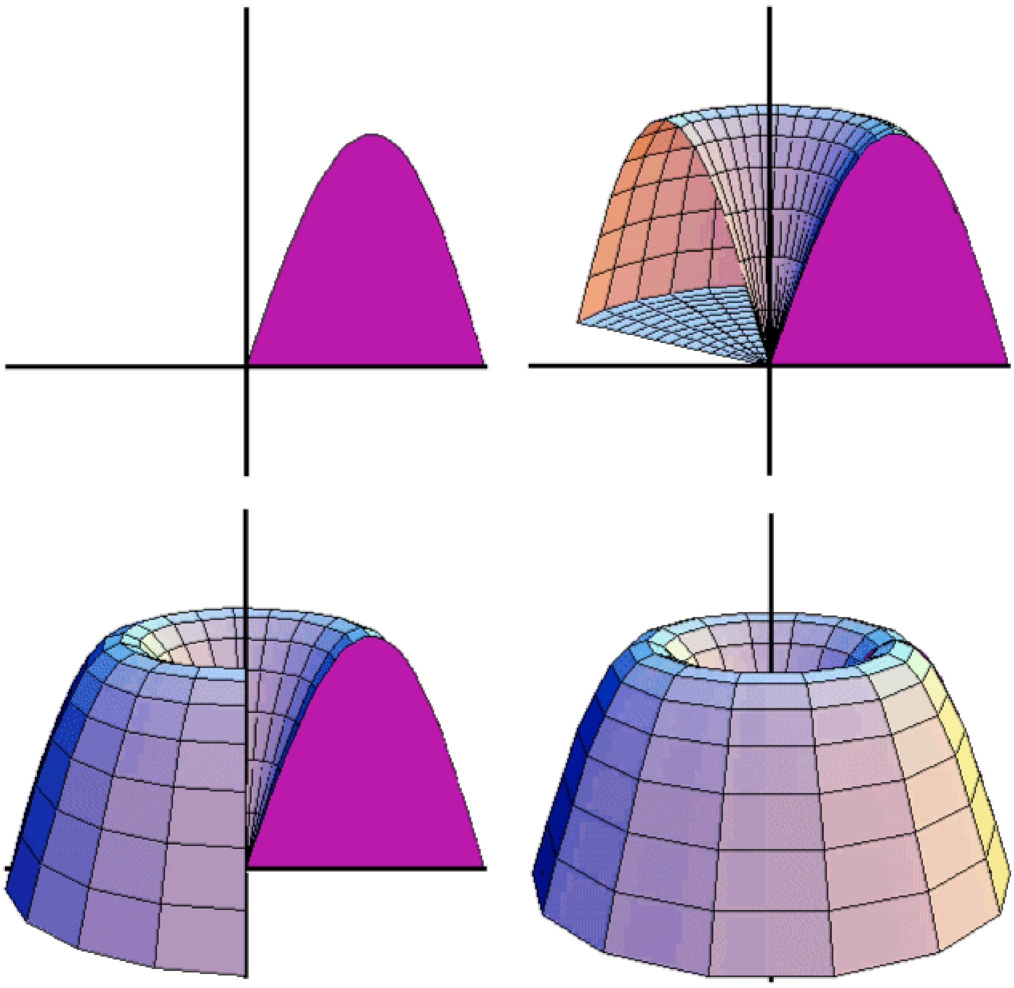
Further Pure A-Level Mathematics
Compulsory Course Component
Core 1

VOLUMES of REVOLUTION



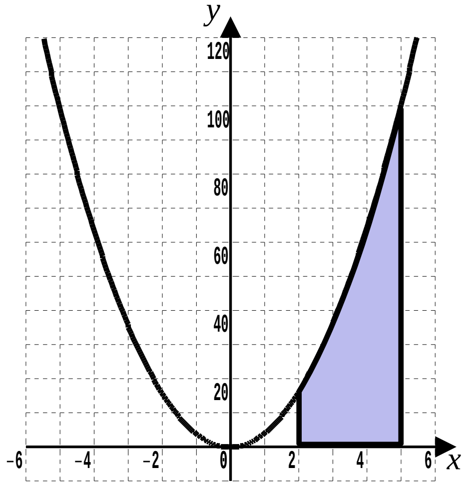
V O L U M E S of R E V O L U T I O N

Lesson 1

Further A-Level Pure Mathematics : Core 1
Volumes of Revolution

1.1 Introduction

Previously, integration was used to determine exactly the area under a curve.



For example, to find the area bounded by the curve $y = 4x^2$, the x-axis and the lines $x = 2$ and $x = 5$, the following piece of mathematics would be set up;

$$
\begin{aligned}
Area &= \int_2^5 4x^2\,dx \\
&= \left[\frac{4x^3}{3}\right]_2^5 \\
&= \left[\frac{4\times 5^3}{3}\right] - \left[\frac{4\times 2^3}{3}\right] \\
&= \left[\frac{500}{3}\right] - \left[\frac{32}{3}\right] \\
&= \left[\frac{468}{3}\right] \\
&= 156
\end{aligned}
$$

Surprisingly, the mathematics to determine the volume swept out by this area as it rotates 360° about the x-axis is not much more complicated.

