lines will meet when, $\sqrt{3}\,x\,+\,4\sqrt{3}\;=\;-\dfrac{1}{\sqrt{3}}\,x$

Multiply through by $\sqrt{3}$

$$3x\,+\,12\;=\;-x$$

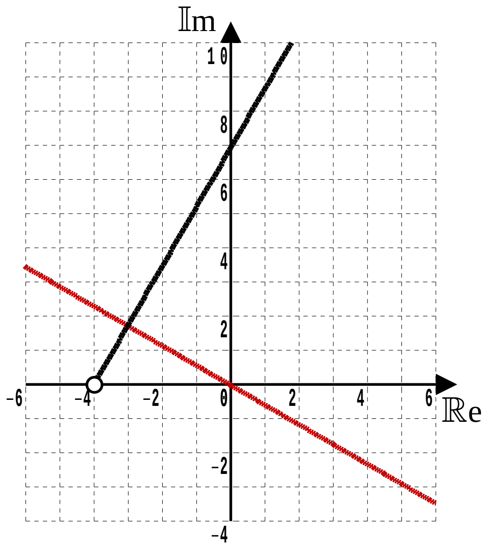
$$4x\;=\;-12$$

$$x\;=\;-3\qquad\text{and so}\qquad y\;=\;\sqrt{3}$$

To get the distance from the origin,

$$|\,z\,|_{\text{MIN}}\;=\;\sqrt{(-3)^2\,+\,\left(\sqrt{3}\,\right)^2}$$

$$=\;2\sqrt{3}$$



[4 marks]

Answer 6

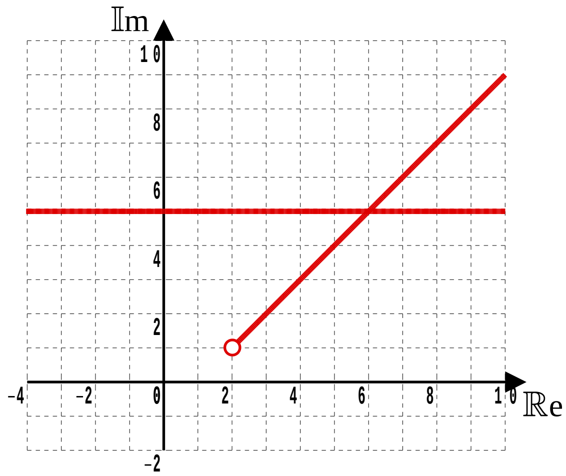
(a) $|z - 2i| = |z - 8i|$ is the perpendicular bisector of the line segment between (0, 2) and (0, 8) which is the line $y = 5$

[2 marks]

(b) $arg(z - 2 - i) = \frac{\pi}{4}$ is a half-line, starting at (2, 1), heading off at 45°

The equation of this line will be $y = x - 1$

[3 marks]



(c)

$$z = 6 + 5i$$

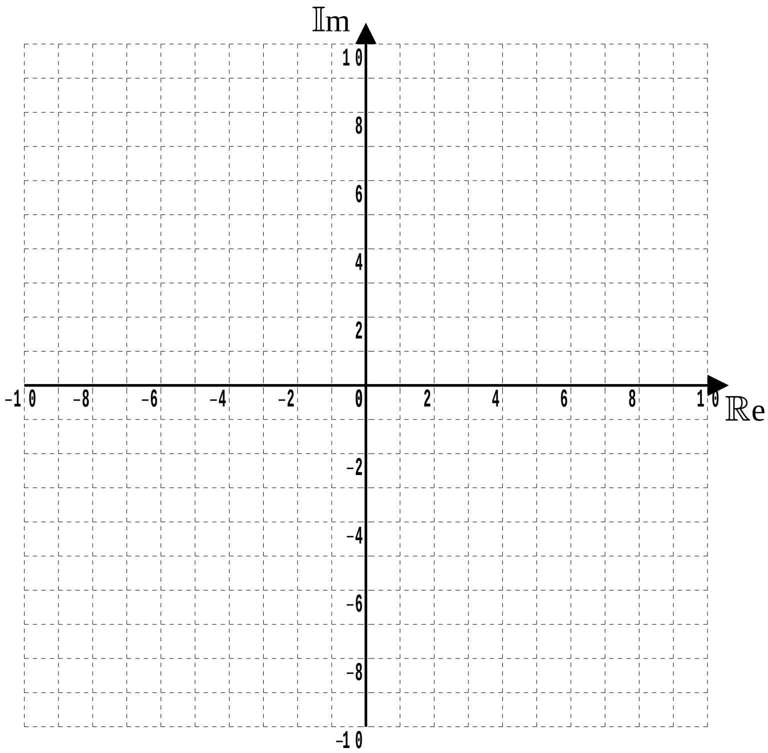
[2 marks]

10.1 Regions and Argand Diagrams

On the Argand diagram, shade in the region specified by,

$$\{\, 3 < |z - 3 + 5\mathrm{i}| \leqslant 5 \,\} \cap \left\{ -\frac{\pi}{2} \leqslant \arg(z - 3 + 5\mathrm{i}) < \frac{3\pi}{4} \right\}$$

Teaching Video : http://www.NumberWonder.co.uk/Video/v9085(10).mp4



[4 marks]

