2.4 Answers

2.4.1 Solution to 2.2 Partial Fractions Revisited

$$Volume = \pi \int y^2 \, dx$$

$$= \pi \int_1^3 \left(\frac{1}{\sqrt{x}\,(5x+1)} \right)^2 dx$$

$$= \pi \int_1^3 \frac{1}{x(5x+1)^2} \, dx$$

$$\frac{1}{x(5x+1)^2} = \frac{A}{x} + \frac{B}{5x+1} + \frac{C}{(5x+1)^2}$$

$$\frac{1 \times [\ \dots \dots\ ]}{[\ x(5x+1)^2\ ]} = \frac{A \times [\ \dots \dots\ ]}{x} + \frac{B \times [\ \dots \dots\ ]}{5x+1} + \frac{C \times [\ \dots \dots\ ]}{(5x+1)^2}$$

$$1 = A(5x+1)^2 + Bx(5x+1) + Cx$$

$$\text{Let } x = 0 \ : \ 1 = A$$

$$\text{Let } x = -\frac{1}{5} \ : \ 1 = -\frac{C}{5} \Rightarrow C = -5$$

$$\text{Let } x = 1 \ : \ 1 = 36 + 6B - 5 \Rightarrow B = -5$$

$$\frac{1}{x(5x+1)^2} = \frac{1}{x} - \frac{5}{5x+1} - \frac{5}{(5x+1)^2}$$

$$\pi \int_1^3 \frac{1}{x(5x+1)^2} \, dx = \pi \int_1^3 \frac{1}{x} \, dx - \pi \int_1^3 \frac{5}{(5x+1)} \, dx - \pi \int_1^3 \frac{5}{(5x+1)^2} \, dx$$

$$= \pi \left[\ln|x| \right]_1^3 - \pi \left[\ln|5x+1| \right]_1^3 + \pi \left[\frac{1}{5x+1} \right]_1^3$$

$$= \pi [\ln 3 - \ln 1] - \pi [\ln 16 - \ln 6] + \pi \left[\frac{1}{16} - \frac{1}{6} \right]$$

$$= \pi \left[\ln \left(\frac{3 \times 6}{16} \right) \right] - \frac{5\pi}{48}$$

$$= \pi \ln \left(\frac{9}{8} \right) - \frac{5\pi}{48}$$

$$= 0.0428$$

[8 marks]

2.4.2 Solution to 2.3 Exercise

Answer 1

$$\begin{aligned} Volume &= \pi \int y^2\, dx \\ &= \pi \int_2^3 \left(\frac{1}{x\sqrt{x-1}} \right)^2 dx \\ &= \pi \int_2^3 \frac{1}{x^2\,(x-1)}\, dx \end{aligned}$$

$$\frac{1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}$$

$$\frac{1 \times [\ \ldots\ldots\ ]}{[\ x^2(x-1)\ ]} = \frac{A \times [\ \ldots\ldots\ ]}{x} + \frac{B \times [\ \ldots\ldots\ ]}{x^2} + \frac{C \times [\ \ldots\ldots\ ]}{x-1}$$

$$1 = Ax(x-1) + B(x-1) + C\,x^2$$

$$\text{Let } x = 0 \ :\ 1 = -B \qquad \Rightarrow \qquad B = -1$$

$$\text{Let } x = 1 \ :\ 1 = C$$

$$\text{Let } x = 2 \ :\ 1 = 2A + B + 4C$$

$$1 = 2A - 1 + 4 \qquad \Rightarrow A = -1$$

$$\frac{1}{x^2(x-1)} = -\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x-1}$$

$$\begin{aligned} \pi \int_2^3 \frac{1}{x^2(x-1)}\, dx &= \pi \int_2^3 \frac{1}{x-1}\, dx - \pi \int_2^3 \frac{1}{x}\, dx - \pi \int_2^3 \frac{1}{x^2}\, dx \\ &= \pi \left[\, ln|\,x-1\,|\,\right]_2^3 - \pi \left[\, ln|\,x\,|\,\right]_2^3 + \pi \left[\frac{1}{x} \right]_2^3 \\ &= \pi [\, ln\, 2 - ln\, 1\,] - \pi [\, ln\, 3 - ln\, 2\,] + \pi \left[\frac{1}{3} - \frac{1}{2} \right] \\ &= \pi \left[ln \left(\frac{2 \times 2}{3} \right) \right] - \frac{\pi}{6} \\ &= \pi\, ln \left(\frac{4}{3} \right) - \frac{\pi}{6} \qquad \square \end{aligned}$$