In an examination, the starting point is the following formula which should be memorised;

$$Volume = \pi \int y^2\,dx$$

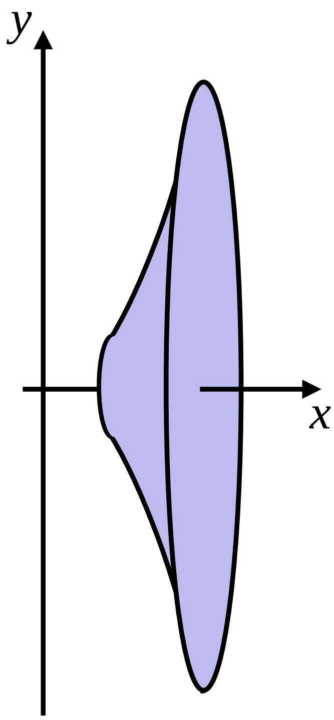
Where this comes from will be investigated shortly, but first let's master how to use it.

1.2 Example

1.2.1 The Question

Find the volume of the solid formed by rotating 360° about the x-axis the curve

$y = 4x^2$ between $x = 2$ and $x = 5$



[4 marks]

1.2.2 The Answer

$$Volume = \pi \int y^2 \, dx$$

$$= \pi \int_2^5 \left(4x^2\right)^2 dx \qquad \text{Common Error is forgetting to square the } y$$

$$= \pi \int_2^5 16x^4 \, dx$$

$$= 16\pi \int_2^5 x^4 \, dx$$

$$= 16\pi \left[\frac{x^5}{5} \right]_2^5$$

$$= 16\pi \left\{ \left[\frac{5^5}{5} \right] - \left[\frac{2^5}{5} \right] \right\}$$

$$= 16\pi \left[\frac{3093}{5} \right]$$

$$= \frac{49488\pi}{5}$$

which, if the units are cm, is about 31 000 cm^3

[4 marks]

1.3 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable

Marks Available : 40

Question 1

Find the volumes of the solids formed by rotating completely about the *x*-axis the areas bounded by the *x*-axis and the given curves and lines.

(i) $y = 5x^2, \quad x = -1, \quad x = 3$

[4 marks]

(ii) $y = 3x - x^2, \quad x = 1, \quad x = 2$

[4 marks]

(iii) $y = 2x - 5, \qquad x = 1, \qquad x = 4$

[4 marks]

(iv) $y = x^3 + 1, \qquad x = -1, \qquad x = 2$

[4 marks]

(v) $y = 1 + \sqrt{x}, \qquad x = 4, \qquad x = 9$

[4 marks]

(vi) $y = x + \dfrac{1}{x}, \qquad x = \dfrac{1}{2}, \qquad x = 2$

[4 marks]

Question 2

Find the volumes of the solids formed by rotating completely about the *x*-axis the areas enclosed by each of the following curves and the *x*-axis.

(i) $y = x^2 - 4$

[4 marks]

(ii) $3y = x^2 (3 - x)$

[4 marks]

Question 3

By first reflecting upon how an integration to find area can be thought of as the summation of an infinite number of infinitely thin rectangles, explain in a similar fashion where the Volume of Revolution formula comes from.

That is, why the formula to find a volume of revolution about the *x*-axis is;

$$Volume = \pi \int y^2 \, dx$$

HINT : Look up a textbook, search online...

[8 marks]

1.4 Answers

Answer 1

(i) 1220π **(ii)** $\dfrac{47\pi}{10}$ **(iii)** 9π

[4, 4, 4 marks]

(iv) $\dfrac{405\pi}{14}$ **(v)** $\dfrac{377\pi}{6}$ **(vi)** $\dfrac{57\pi}{8}$

[4, 4, 4 marks]

Answer 2

(i) $\dfrac{512\,\pi}{15}$ **(ii)** $\dfrac{81\,\pi}{35}$

[4, 4 marks]

Answer 3

When finding area;
Integration is the sum of an infinite number of infinitely thin rectangles.
Their height is y and their width is dx
As the area of a rectangle is height $\times$ width, the rectangles to be summed each have an area of $y \times dx$.
Thus;

$$Area \;=\; \int y\,dx$$

with the slight complication that integration sees areas below the x-axis as negative.
So, more correctly;

$$Area \;=\; \int \;|\,y\,|\;dx$$

When finding volume;
Integration is the sum of an infinite number of infinitely thin cylinders.
Their radius is y and their height is dx
As the area of a cylinder is $\pi \times radius^2$, the cylinders to be summed each have a volume of $\pi \times y^2 \times dx$
Thus;

$$Volume \;=\; \pi \int y^2\,dx$$

As y^2 is always positive, there is no need to worry about negative volumes.