10.2 Exercise

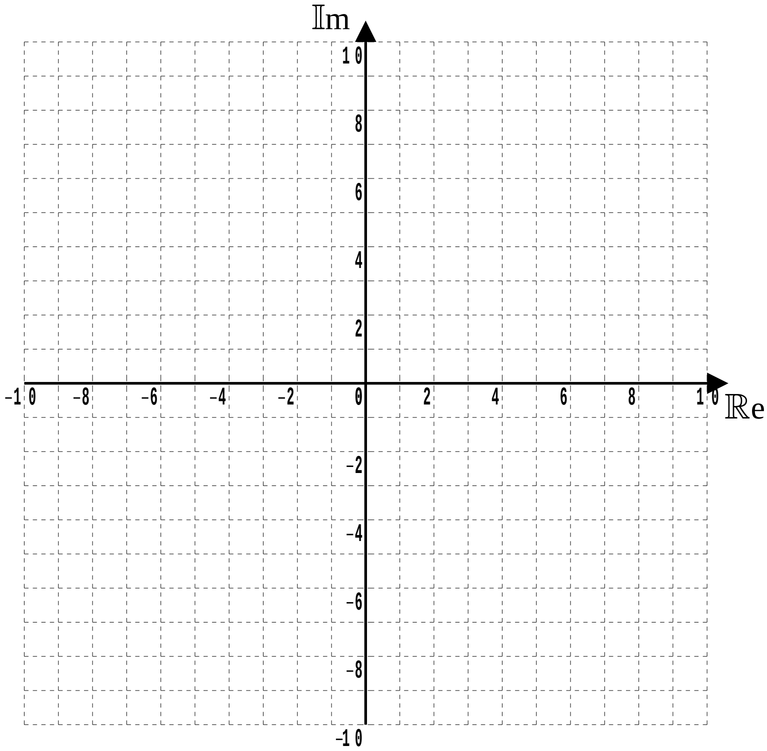
Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 32

Question 1

(i) On the Argand diagram, shade in the region, R, specified by,

$$\{ 3 \leqslant |z + 3 - 4\mathrm{i}| < 5 \} \cap \left\{ -\frac{\pi}{3} < arg\,(z + 3 - 4\mathrm{i}) \leqslant \frac{\pi}{3} \right\}$$



[4 marks]

(ii) Show how to use the formula $Arc\ length = r\theta^{c}$ to determine the ***exact*** perimeter of the region R

[2 marks]

(iii) Show how to use the formula $Sector\ area = \dfrac{1}{2} r^2 \theta^{c}$ to determine the ***exact*** area of the region R

[2 marks]

Question 2

Further A-Level SAM Question for Core Pure Mathematics 2, June 2020, Q6 (b) (i)

The set of points A is defined by

$$A = \left\{ z \ : \ 0 \leqslant arg\, z \leqslant \frac{\pi}{3} \right\} \cap \{ z : |z - 4 - 3\mathrm{i}| \leqslant 5 \}$$

Show, by shading on an Argand diagram, the set of points A

[2 marks]

Question 3

The complex number z is represented by a point P on an Argand diagram.

Given that $|z + 4 - 4\mathrm{i}| \leqslant 4$ and $\frac{\pi}{2} \leqslant arg(z) \leqslant \frac{3\pi}{4}$ shade the locus of P

[2 marks]

Question 4

Further A-Level Examination Question from June 2003, FP2, Q1 (edited)

(a) On the same Argand diagram sketch the loci given by the following equations,

$$|z - 1| = 1 \qquad arg(z + 1) = \frac{\pi}{12} \qquad arg(z + 1) = \frac{\pi}{2}$$

[4 marks]

(b) Shade on your diagram the region for which

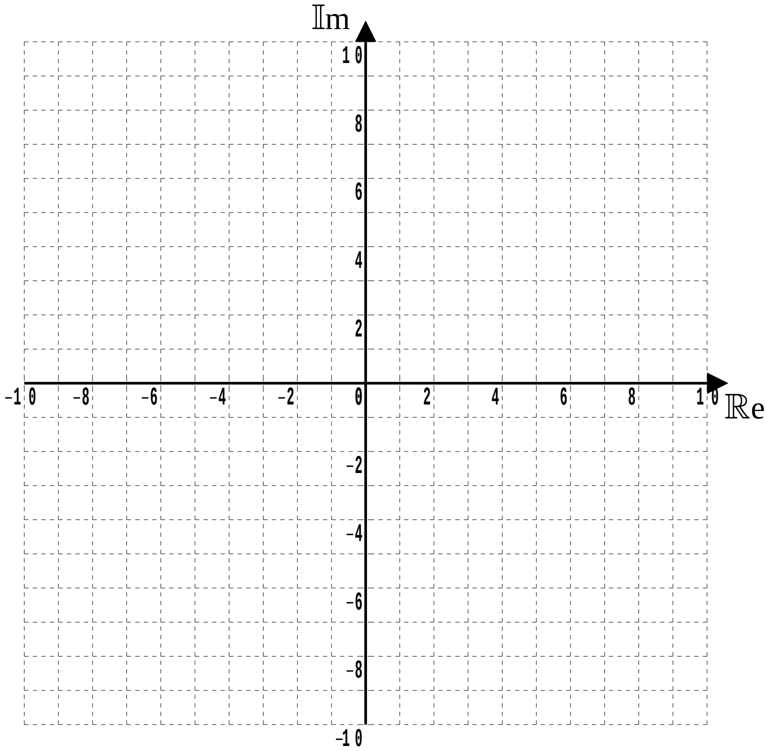
$$|z - 1| \leqslant 1 \quad \text{and} \quad \frac{\pi}{12} \leqslant arg(z + 1) \leqslant \frac{\pi}{2}$$

[1 mark]

Question 5

(i) On the Argand diagram, shade in the region, *R*, specified by,

$$\{ z \in \mathbb{C} : 3 < |z| \leqslant 5 \} \cap \{ z \in \mathbb{C} : |z| \leqslant |z - 6\mathrm{i}| \}$$



[4 marks]