(As a decimal this is about 0.380)

[8 marks]

Answer 2

(i)

$$\int_1^2 \frac{1}{(4x-3)^2}\,dx = \frac{1}{4}\int_1^2 \frac{4}{(4x-3)^2}\,dx$$

$$= \frac{1}{4}\left[-\frac{1}{4x-3}\right]_1^2$$

$$= -\frac{1}{4}\left[\frac{1}{5}-\frac{1}{1}\right]$$

$$= -\frac{1}{4}\left[-\frac{4}{5}\right]$$

$$= \frac{1}{5} \qquad \square$$

[4 marks]

(ii)

$$Volume = \pi\int y^2\,dx$$

$$= \pi\int_1^2 \left(\frac{4\sqrt{x}}{4x-3}\right)^2 dx$$

$$= \pi\int_1^2 \frac{16x}{(4x-3)^2}\,dx$$

$$\frac{16x}{(4x-3)^2} = \frac{A}{4x-3} + \frac{B}{(4x-3)^2}$$

$$\frac{16x \times [\,\dots\dots\,]}{[\,(4x-3)^2\,]} = \frac{A \times [\,\dots\dots\,]}{4x-3} + \frac{B \times [\,\dots\dots\,]}{(4x-3)^2}$$

$$16x = A(4x-3) + B$$

$$\text{Let } x = \frac{3}{4} \;:\; 12 = B \quad\Rightarrow\quad B = 12$$

$$\text{Let } x = 1 \;:\; 16 = A + B \quad\Rightarrow\quad A = 4$$

$$\frac{16x}{(4x-3)^2} = \frac{4}{4x-3} + \frac{12}{(4x-3)^2}$$

$$\pi\int_1^2 \frac{16x}{(4x-3)^2}\,dx = \pi\int_1^2 \frac{4}{4x-3}\,dx + 3\pi\int_1^2 \frac{4}{(4x-3)^2}\,dx$$

$$= \pi\left[\,ln|4x-3|\,\right]_1^2 + 3\pi\left[-\frac{1}{5}+\frac{1}{1}\right]$$

$$= \pi\left[\,ln\,5 - ln\,1\,\right] + \frac{12\pi}{5}$$

$$= \pi\,ln\,5 + \frac{12\pi}{5} \qquad \square$$

(As a decimal this is about 12.6)

Note : This question could also be done using the substitution $t = 4x - 3$

[8 marks]

Answer 3

(i)

$$\int_1^3 \frac{1}{(2x-1)^3}\,dx = \frac{1}{2}\int_1^3 \frac{2}{(2x-1)^3}\,dx$$

$$= \frac{1}{2}\left[\frac{1}{(2x-1)^2} \times \frac{1}{(-2)}\right]_1^3$$

$$= -\frac{1}{4}\left[\frac{1}{25} - \frac{1}{1}\right]$$

$$= -\frac{1}{4}\left[-\frac{24}{25}\right]$$

$$= \frac{6}{25} \quad \square$$

[4 marks]

(ii)

$$Volume = \pi \int y^2\,dx$$

$$= \pi \int_1^3 \left(\frac{2\sqrt{2}\,x}{(2x-1)^{1.5}}\right)^2 dx$$

$$= \pi \int_1^3 \frac{8x^2}{(2x-1)^3}\,dx$$

$$\frac{8x^2}{(2x-1)^3} = \frac{A}{2x-1} + \frac{B}{(2x-1)^2} + \frac{C}{(2x-1)^3}$$

$$\frac{8x^2 \times [\,\ldots\ldots\,]}{[\,(2x-1)^3\,]} = \frac{A \times [\,\ldots\ldots\,]}{2x-1} + \frac{B \times [\,\ldots\ldots\,]}{(2x-1)^2} + \frac{C \times [\,\ldots\ldots\,]}{(2x-1)^3}$$