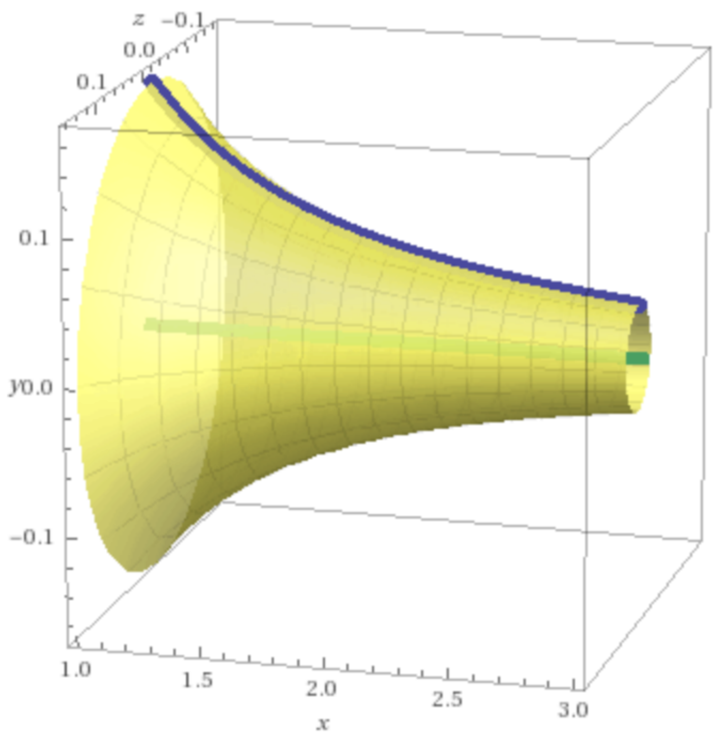
[8 marks]

2.1 Integration Techniques

As a topic, Volumes of Revolution provides an opportunity to revise and consolidate the integration techniques taught earlier in the Year 2 A-Level course.

2.2 Partial Fractions Revisited

Find the volume of the solid generated when the profile curve $y = \dfrac{1}{\sqrt{x}\,(5x + 1)}$ between $x = 1$ and $x = 3$ is rotated 2π radians about the x-axis.



[8 marks]

An answer to this question would begin by quoting the result,

$$Volume = \pi \int y^2 \, dx$$

For this particular problem,

$$Volume = \pi \int_1^3 \left(\frac{1}{\sqrt{x}\,(5x + 1)} \right)^2 dx$$

$$= \pi \int_1^3 \frac{1}{x\,(5x + 1)^2} \, dx$$

This tricky integration requires the use of partial fractions.

Teaching Video (Part I) : http://www.NumberWonder.co.uk/v9087/1a.mp4

Teaching Video (Part II) : http://www.NumberWonder.co.uk/v9087/1b.mp4

[8 marks]

2.3 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable

Marks Available : 40

Question 1

Show that the volume of the solid generated when the profile curve $y = \dfrac{1}{x\sqrt{x - 1}}$ between $x = 2$ and $x = 3$ is rotated 2π radians about the x-axis is $\pi\, ln\left(\dfrac{4}{3}\right) - \dfrac{\pi}{6}$

[8 marks]

Question 2

(i) Show that $\int_1^2 \frac{1}{(4x - 3)^2}\, dx = \frac{1}{5}$

[4 marks]

(ii) Show that the volume swept out when the profile curve $y = \frac{4\sqrt{x}}{4x - 3}$

between $x = 1$ and $x = 2$ is rotated $2\pi^c$ about the x-axis is $\pi \ln 5 + \frac{12\pi}{5}$

[8 marks]

Question 3

(i) Show that $\int_1^3 \frac{1}{(2x - 1)^3}\, dx = \frac{6}{25}$

[4 marks]

(ii) Show that the volume swept out when the profile curve $y = \frac{2\sqrt{2}\, x}{(2x - 1)^{1.5}}$

between $x = 1$ and $x = 3$ is rotated $2\pi^c$ about the x-axis is given by,

$$Volume = \pi \ln 5 + \frac{52\pi}{25}$$

[8 marks]

Question 4

Find the volume of the solid generated when the profile curve $y = \dfrac{2}{\sqrt{x}\,(3x - 2)}$ between $x = 1$ and $x = 2$ is rotated 2π radians about the x-axis.

[8 marks]

2.4 Answers

2.4.1 Solution to 2.2 Partial Fractions Revisited

$$Volume = \pi \int y^2 \, dx$$

$$= \pi \int_1^3 \left(\frac{1}{\sqrt{x}\,(5x+1)} \right)^2 dx$$

$$= \pi \int_1^3 \frac{1}{x(5x+1)^2} \, dx$$

$$\frac{1}{x(5x+1)^2} = \frac{A}{x} + \frac{B}{5x+1} + \frac{C}{(5x+1)^2}$$

$$\frac{1 \times [\,\dots\dots\,]}{[\,x(5x+1)^2\,]} = \frac{A \times [\,\dots\dots\,]}{x} + \frac{B \times [\,\dots\dots\,]}{5x+1} + \frac{C \times [\,\dots\dots\,]}{(5x+1)^2}$$