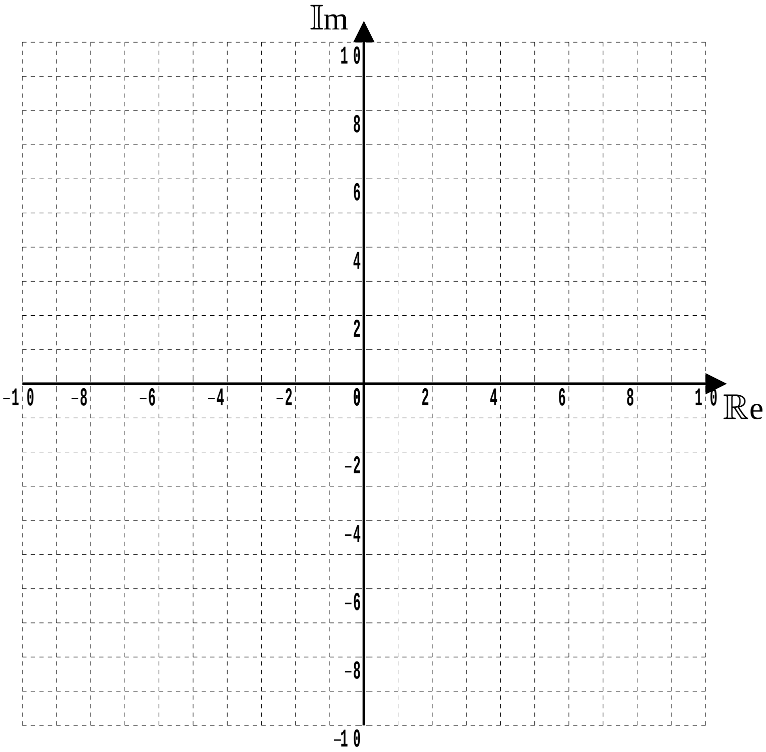
(ii) Determine the area of the region *R*, correct to three significant figures.

[3 marks]

Question 6

(i) On the Argand diagram, shade in the region, *R*, specified by,

$$\{ z \in \mathbb{C} : |z| \leqslant 6 \} \cap \left\{ z \in \mathbb{C} : 0 \leqslant arg(z+6) \leqslant \frac{\pi}{4} \right\}$$



[4 marks]

(ii) Determine the ***exact*** perimeter of the region *R*

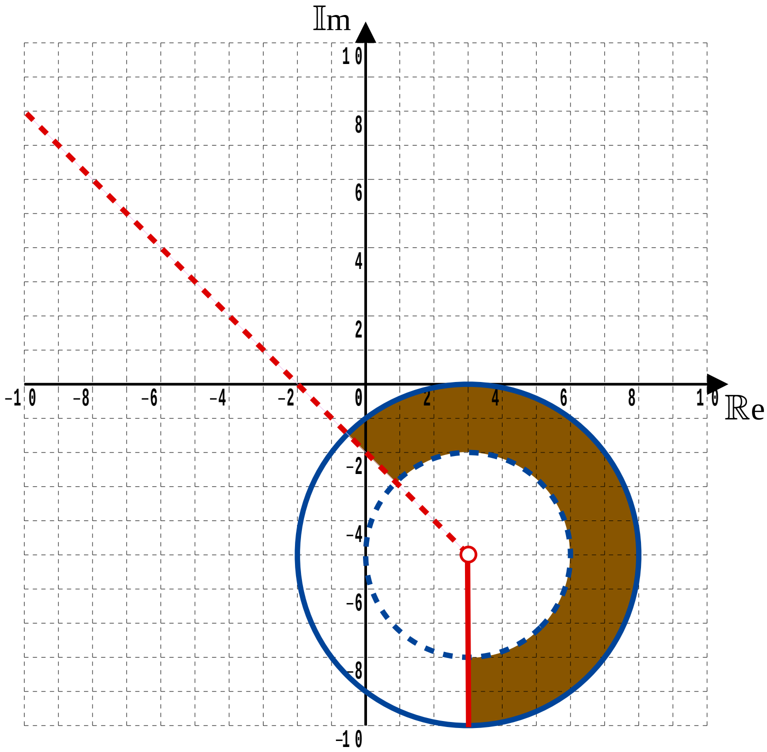
[2 marks]

(iii) Determine the ***exact*** area of the region *R*

[2 marks]

10.3 Answers

10.3.1 Solution to 10.1 Teaching Video Example

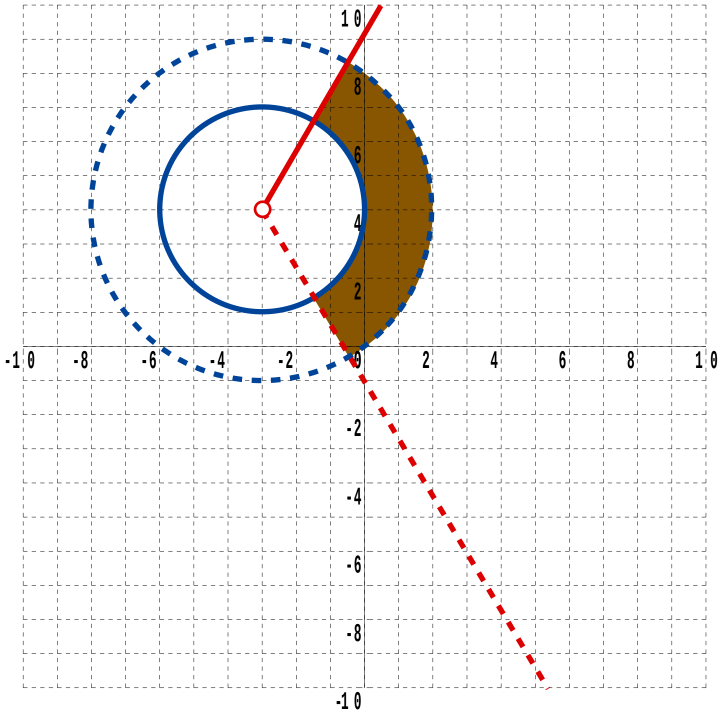


[4 marks]

10.3.2 Solutions to 10.2 Exercise

Answer 1

(i)



[4 marks]

(ii) The perimeter is made up of one 120° arc of a circle of radius 3, another 120° arc of a circle of radius 5 and two straight radial lines between the differences in the radii of the concentric circles.

$$Perimeter = 3 \times \frac{2\pi}{3} + 5 \times \frac{2\pi}{3} + 2 + 2$$

$$= \frac{16\pi}{3} + 4 \qquad \text{(20.8 units)}$$

[2 marks]

(iii) The area is made up of one 120° segment of a circle of radius 3, subtracted from another 120° segment of a circle of radius 5.