$$8x^2 = A(2x-1)^2 + B(2x-1) + C$$

$$\text{Let } x = \frac{1}{2} : 2 = C \quad \Rightarrow \quad C = 2$$

$$\text{Let } x = 0 : 0 = A - B + C \quad \Rightarrow \quad A - B = -2$$

$$\text{Let } x = 1 : 8 = A + B + C \quad \Rightarrow \quad A + B = 6$$

$$\therefore A = 2, \quad B = 4$$

$$\frac{8x^2}{(2x-1)^3} = \frac{2}{2x-1} + \frac{4}{(2x-1)^2} + \frac{2}{(2x-1)^3}$$

$$\pi \int_1^3 \frac{8x^2}{(2x-1)^3}\,dx = \pi \int_1^3 \frac{2}{2x-1}\,dx + 2\pi \int_1^3 \frac{2}{(2x-1)^2}\,dx + 2\pi \int_1^3 \frac{1}{(2x-1)^3}\,dx$$

$$= \pi\,\Big[\, ln\,|2x-1|\,\Big]_1^3 + 2\pi\left[-\frac{1}{2x-1}\right]_1^3 + 2\pi\left[\frac{6}{25}\right]$$

$$= \pi\big[\, ln\,5 - ln\,1\,\big] - 2\pi\left[\frac{1}{5} - \frac{1}{1}\right] + \frac{12\pi}{25}$$

$$= \pi\, ln\,5 + \frac{8\pi}{5} + \frac{12\pi}{25}$$

$$= \pi\, ln\,5 + \frac{52\pi}{25} \qquad \square$$

(As a decimal, this is about = 11.6)

Note : This question could also be done using the substitution $t = 2x - 1$

[8 marks]

Answer 4

$$Volume \;=\; \pi \int y^2\, dx$$

$$= \pi \int_1^2 \left(\frac{2}{\sqrt{x}\,(3x - 2)} \right)^2 dx$$

$$= \pi \int_1^2 \frac{4}{x\,(3x - 2)^2}\, dx$$

$$\frac{4}{x\,(3x - 2)^2} = \frac{A}{x} + \frac{B}{3x - 2} + \frac{C}{(3x - 2)^2}$$

$$\frac{4 \times [\,\dots\dots\,]}{[\,x\,(3x - 2)^2\,]} = \frac{A \times [\,\dots\dots\,]}{x} + \frac{B \times [\,\dots\dots\,]}{3x - 2} + \frac{C \times [\,\dots\dots\,]}{(3x - 2)^2}$$

$$4 = A\,(3x - 2)^2 + Bx\,(3x - 2) + Cx$$

$$\text{Let } x = 0 \;:\; 4 = 4A \qquad \Rightarrow A = 1$$

$$\text{Let } x = \frac{2}{3} \;:\; 4 = \frac{2C}{3} \qquad \Rightarrow C = 6$$

$$\text{Let } x = 1 \;:\; 4 = A + B + C \qquad \Rightarrow B = -3$$

$$\frac{4}{x\,(3x - 2)^2} = \frac{1}{x} - \frac{3}{3x - 2} + \frac{6}{(3x - 2)^2}$$

$$\pi \int_1^2 \frac{1}{x\,(3x - 2)^2}\, dx = \pi \int_1^2 \frac{1}{x}\, dx - \pi \int_1^2 \frac{3}{(3x - 2)}\, dx + 2\pi \int_1^2 \frac{3}{(3x - 2)^2}\, dx$$

$$= \pi \Big[\, ln\,|x|\, \Big]_1^2 - \pi \Big[\, ln\,|3x - 2|\, \Big]_1^2 - 2\pi \left[\frac{1}{3x - 2} \right]_1^2$$

$$= \pi \big[\, ln\,2 - ln\,1 \,\big] - \pi \big[\, ln\,4 - ln\,1 \,\big] - 2\pi \left[\frac{1}{4} - \frac{1}{1} \right]$$

$$= \pi \left[ln \left(\frac{1}{2} \right) \right] + \frac{3\pi}{2}$$

$$= \frac{3\pi}{2} - \pi\, ln\,2$$

(As a decimal this is about 2.53)

[8 marks]