$$1 = A(5x+1)^2 + Bx(5x+1) + Cx$$

$$\text{Let } x = 0 \ : \ 1 = A$$

$$\text{Let } x = -\frac{1}{5} \ : \ 1 = -\frac{C}{5} \Rightarrow C = -5$$

$$\text{Let } x = 1 \ : \ 1 = 36 + 6B - 5 \Rightarrow B = -5$$

$$\frac{1}{x(5x+1)^2} = \frac{1}{x} - \frac{5}{5x+1} - \frac{5}{(5x+1)^2}$$

$$\pi \int_1^3 \frac{1}{x(5x+1)^2} \, dx = \pi \int_1^3 \frac{1}{x} \, dx - \pi \int_1^3 \frac{5}{(5x+1)} \, dx - \pi \int_1^3 \frac{5}{(5x+1)^2} \, dx$$

$$= \pi \left[\ln|x| \right]_1^3 - \pi \left[\ln|5x+1| \right]_1^3 + \pi \left[\frac{1}{5x+1} \right]_1^3$$

$$= \pi [\ln 3 - \ln 1] - \pi [\ln 16 - \ln 6] + \pi \left[\frac{1}{16} - \frac{1}{6} \right]$$

$$= \pi \left[\ln \left(\frac{3 \times 6}{16} \right) \right] - \frac{5\pi}{48}$$

$$= \pi \ln \left(\frac{9}{8} \right) - \frac{5\pi}{48}$$

$$= 0.0428$$

[8 marks]

2.4.2 Solution to 2.3 Exercise

Answer 1

$$\begin{aligned}
Volume &= \pi \int y^2 \, dx \\
&= \pi \int_2^3 \left(\frac{1}{x\sqrt{x-1}} \right)^2 dx \\
&= \pi \int_2^3 \frac{1}{x^2\,(x-1)} \, dx
\end{aligned}$$

$$\frac{1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}$$

$$\frac{1 \times [\ldots\ldots]}{[\,x^2(x-1)\,]} = \frac{A \times [\ldots\ldots]}{x} + \frac{B \times [\ldots\ldots]}{x^2} + \frac{C \times [\ldots\ldots]}{x-1}$$

$$1 = Ax(x-1) + B(x-1) + Cx^2$$

$$\text{Let } x = 0 \;:\; 1 = -B \quad \Rightarrow \quad B = -1$$

$$\text{Let } x = 1 \;:\; 1 = C$$

$$\text{Let } x = 2 \;:\; 1 = 2A + B + 4C$$

$$1 = 2A - 1 + 4 \quad \Rightarrow A = -1$$

$$\frac{1}{x^2(x-1)} = -\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x-1}$$

$$\begin{aligned}
\pi \int_2^3 \frac{1}{x^2(x-1)} \, dx &= \pi \int_2^3 \frac{1}{x-1} \, dx - \pi \int_2^3 \frac{1}{x} \, dx - \pi \int_2^3 \frac{1}{x^2} \, dx \\
&= \pi \left[ln|x-1| \right]_2^3 - \pi \left[ln|x| \right]_2^3 + \pi \left[\frac{1}{x} \right]_2^3 \\
&= \pi [\, ln\,2 - ln\,1 \,] - \pi [\, ln\,3 - ln\,2 \,] + \pi \left[\frac{1}{3} - \frac{1}{2} \right] \\
&= \pi \left[ln \left(\frac{2 \times 2}{3} \right) \right] - \frac{\pi}{6} \\
&= \pi \, ln \left(\frac{4}{3} \right) - \frac{\pi}{6} \qquad \square
\end{aligned}$$

(As a decimal this is about 0.380)

[8 marks]

Answer 2

(i)

$$\int_1^2 \frac{1}{(4x-3)^2}\,dx = \frac{1}{4}\int_1^2 \frac{4}{(4x-3)^2}\,dx$$

$$= \frac{1}{4}\left[-\frac{1}{4x-3}\right]_1^2$$

$$= -\frac{1}{4}\left[\frac{1}{5}-\frac{1}{1}\right]$$

$$= -\frac{1}{4}\left[-\frac{4}{5}\right]$$

$$= \frac{1}{5} \qquad \square$$

[4 marks]

(ii)

$$Volume = \pi \int y^2\,dx$$

$$= \pi \int_1^2 \left(\frac{4\sqrt{x}}{4x-3}\right)^2 dx$$

$$= \pi \int_1^2 \frac{16x}{(4x-3)^2}\,dx$$

$$\frac{16x}{(4x-3)^2} = \frac{A}{4x-3} + \frac{B}{(4x-3)^2}$$

$$\frac{16x \times [\ \dots\dots\ ]}{[\ (4x-3)^2\ ]} = \frac{A \times [\ \dots\dots\ ]}{4x-3} + \frac{B \times [\ \dots\dots\ ]}{(4x-3)^2}$$

$$16x = A(4x-3) + B$$

$$\text{Let } x = \frac{3}{4} \;:\; 12 = B \quad \Rightarrow \quad B = 12$$

$$\text{Let } x = 1 \;:\; 16 = A + B \quad \Rightarrow \quad A = 4$$

$$\frac{16x}{(4x-3)^2} = \frac{4}{4x-3} + \frac{12}{(4x-3)^2}$$

$$\pi\int_1^2 \frac{16x}{(4x-3)^2}\,dx = \pi\int_1^2 \frac{4}{4x-3}\,dx + 3\pi\int_1^2 \frac{4}{(4x-3)^2}\,dx$$

$$= \pi\left[\,ln|4x-3|\,\right]_1^2 + 3\pi\left[-\frac{1}{5}+\frac{1}{1}\right]$$

$$= \pi\left[\,ln\,5 - ln\,1\,\right] + \frac{12\pi}{5}$$

$$= \pi\,ln\,5 + \frac{12\pi}{5} \qquad \square$$

(As a decimal this is about 12.6)