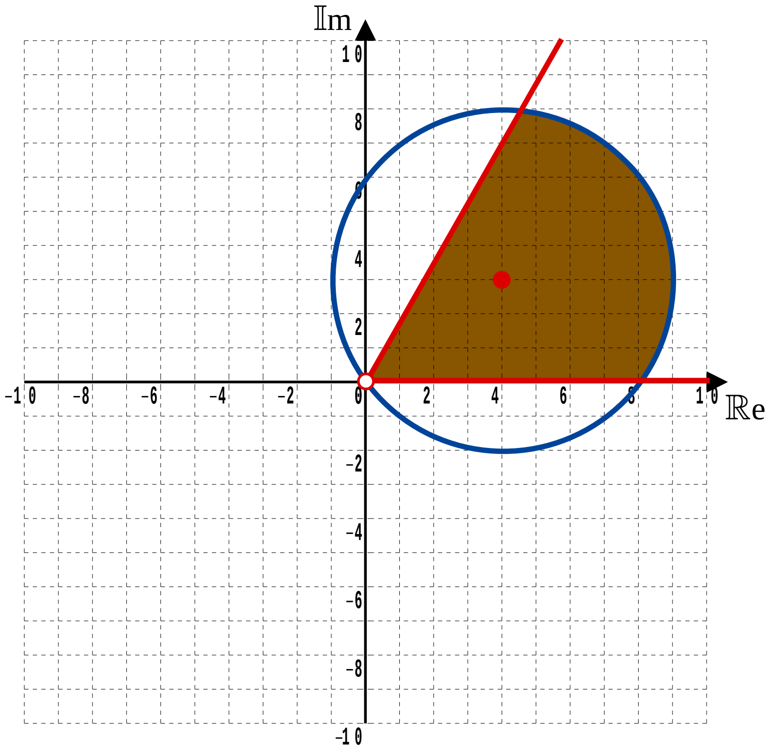
$$Area = \frac{1}{2} \times 5^2 \times \frac{2\pi}{3} - \frac{1}{2} \times 3^2 \times \frac{2\pi}{3}$$

$$= \frac{25\pi - 9\pi}{3}$$

$$= \frac{16\pi}{3} \qquad (16.8 \text{ units}^2)$$

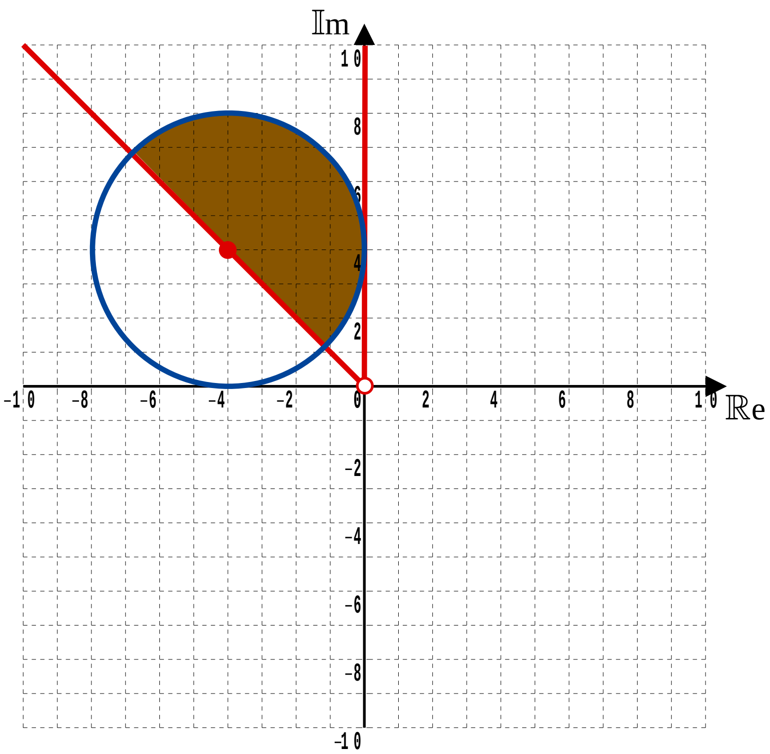
[2 marks]

Answer 2



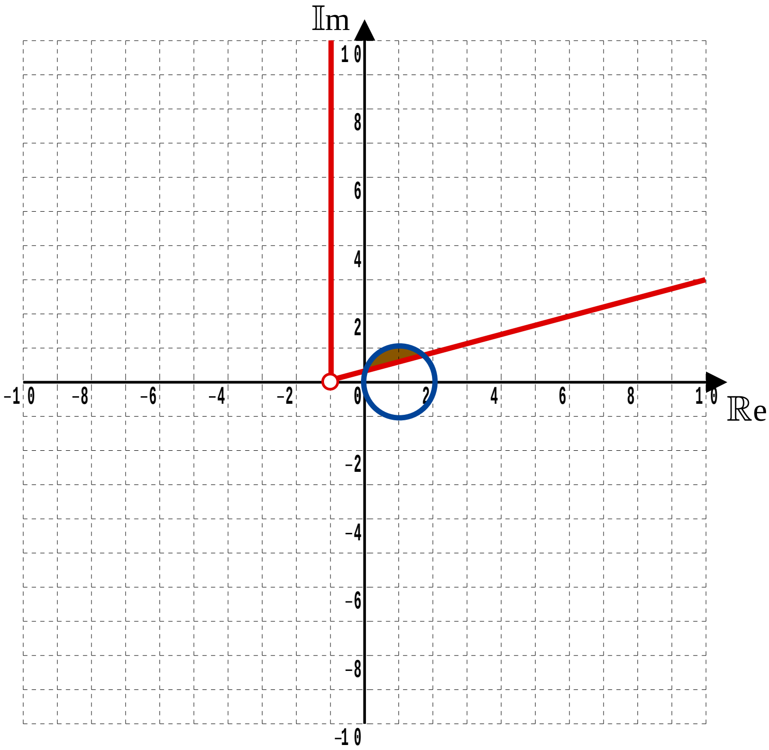
[2 marks]

