4.3 Answers

Further A-Level Pure Mathematics : Core 1

Volumes of Revolution

4.3.1 Solution to 4.1 Teaching Video Example

$$Volume = \pi \int x^2 \, dy$$

$$= \pi \int_0^{\frac{\pi}{4}} \left(\frac{\sqrt{y}}{\cos y} \right)^2 dy$$

$$= \pi \int_0^{\frac{\pi}{4}} y \sec^2 y \, dy$$

$$\boldsymbol{L} \quad \boldsymbol{I} \quad \boldsymbol{D} \quad \boldsymbol{I}$$

$$= \pi \left[y \tan y \right]_0^{\frac{\pi}{4}} - \pi \int_0^{\frac{\pi}{4}} 1 \ \tan y \ dy$$

$$= \pi \left[\frac{\pi}{4} \tan\left(\frac{\pi}{4} \right) - 0 \right] - \pi \left[\ln \left| \sec x \right| \right]_0^{\frac{\pi}{4}}$$

$$= \frac{\pi^2}{4} - \pi \left[\ln \left| \frac{1}{\cos\left(\frac{\pi}{4} \right)} \right| - \ln \left| \frac{1}{\cos(0)} \right| \right]$$

$$= \frac{\pi^2}{4} - \pi \left[\ln \sqrt{2} - \ln 1 \right]$$

$$= \frac{\pi^2}{4} - \pi \ln 2^{\frac{1}{2}}$$

$$= \frac{\pi^2}{4} - \frac{\pi}{2} \ln 2$$

(As a decimal this is about 1.3786)

[10 marks]

4.3.2 Solution to 4.2 Exercise

Answer 1

$$\begin{aligned}
Volume &= \pi \int y^2\,dx = \pi \int_0^{\ln 2} \left(\sqrt{x}\ e^{\frac{x}{2}}\right)^2 dx \\
&= \pi \int_0^{\ln 2} x\,e^x\,dx \\
&\quad \boldsymbol{L} \quad \boldsymbol{I} \quad \boldsymbol{D} \quad \boldsymbol{I} \\
&= \pi \Big[\, x\,e^x \,\Big]_0^{\ln 2} - \pi \int_0^{\ln 2} e^x\,dx \\
&= \pi \big[\, \ln 2\ e^{\ln 2} - 0 \,\big] - \pi \big[\, e^x \,\big]_0^{\ln 2} \\
&= 2\pi \ln 2 - \pi \big[\, e^{\ln 2} - e^0 \,\big] \\
&= 2\pi \ln 2 - \pi \big[\, 2 - 1 \,\big] \\
&= 2\pi \ln 2 - \pi \\
&= \pi\,(2 \ln 2 - 1) \qquad \square
\end{aligned}$$

[10 marks]

Answer 2

$$\begin{aligned}
Volume &= \pi \int x^2\,dy \\
&\quad \pi \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(\frac{\sqrt{y}}{\sin y}\right)^2 dy \\
&= \pi \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} y \csc^2 y\ \ dy \\
&\quad \boldsymbol{L} \quad \boldsymbol{I} \quad \boldsymbol{D} \quad \boldsymbol{I} \\
&= \pi \Big[\, y\,(-1)\cot y \,\Big]_{\frac{\pi}{6}}^{\frac{5\pi}{6}} - \pi \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (-1)\ \cot y\ dy \\
&= -\pi \left[\, \frac{5\pi}{6 \tan\left(\frac{5\pi}{6}\right)} - \frac{\pi}{6 \tan\left(\frac{\pi}{6}\right)} \right] + \pi \big[\, \ln |\sin y| \,\big]_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \\
&= -\pi \left[-\frac{5\pi\sqrt{3}}{6} - \frac{\pi\sqrt{3}}{6} \right] + \pi \left[\, \ln \left|\sin\left(\frac{5\pi}{6}\right)\right| - \ln \left|\sin\left(\frac{\pi}{6}\right)\right| \right] \\
&= \pi^2 \sqrt{3} + \pi \left[\, \ln\left(\frac{1}{2}\right) - \ln\left(\frac{1}{2}\right) \right] \\
&= \pi^2 \sqrt{3} \qquad \square
\end{aligned}$$

[10 marks]

Answer 3

$$
\begin{aligned}
Volume &= \pi \int x^2 \, dy \\
&= \pi \int_0^{\ln 4} \left(\sqrt{y} \; e^{-y} \right)^2 \, dy \\
&= \pi \int_0^{\ln 4} y \, e^{-2y} \, dy \\
&\quad \boldsymbol{L} \quad \boldsymbol{I} \quad \boldsymbol{D} \quad \boldsymbol{I} \\
&= \pi \left[\frac{y \, e^{-2y}}{-2} \right]_0^{\ln 4} - \pi \int_0^{\ln 4} \left(\frac{e^{-2y}}{-2} \right) dy \\
&= -\frac{\pi}{2} \left[\ln 4 \; e^{-2 \ln 4} - 0 \right] + \frac{\pi}{2} \int_0^{\ln 4} e^{-2y} \, dy \\
&= -\frac{\pi}{2} \left[\ln 4 \, e^{\ln 4^{-2}} \right] + \frac{\pi}{2} \left[\frac{e^{-2y}}{-2} \right]_0^{\ln 4} \\
&= -\frac{\pi}{2} \ln 4 \; \times 4^{-2} - \frac{\pi}{4} \left[e^{-2 \ln 4} - e^0 \right] \\
&= -\frac{\pi \ln 4}{32} - \frac{\pi}{4} \left[e^{\ln 4^{-2}} - 1 \right] \\
&= -\frac{\pi \ln 4}{32} - \frac{\pi}{4} \left[4^{-2} - 1 \right] \\
&= -\frac{\pi \ln 4}{32} - \frac{\pi}{4} \left[\frac{1}{16} - \frac{16}{16} \right] \\
&= -\frac{\pi \ln 4}{32} - \frac{\pi}{4} \left[-\frac{15}{16} \right] \\
&= -\frac{\pi \ln 4}{32} + \frac{15 \pi}{64} \\
&= \frac{\pi}{64} \left(15 - 2 \ln 4 \right) \qquad \square
\end{aligned}
$$

[10 marks]