Note : This question could also be done using the substitution $t = 4x - 3$

[8 marks]

Answer 3

(i)

$$\int_1^3 \frac{1}{(2x-1)^3}\,dx = \frac{1}{2}\int_1^3 \frac{2}{(2x-1)^3}\,dx$$

$$= \frac{1}{2}\left[\frac{1}{(2x-1)^2} \times \frac{1}{(-2)}\right]_1^3$$

$$= -\frac{1}{4}\left[\frac{1}{25} - \frac{1}{1}\right]$$

$$= -\frac{1}{4}\left[-\frac{24}{25}\right]$$

$$= \frac{6}{25} \qquad \square$$

[4 marks]

(ii)

$$Volume = \pi \int y^2\,dx$$

$$= \pi \int_1^3 \left(\frac{2\sqrt{2}\,x}{(2x-1)^{1.5}}\right)^2 dx$$

$$= \pi \int_1^3 \frac{8x^2}{(2x-1)^3}\,dx$$

$$\frac{8x^2}{(2x-1)^3} = \frac{A}{2x-1} + \frac{B}{(2x-1)^2} + \frac{C}{(2x-1)^3}$$

$$\frac{8x^2 \times [\,\dots\dots\,]}{[\,(2x-1)^3\,]} = \frac{A \times [\,\dots\dots\,]}{2x-1} + \frac{B \times [\,\dots\dots\,]}{(2x-1)^2} + \frac{C \times [\,\dots\dots\,]}{(2x-1)^3}$$

$$8x^2 = A(2x-1)^2 + B(2x-1) + C$$

Let $x = \frac{1}{2}$: $2 = C \quad \Rightarrow \quad C = 2$

Let $x = 0$: $0 = A - B + C \quad \Rightarrow \quad A - B = -2$

Let $x = 1$: $8 = A + B + C \quad \Rightarrow \quad A + B = 6$

$$\therefore A = 2, \quad B = 4$$

$$\frac{8x^2}{(2x-1)^3} = \frac{2}{2x-1} + \frac{4}{(2x-1)^2} + \frac{2}{(2x-1)^3}$$

$$\pi \int_1^3 \frac{8x^2}{(2x-1)^3}\,dx = \pi \int_1^3 \frac{2}{2x-1}\,dx + 2\pi \int_1^3 \frac{2}{(2x-1)^2}\,dx + 2\pi \int_1^3 \frac{1}{(2x-1)^3}\,dx$$

$$= \pi \left[\ln|2x-1| \right]_1^3 + 2\pi \left[-\frac{1}{2x-1} \right]_1^3 + 2\pi \left[\frac{6}{25} \right]$$

$$= \pi \left[\ln 5 - \ln 1 \right] - 2\pi \left[\frac{1}{5} - \frac{1}{1} \right] + \frac{12\pi}{25}$$

$$= \pi \ln 5 + \frac{8\pi}{5} + \frac{12\pi}{25}$$

$$= \pi \ln 5 + \frac{52\pi}{25} \qquad \square$$

(As a decimal, this is about = 11.6)

Note : This question could also be done using the substitution $t = 2x - 1$

[8 marks]

Answer 4

$$Volume = \pi \int y^2 \, dx$$

$$= \pi \int_1^2 \left(\frac{2}{\sqrt{x}\ (3x - 2)} \right)^2 dx$$

$$= \pi \int_1^2 \frac{4}{x(3x - 2)^2} \, dx$$

$$\frac{4}{x(3x - 2)^2} = \frac{A}{x} + \frac{B}{3x - 2} + \frac{C}{(3x - 2)^2}$$

$$\frac{4 \times [\,\dots\dots\,]}{[\,x(3x - 2)^2\,]} = \frac{A \times [\,\dots\dots\,]}{x} + \frac{B \times [\,\dots\dots\,]}{3x - 2} + \frac{C \times [\,\dots\dots\,]}{(3x - 2)^2}$$

$$4 = A(3x - 2)^2 + Bx(3x - 2) + Cx$$

Let $x = 0$: $4 = 4A$ $\quad\Rightarrow A = 1$

Let $x = \frac{2}{3}$: $4 = \frac{2C}{3}$ $\quad\Rightarrow C = 6$

Let $x = 1$: $4 = A + B + C$ $\quad\Rightarrow B = -3$

$$\frac{4}{x(3x - 2)^2} = \frac{1}{x} - \frac{3}{3x - 2} + \frac{6}{(3x - 2)^2}$$