Answer 3



[2 marks]

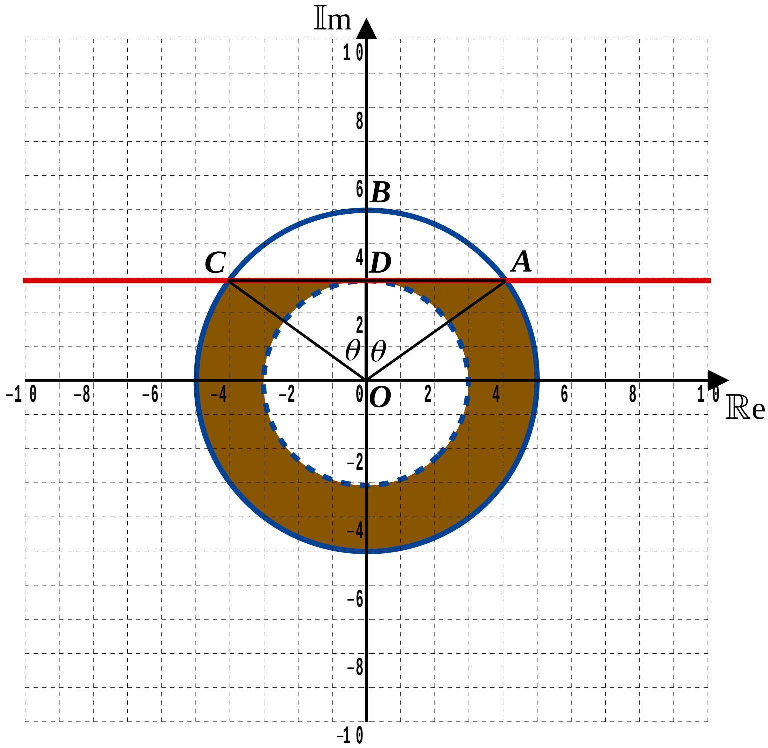
Answer 4



[4, 1 marks]

Answer 5

(i) The $|z|$ is a circle centre (0, 0) : Write it as $|z - (0 + 0i)|$ if it helps.
So this circle's radius varies between 3 and 5 giving an annulus (ring).
$|z| = |z - 6i|$ could be written as $|z - (0 + 0i)| = |z - (0 + 6i)|$
to make it clear that it is the perpendicular bisector of the line segment between (0, 0) and (0, 6) which is 'obviously' the line with equation $y = 3$.



[4 marks]

(ii)

$$|AD| = \sqrt{5^2 - 3^2}$$
$$= 4$$
$$\theta = arctan\left(\frac{4}{3}\right)$$
$$= 0.9273$$

Area Segment *ABCD* = Area Sector *OABC* – Area Triangle *OADC*

$$= \frac{1}{2} \times 5^2 \times 2 \times 0.9273 - \frac{1}{2} \times 8 \times 3$$
$$= 23.18 - 12$$
$$= 11.18$$

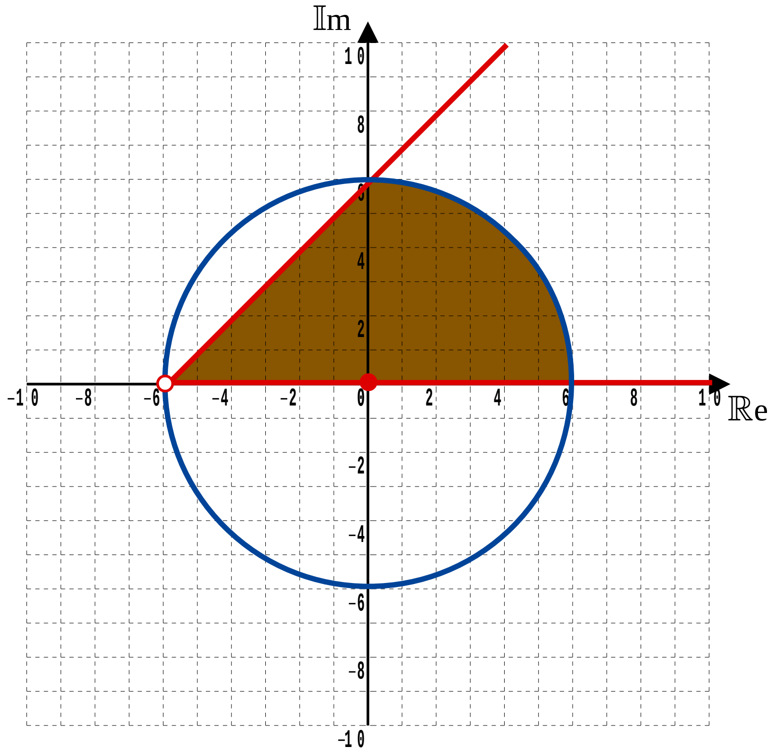
Shaded Area = Area Large Circle – Area Small Circle – Area Segment *ABCD*

$$= \pi \times 5^2 - \pi \times 3^2 - 11.18$$
$$= 78.5 - 28.3 - 11.2$$
$$= 39.0$$

[3 marks]

Answer 6

(i)



[4 marks]

(ii)

Perimeter = Diameter + Quarter Circumference + Hypotenuse of Triangle

$$= 12 + \frac{1}{4} \times 2 \times \pi \times 6 + \sqrt{6^2 + 6^2}$$

$$= 12 + 3\pi + 6\sqrt{2}$$

[2 marks]

(iii)

Area shaded = Area Quarter Circle + Area Triangle

$$= \frac{1}{4} \times \pi \times 6^2 + \frac{1}{2} \times 6 \times 6$$

$$= 9\pi + 18$$

[2 marks]

Lesson 11

11.1 Test

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 45

Question 1

In general a complex number z is often defined as being a number of the form $x + y\mathrm{i}$.
In this definition which one of the following describes x and y ?

