Year 1 Pure Mathematics Examination Revision
Health Check N° 3

Answer 1 (Version 1)

Statement: $\cos^4 x - \sin^4 x = 1 - 2\sin^2 x$

Proof : $\quad$ LHS $= \cos^4 x - \sin^4 x$

$$= (\cos^2 x)^2 - (\sin^2 x)^2$$

$$= (\cos^2 x + \sin^2 x)(\cos^2 x - \sin^2 x) \quad \text{difference of two squares}$$

$$= \cos^2 x - \sin^2 x$$

$$= (1 - \sin^2 x) - \sin^2 x$$

$$= 1 - 2\sin^2 x$$

$$= \text{RHS} \qquad \square$$

[4 marks]

Answer 1 (Version 2)

Statement: $\cos^4 x - \sin^4 x = 1 - 2\sin^2 x$

Proof : $\quad$ LHS $= \cos^4 x - \sin^4 x$

$$= (1 - \sin^2 x)^2 - \sin^4 x$$

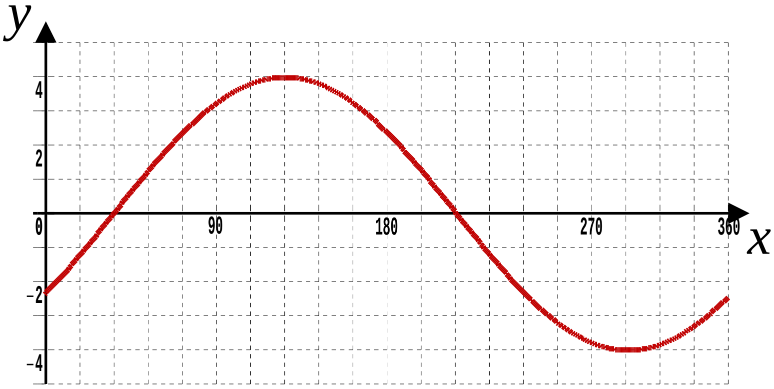
$$= 1 - 2\sin^2 x + \sin^4 x - \sin^4 x$$

$$= 1 - 2\sin^2 x$$

$$= \text{RHS} \qquad \square$$

[4 marks]

Answer 2



[3 marks]

Answer 3

Using the binomial theorem,

$$(1 + x)^8 = 1 + 8x + 28x^2 + 56x^3 + 70x^4 + 56x^5 + 28x^6 + 8x^7 + x^8$$

$$(1 - x)^8 = 1 - 8x + 28x^2 - 56x^3 + 70x^4 - 56x^5 + 28x^6 - 8x^7 + x^8$$

$$\therefore \ (1 + x)^8 - (1 - x)^8 = 16x + 112x^3 + 112x^5 + 16x^7$$

$$= 16\left(x + 7x^3 + 7x^5 + x^7\right)$$

$$\frac{(1 + x)^8 - (1 - x)^8}{16} = x + 7x^3 + 7x^5 + x^7$$

[5 marks]

Answer 4

(i) The graph of $y = sin\,x$ has values (outputs) between -1 and $+1$

It is said that the amplitude of the sine function is 1.

The graph plotted has an amplitude of 3, achieved by replacing *y* with $\frac{y}{3}$ and both sides have then been multiplied by 3. $\therefore\ a = 3$

$\therefore\ a = 3$

[2 marks]

(ii) The graph of $y = sin\,x$ passes through the origin. That point, which would not be moved by the scaling of part (i), can be considered to now be at $(-60,\ 0)$ after a translation of $\begin{pmatrix} -60 \\ 0 \end{pmatrix}$

Algebraically this is achieved by replacing *x* with $(x + 60)$

$\therefore\ b = 60$

[2 marks]

(iii) $a\,sin(x + b) = k \Leftrightarrow sin(x + b) = \frac{k}{a}$

However, $-1 \leqslant sin\,\omega \leqslant 1$ for any real ω whatsoever.

$\therefore$ To have solutions; $-a \leqslant k \leqslant a$

[2 marks]

Answer 5

$$\frac{10x^2}{x^2 + 1} = 3x + 2$$

$$10x^2 = (3x + 2)(x^2 + 1)$$

$$10x^2 = 3x^3 + 2x^2 + 3x + 2$$

$$3x^3 - 8x^2 + 3x + 2 = 0$$

It's fairly obvious that $x = 1$ is a root by the factor theorem.