$$\pi \int_1^2 \frac{1}{x(3x - 2)^2} \, dx = \pi \int_1^2 \frac{1}{x} \, dx - \pi \int_1^2 \frac{3}{(3x - 2)} \, dx + 2\pi \int_1^2 \frac{3}{(3x - 2)^2} \, dx$$

$$= \pi \left[\ln|x| \right]_1^2 - \pi \left[\ln|3x - 2| \right]_1^2 - 2\pi \left[\frac{1}{3x - 2} \right]_1^2$$

$$= \pi [\ln 2 - \ln 1] - \pi [\ln 4 - \ln 1] - 2\pi \left[\frac{1}{4} - \frac{1}{1} \right]$$

$$= \pi \left[\ln\left(\frac{1}{2} \right) \right] + \frac{3\pi}{2}$$

$$= \frac{3\pi}{2} - \pi \ln 2$$

(As a decimal this is about 2.53)

[8 marks]

3.1 Spin About Y

The volume of revolution formed when $x = f(y)$ is rotated through 2π radians about the y-axis between $y = a$ and $y = b$ is given by

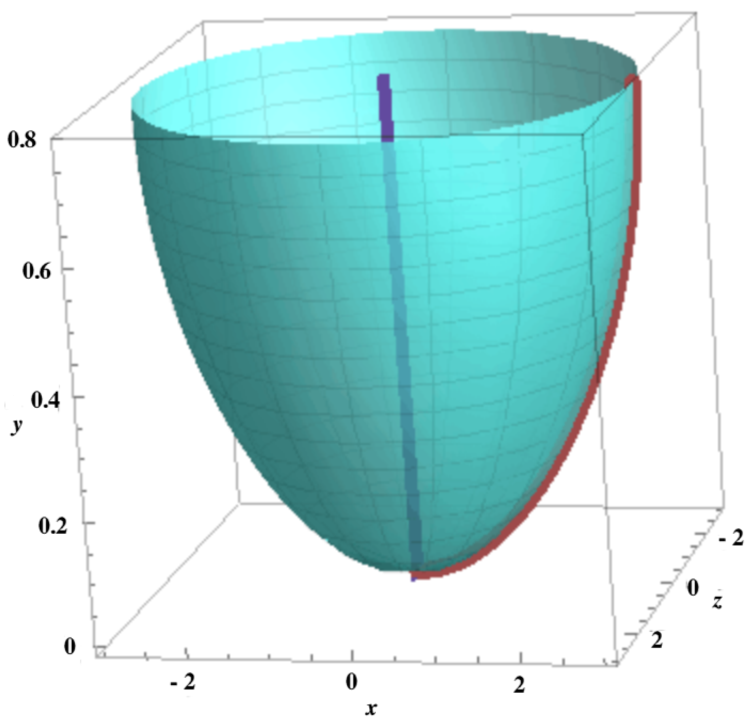
$$Volume = \pi \int_a^b x^2 \, dy$$

3.2.1 The Question

Find the exact volume swept out by the part of the following profile curve between the bounding lines given when it is rotated by $2\pi^c$ about the y-axis.

$$x = 3\sqrt{\sin(2y)}, \qquad y = 0, \qquad y = \frac{\pi}{4}$$

Give your answer as a multiple of π



You may like to try answering this question yourself before taking a look at the answer over the page.

NOTE : RADIANS must be used whenever trigonometry and calculus mix.

[5 marks]

3.2.2 The Answer

$$
\begin{aligned}
Volume &= \pi \int x^2 \, dy \\
&= \pi \int_0^{\frac{\pi}{4}} 9 \sin(2y) \, dy \\
&= \frac{9\pi}{2} \int_0^{\frac{\pi}{4}} 2 \sin(2y) \, dy \qquad \text{Setting up a “Chain Rule Backwards”} \\
&= \frac{9\pi}{2} \Big[-\cos(2y) \Big]_0^{\frac{\pi}{4}} \\
&= \frac{9\,\pi}{2} \left[-\cos\left(\frac{\pi}{2}\right) + \cos(0) \right] \\
&= \frac{9\,\pi}{2} \left[-0 + 1 \right] \\
&= \frac{9\pi}{2}
\end{aligned}
$$

[5 marks]

3.3 A Handy Table is Trigonometric Derivatives and Integrals

$f(x)$	$f'(x)$	Given ?
$\sin x$	$\cos x$	
$\cos x$	$-\sin x$	
$\tan x$	$\sec^2 x$	*
$\sec x$	$\sec x \ \tan x$	*
$\csc x$	$-\csc x \ \cot x$	*
$\cot x$	$-\csc^2 x$	*
$\ln x$	$\frac{1}{x}$	
$\ln\lvert \sec x \rvert$	$\tan x$	*
$\ln\lvert \sin x \rvert$	$\cot x$	*
$\ln \lvert \sec x + \tan x \rvert$	$\sec x$	*
$\ln \left\lvert \tan\left(\frac{1}{2} x + \frac{1}{4} \pi \right) \right\rvert$	$\sec x$	*
$-\ln \lvert \csc x + \cot x \rvert$	$\csc x$	*
$\ln \left\lvert \tan\left(\frac{1}{2} x \right) \right\rvert$	$\csc x$	*
e^x	e^x	