(A) Integers, $x,\ y \in \mathbb{Z}$

(B) Rational Numbers, $x,\ y \in \mathbb{Q}$

(C) Real Numbers, $x, y \in \mathbb{R}$

(D) Complex Numbers, $x,\ y \in \mathbb{C}$

[1 mark]

Question 2

Further A-Level Examination Question from May 2017, FP1, Q6

Given that 4 and $2\mathrm{i} - 3$ are roots of the equation

$$x^3 + ax^2 + bx - 52 = 0$$

where a and b are real constants,

(a) write down the third root of the equation

[1 mark]

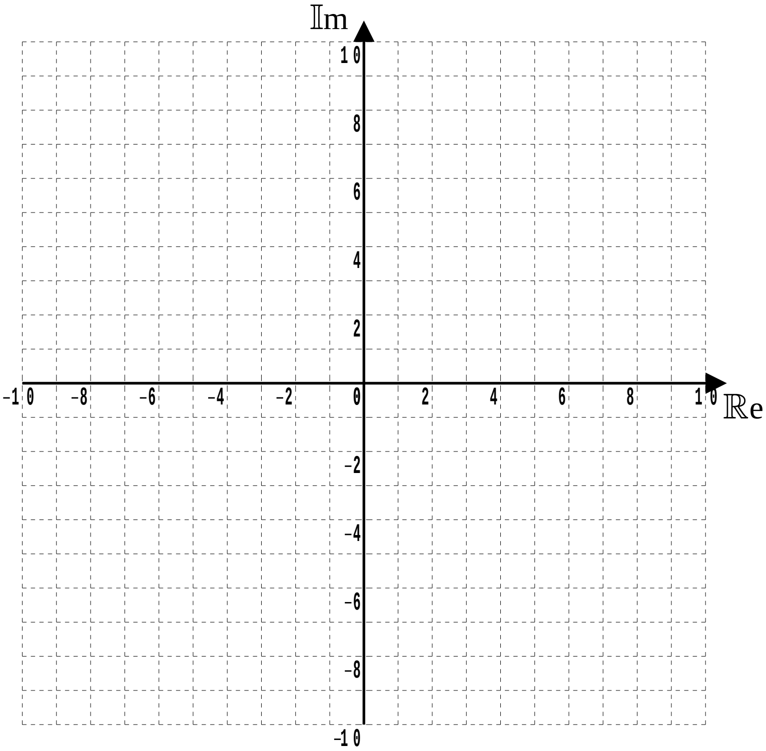
(b) Find the value of a and the value of b

[5 marks]

Question 3

(i) On the Argand diagram, shade in the region, R, specified by,

$$\{ z \in \mathbb{C} : 2.5 \leqslant |z| < 6.5 \} \cap \{ z \in \mathbb{C} : |z| \leqslant |z + 5i| \}$$



[5 marks]

(ii) Determine the area of the region R, correct to three significant figures.

[5 marks]

Question 4

Further A-Level Examination Question from January 2018, F1, Q5

(i) Given that

$$\frac{2z + 3}{z + 5 - 2\text{i}} = 1 + \text{i}$$

find z, giving your answer in the form $a + b\text{i}$, where a and b are real constants.

[5 marks]

(ii) Given that

$$w = (3 + \lambda\text{i})(2 + \text{i})$$

where λ is a real constant, and that

$$|w| = 15$$

find the possible values of λ

[4 marks]

Question 5

Further A-Level Examination Question from June 2012, FP2, Q8

The point P represents a complex number z on an Argand diagram such that

$$|z - 6\mathrm{i}| = 2|z - 3|$$

(a) Show that, as z varies, the locus of P is a circle, stating the radius and the coordinates of the centre of this circle

[6 marks]

The point Q represents a complex number z on an Argand diagram such that

$$arg\ (z - 6) = -\frac{3\pi}{4}$$

(b) Sketch, on the same Argand diagram, the locus of P and the locus of Q as z varies

[4 marks]

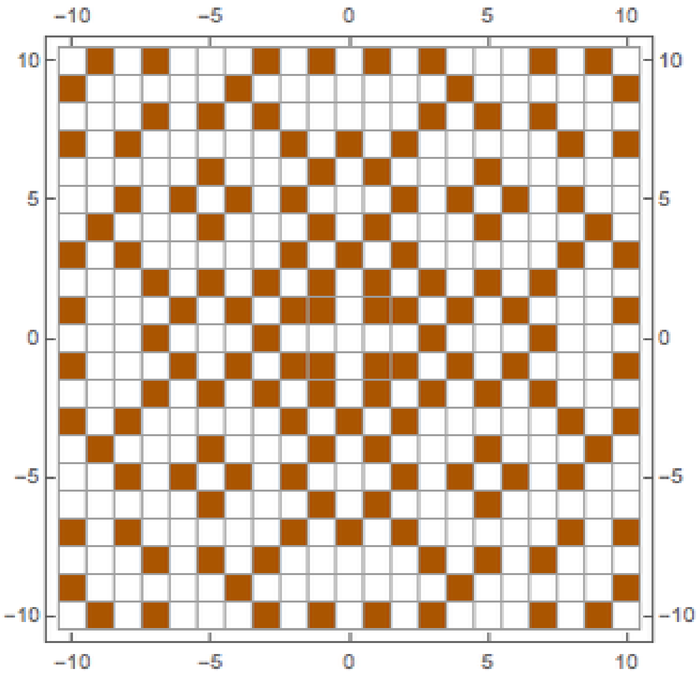
(c) Find the complex number for which both

$$|z - 6i| = 2|z - 3| \quad \text{and} \quad arg\ (z - 6) = -\frac{3\pi}{4}$$

[4 marks]

Question 6

A section of the grid of Gaussian Integers, $\mathbb{Z}[i]$, is shown.
In the grid, a dark square indicates a Gaussian Prime.



(i) Which one of the following four Gaussian Integers is a Gaussian Prime ?