$\therefore \ (x - 1)$ is a factor.

$$\begin{array}{r|l} & 3x^2 \quad - \quad 5x \quad - \quad 2 \\ \hline x - 1 & 3x^3 - 8x^2 + 3x + 2 \\ & \underline{3x^3 - 3x^2} \\ & \quad -5x^2 + 3x \\ & \quad \underline{-5x^2 + 5x} \\ & \qquad -2x + 2 \\ & \qquad \underline{-2x + 2} \\ & \qquad\qquad 0 \end{array}$$

$0 \leftarrow$ As expected !

$$\therefore \ 3x^3 - 8x^2 + 3x + 2 = (x - 1)(3x^2 - 5x - 2)$$

$$= (x - 1)(3x + 1)(x - 2)$$

$\therefore$ The roots are $x \in \left\{-\frac{1}{3},\ 1,\ 2\right\}$

To get the y parts of the coordinates of intersections use the $y = 3x + 2$

to get $\left(-\frac{1}{3},\ 1\right),\ (1,\ 5),\ (2,\ 8)$

[6 marks]

Answer 6

As y is inversely proportional to x; $y = \frac{k}{x}$ where k is the constant of proportionality.

After a translation of $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ the algebra to go with the graph will be,

$$y - 2 = \frac{k}{(x - 3)}$$

As it passes through the origin, $-2 = \frac{k}{-3}$ and so $k = 6$

$\therefore \ y - 2 = \frac{6}{x - 3}$ leading to $y = \frac{2x}{x - 3}$

That is, $a = 2,\ b = 0,\ c = 1$ and $d = -3$

[6 marks]

Answer 7

(i)

$$
\begin{aligned}
f(x) &= 9\cos^4 x - 12\cos^2 x + 7 \\
&= \left[\, 9\cos^4 x - 12\cos^2 x \,\right] + 7 \\
&= 9\left[\, \cos^4 x - \frac{4}{3}\cos^2 x \,\right] + 7 \\
&= 9\left[\, \left(\cos^2 x - \frac{2}{3}\right)^2 - \frac{4}{9} \,\right] + 7 \\
&= 9\left(\cos^2 x - \frac{2}{3}\right)^2 - 4 + 7 \\
&= 9\left(\cos^2 x - \frac{2}{3}\right)^2 + 3
\end{aligned}
$$

$\therefore$ The minimum value is 3

[6 marks]

(ii) The minimum value will occur when,

$$
\begin{aligned}
\cos^2 x &= \frac{2}{3} \\
\cos x &= \pm\sqrt{\frac{2}{3}} \\
x &\in \{ 35.3°,\ 144.7°,\ 215.3°,\ 324.7° \}
\end{aligned}
$$

[4 marks]

Answer 8

$$
\begin{aligned}
1 + \frac{x}{3} &= 1.01 \\
\frac{x}{3} &= 0.01 \\
x &= 0.03
\end{aligned}
$$

$$
\begin{aligned}
\therefore\ 1.01^{18} &\approx 1 + 6 \times 0.03 + 17 \times 0.03^2 \\
1.01^{18} &\approx 1 + 0.18 + 0.0153 \\
1.01^{18} &\approx 1.1953 \qquad \left(\text{Compare with}\, 1.01^{18} = 1.196147476...\right)
\end{aligned}
$$

[4 marks]

Answer 9

Points of intersection will occur when the following has solutions,

$$(x - 1)^2 = kx$$

$$x^2 - 2x + 1 = kx$$

$$x^2 - (2 + k)x + 1 = 0$$

For solutions the discrimiant, $D \geqslant 0$

$$(-(2 + k))^2 - 4 \times 1 \times 1 \geqslant 0$$

$$4 + 4k + k^2 - 4 \geqslant 0$$

$$k^2 + 4k \geqslant 0$$

$$k(k + 4) \geqslant 0$$

$$\therefore\ k \leqslant -4 \text{ or } k \geqslant 0$$

Geometrically this is interesting and illuminating to try and understand.
Try drawing a picture with the parabola touching the x-axis at $x = 1$ and crossing the y-axis at $y = 1$
Remember that lines in the first quadrant have gradients, m, that satisfy,

$$0 \leqslant m < \infty$$

Can you explain what the maths is telling you about which straight lines through the origin will intersect the parabola ?

[6 marks]