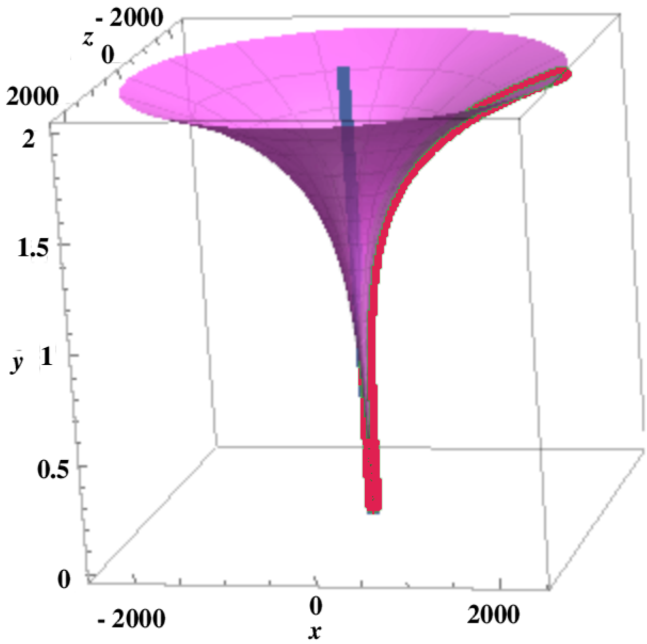
* Formulae marked with an asterisk are provided
in the examination in a book of formulae.

3.4 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 40

Question 1



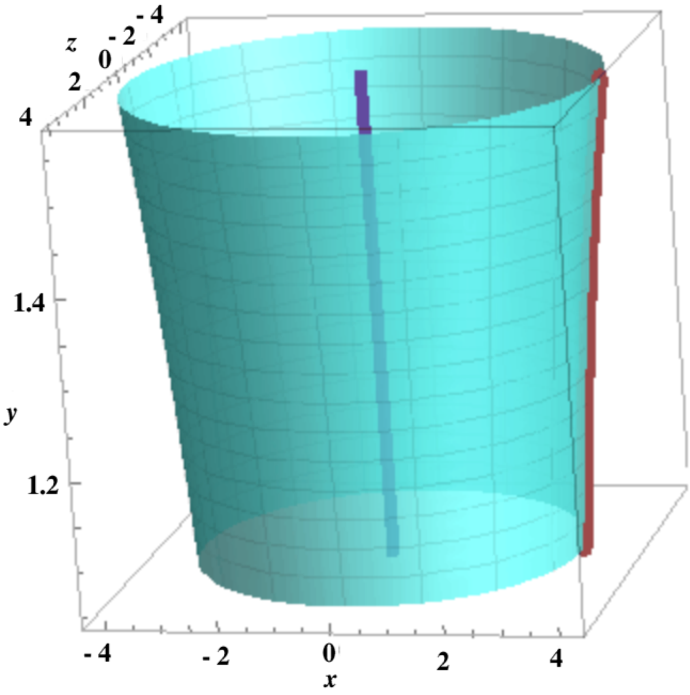
(i) Show that the volume swept out by the curve $x = 6\,e^{3y}$ between $y = 0$ and $y = 2$ when it is rotated by $2\pi^c$ about the y-axis is exactly $6\pi\,(\,e^{12} - 1\,)$

[5 marks]

(ii) Give this volume as a decimal correct to three decimal places.

[1 mark]

Question 2



(i) Show that the volume swept out by the curve $x = \dfrac{3}{cos\,(0.5y)}$ between $y = \dfrac{\pi}{3}$ and $y = \dfrac{\pi}{2}$ when it is rotated by $2\pi^{c}$ about the y-axis is exactly $6\pi\,(3 - \sqrt{3}\,)$

[5 marks]

(ii) Give this volume as a decimal correct to three decimal places.

[1 mark]

Question 3

The volume of revolution of a shot glass is 4π cm^3 exactly.
The profile curve is $x = \sqrt{y}$ and the rotation is about the y-axis.
The lower limit of the profile curve is $y = 1$ cm.
The upper limit is not known, call it a cm.
Calculate the upper limit, a, of the profile curve.
Clearly showing your method and working.

[5 marks]

Question 4

Find the exact volume swept out by the part of the following profile curve between the bounding lines given when it is rotated by $2\pi^c$ about the y-axis.

$$x = y + \frac{1}{\sqrt{y}}, \qquad x = 1, \qquad x = 4$$

Write your answer in the form $\pi\,(K + ln\ 4)$ where K is a constant, the value of which you should determine.