$2i \qquad 2 + 3i \qquad 3 + 4i \qquad 5$

[1 mark]

(ii) Find z such that $z^2 - 3 - 4i = 0, \qquad z \in \mathbb{Z}[i]$

[4 marks]

11.2 Answers to 11.1 Test

Answer 1

(C) Real Numbers, $x, y \in \mathbb{R}$

[1 mark]

Answer 2

(a) As all of the coefficients of the cubic are real numbers, the complex roots come in conjugate pairs. Thus the third root is $-3 - 2i$

[1 mark]

(b) The roots are $x_1 = 4$, $x_2 = -3 + 2i$ and $x_3 = -3 - 2i$

So the factors are,

$$(x - 4)(x + 3 - 2i)(x + 3 + 2i)$$

$$= (x - 4)\left((x + 3)^2 - (2i)^2\right)$$

$$= (x - 4)\left(x^2 + 6x + 9 - 4i^2\right)$$

$$= (x - 4)\left(x^2 + 6x + 13\right)$$

Multiplication grid :

	x^2	$6x$	13
x	x^3	$6x^2$	$13x$
-4	$-4x^2$	$-24x$	-52

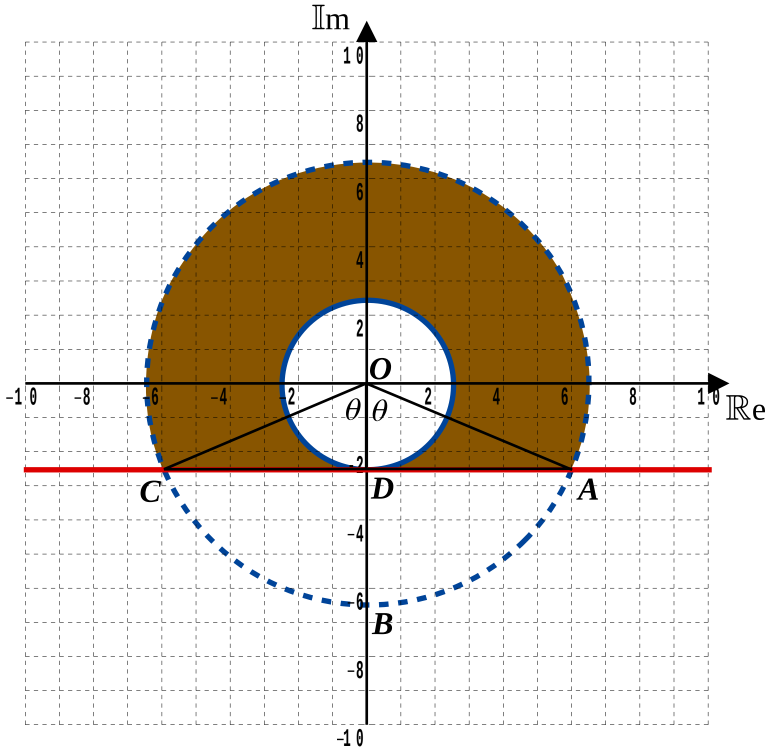
$$= x^3 + 2x^2 - 11x - 52$$

$$\therefore \quad a = 2 \quad \text{and} \quad b = -11$$

[5 marks]

Answer 3

(i) The $|z|$ is a circle centre (0, 0) : Write is as $|z - (0 + 0i)|$ if it helps.
So this circle's radius varies between 2.5 and 6.5 giving an annulus (ring).
$|z| = |z + 5i|$ could be written as $|z - (0 + 0i)| = |z - (0 - 5i)|$
to make it clear that it is the perpendicular bisector of the line segment
between (0, 0) and (0, – 5) ; 'obviously' the line with equation $y = -2.5$.



[5 marks]

(ii)

$$|AD| = \sqrt{6.5^2 - 2.5^2}$$
$$= 6$$
$$\theta = arccos\left(\frac{2.5}{6.5}\right)$$
$$= 1.176$$

Area Segment *ABCD* = Area Sector *OABC* – Area Triangle *OADC*

$$= \frac{1}{2} \times 6.5^2 \times 2 \times 1.176 - \frac{1}{2} \times 12 \times 2.5$$
$$= 49.69 - 15$$
$$= 34.69$$

Shaded Area = Area Large Circle – Area Small Circle – Area Segment *ABCD*

$$= \pi \times 6.5^2 - \pi \times 2.5^2 - 34.69$$
$$= 132.73 - 19.63 - 34.69$$
$$= 78.4$$

[5 marks]

Answer 4

(i) Various ways through this are to be found...

$$\frac{2z + 3}{z + 5 - 2\mathrm{i}} = 1 + \mathrm{i}$$

$$(2z + 3) = (z + 5 - 2\mathrm{i})(1 + \mathrm{i})$$

Multiplication grid :

	z	$+5$	$-2\mathrm{i}$
1	z	$+5$	$-2\mathrm{i}$
$+\mathrm{i}$	$z\mathrm{i}$	$+5\mathrm{i}$	$-2\mathrm{i}^2$

$$\begin{aligned} 2z + 3 &= z + 5 - 2\mathrm{i} + z\mathrm{i} + 5\mathrm{i} - 2\mathrm{i}^2 \\ &= z + 7 + 3\mathrm{i} + z\mathrm{i} \\ 2z - z - z\mathrm{i} &= 7 + 3\mathrm{i} - 3 \\ z - z\mathrm{i} &= 4 + 3\mathrm{i} \\ z(1 - \mathrm{i}) &= 4 + 3\mathrm{i} \\ z &= \frac{4 + 3\mathrm{i}}{1 - \mathrm{i}} \times \frac{1 + \mathrm{i}}{1 + \mathrm{i}} \\ &= \frac{4 + 4\mathrm{i} + 3\mathrm{i} + 3\mathrm{i}^2}{1 - \mathrm{i}^2} \\ &= \frac{1 + 7\mathrm{i}}{2} \\ &= \frac{1}{2} + \frac{7}{2}\mathrm{i} \end{aligned}$$

[5 marks]

(ii)

$$\begin{aligned} |w| &= 15 \\ |(3 + \lambda\mathrm{i})(2 + \mathrm{i})| &= 15 \\ |6 + 3\mathrm{i} + 2\lambda\mathrm{i} + \lambda\mathrm{i}^2| &= 15 \\ |(6 - \lambda) + (3 + 2\lambda)\mathrm{i}| &= 15 \\ (6 - \lambda)^2 + (3 + 2\lambda)^2 &= 225 \\ 36 - 12\lambda + \lambda^2 + 9 + 12\lambda + 4\lambda^2 &= 225 \\ 5\lambda^2 &= 180 \\ \lambda^2 &= 36 \\ \lambda &= \pm 6 \end{aligned}$$

[4 marks]

Answer 5

(a)

$$|z - 6\text{i}| = 2\,|z - 3|$$
$$|x + y\text{i} - 6\text{i}| = 2\,|x + y\text{i} - 3|$$
$$|x + (y - 6)\text{i}| = 2|\,(x - 3) + y\text{i}\,|$$
$$x^2 + (y - 6)^2 = 4\left(\,(x - 3)^2 + y^2\right)$$
$$x^2 + y^2 - 12y + 36 = 4\left(x^2 - 6x + 9 + y^2\right)$$
$$x^2 + y^2 - 12y + 36 = 4x^2 - 24x + 36 + 4y^2$$
$$3x^2 + 3y^2 - 24x + 12y = 0$$

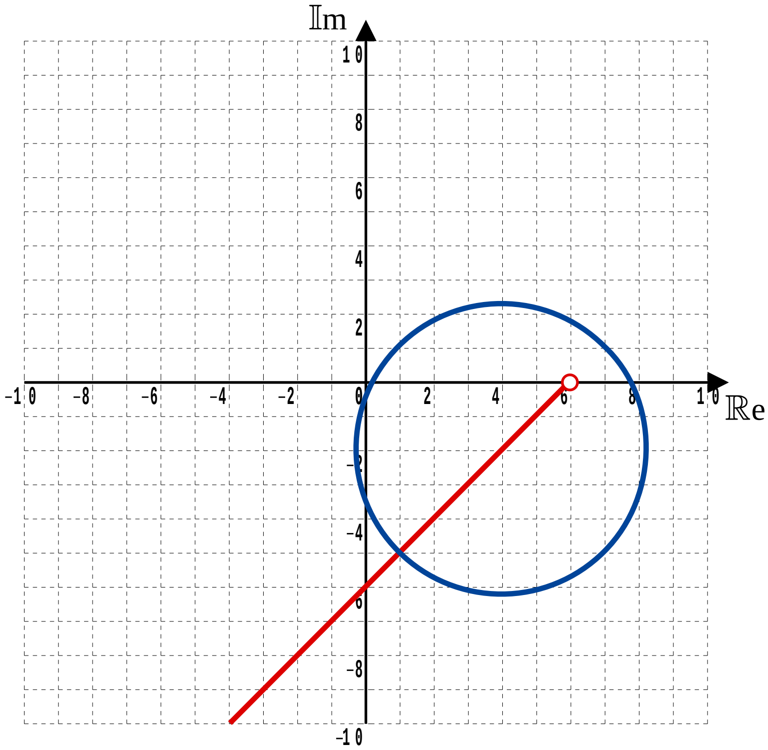
Divide through by 3

$$x^2 - 8x + y^2 + 4y = 0$$
$$\left[x^2 - 8x\right] + \left[y^2 + 4y\right] = 0$$
$$\left[\,(x - 4)^2 - 16\right] + \left[\,(y + 2)^2 - 4\right] = 0$$
$$(x - 4)^2 + (y + 2)^2 = 20$$

which is a circle with centre $(4, -2)$ and radius $2\sqrt{5}$ (4.47)

[6 marks]

(b) $arg\,(z - 6) = -\dfrac{3\pi}{4}$ is a half-line starting (6, 0) heading off $-\ 135°$



[4 marks]

(c)

$$\text{Circle : } (x-4)^2+(y+2)^2=20$$

$$\text{Line : } y=x-6 \text{ with } x<6$$

$$\begin{aligned}(x-4)^2+((x-6)+2)^2&=20\\(x-4)^2+(x-4)^2&=20\\2(x-4)^2&=20\\(x-4)^2&=10\\x-4&=\pm\sqrt{10}\\x&=4\pm\sqrt{10}\\x&=4-\sqrt{10}\quad\text{as } x<6\end{aligned}$$

$$\begin{aligned}y&=x-6\\&=4-\sqrt{10}-6\\&=-2-\sqrt{10}\end{aligned}$$

$$\therefore z=\left(4-\sqrt{10}\right)+\left(-2-\sqrt{10}\right)\mathrm{i}$$

[4 marks]

Answer 6

(i) $2 + 3\mathrm{i}$ is a Gaussian prime

[1 mark]

(ii)

$$z^2 - 3 - 4\mathrm{i} = 0$$

$$z^2 = 3 + 4\mathrm{i}$$

$$(a + b\mathrm{i})^2 = 3 + 4\mathrm{i}$$

$$a^2 + 2ab\mathrm{i} + b^2\mathrm{i}^2 = 3 + 4\mathrm{i}$$

$$a^2 - b^2 + 2ab\mathrm{i} = 3 + 4\mathrm{i}$$

Two complex numbers (the one on the left and the one on the right) are equal if and only if the real parts are equal and the imaginary parts are equal.

$$\therefore\ a^2 - b^2 = 3 \quad \text{and} \quad 2ab = 4$$

$$ab = 2$$

As a, b are integers the only possibility with $|a| > |b|$ (to have $a^2 - b^2$ positive) are, $(a, b) = (-2, -1)$, and $(2, 1)$

Both of these also satisfy $a^2 - b^2 = 3$

Thus, the solutions to

$$z^2 - 5 - 12\mathrm{i} = 0$$

are $-2 - \mathrm{i}$ or $2 + \mathrm{i}$

These could be presented as $\pm(2 + \mathrm{i})$

which has an obvious similarity with previous work with the real numbers

[4 marks]