[5 marks]

Question 5

(i) Use the product rule to differentiate with respect to y,

$$x = y \ln y$$

[2 marks]

(ii) Hence show that,

$$\int_1^8 (1 + \ln y)\, dy = 24 \ln 2$$

[3 marks]

(iii) Hence state the volume of the solid formed when the profile curve

$$x = \sqrt{1 + \ln y}$$

is rotated 2π radians about the y-axis between $y = 1$ and $y = 8$

[1 marks]

Question 6

Show that the volume swept out by the curve

$$x = \frac{1}{4} e^{\frac{y}{2}}$$

between $y = 0$ and $y = 4 \ln 3$ when it is rotated by $2\pi^c$ about the y-axis is exactly 5π

[6 marks]

Question 7

Find the volume swept out by the part of the following profile curve between the bounding lines given when it is rotated by $2\pi^c$ about the y-axis.

$$x = \frac{1}{3}\sqrt{sin\left(\frac{y}{2}\right)}, \qquad y = 0, \qquad y = \frac{\pi}{6}$$

Give your answer correct to 3 significant figures.

[6 marks]

3.5 Answers to 3.4 Exercise

Answer 1

(i)

$$\begin{aligned} Volume &= \pi \int x^2 \, dy \\ &= \pi \int_0^2 \left(6\, e^{3y} \right)^2 dy \\ &= \pi \int_0^2 36\, e^{6y} \, dy \\ &= 6\pi \int_0^2 6\, e^{6y} \, dy \\ &= 6\pi \left[e^{6y} \right]_0^2 \\ &= 6\pi \left(e^{12} - e^0 \right) \\ &= 6\pi \left(e^{12} - 1 \right) \qquad \square \end{aligned}$$

[5 marks]

(ii) 3 067 836.693

[1 mark]

Answer 2

(i)

$$\begin{aligned} Volume &= \pi \int x^2 \, dy \\ &= \pi \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \left(\frac{3}{\cos(0.5y)} \right)^2 dy \\ &= 9\pi \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sec^2\left(\frac{y}{2} \right) dy \\ &= 18\pi \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \left(\frac{1}{2} \right) \sec^2\left(\frac{y}{2} \right) dy \\ &= 18\pi \left[\tan\left(\frac{y}{2} \right) \right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} \\ &= 18\pi \left[\tan\left(\frac{\pi}{4} \right) - \tan\left(\frac{\pi}{6} \right) \right] \\ &= 18\pi \left[1 - \frac{\sqrt{3}}{3} \right] \\ &= 6\pi \left(3 - \sqrt{3} \right) \end{aligned}$$

[5 marks]

(ii) 23.900

[1 mark]

Answer 3

$$Volume = \pi \int x^2\, dy$$

$$4\pi = \pi \int_1^a \left(\sqrt{y}\right)^2 dy$$

$$4 = \int_1^a y\ dy$$

$$4 = \left[\frac{y^2}{2}\right]_1^a$$

$$4 = \frac{a^2}{2} - \frac{1}{2}$$

$$\frac{a^2}{2} = \frac{9}{2}$$

$$a^2 = 9$$

$a = 3$ (Ignore $a = -3$ as it's to be an upper limit)

[5 marks]

Answer 4

$$Volume = \pi \int x^2\, dy$$

$$= \pi \int_1^4 \left(y + \frac{1}{\sqrt{y}}\right)^2 dy$$

$$= \pi \int_1^4 \left(y^2 + 2y^{\frac{1}{2}} + \frac{1}{y}\right) dy$$

$$= \pi \left[\frac{y^3}{3} + \frac{4y^{\frac{3}{2}}}{3} + \ln y\right]_1^4$$

$$= \pi \left\{\left[\frac{64}{3} + \frac{32}{3} + \ln 4\right] - \left[\frac{1}{3} + \frac{4}{3} + \ln 1\right]\right\}$$

$$= \pi \left\{\left[\frac{96}{3} + \ln 4\right] - \left[\frac{5}{3}\right]\right\}$$

$$= \pi \left\{\frac{91}{3} + \ln 4\right\}$$

That is, $K = \frac{91}{3}$

[5 marks]

Answer 5

(i)

$$\frac{dx}{dy} = ln\, y + 1$$

[2 marks]

(ii)

$$\int_1^8 (1 + ln\, y)\, dy = [\, y\, ln\, y \,]_1^8$$

$$= 8\, ln\, 8 - ln\, 1$$

$$= 8\, ln\, 2^3 - 0$$

$$= 24\, ln\, 2 \qquad \square$$

[3 marks]

(iii)

$$24\, \pi\, ln\, 2$$

[1 mark]

Answer 6

$$Volume = \pi \int x^2\, dy$$

$$= \pi \int_0^{4\, ln\, 3} \left(\frac{1}{4}\, e^{\frac{y}{2}} \right)^2 dy$$

$$= \frac{\pi}{16} \int_0^{4\, ln\, 3} e^y\, dy$$

$$= \frac{\pi}{16} [\, e^y \,]_0^{4\, ln\, 3}$$

$$= \frac{\pi}{16} \left[\, e^{4\, ln\, 3} - e^0 \,\right]$$

$$= \frac{\pi}{16} \left[\, e^{ln\, 3^4} - 1 \,\right]$$

$$= \frac{\pi}{16} [\, 81 - 1 \,]$$

$$= 5\, \pi \qquad \square$$

[6 marks]

Answer 7

$$Volume = \pi \int x^2 \, dy$$

$$= \pi \int_0^{\frac{\pi}{6}} \frac{1}{9} sin \left(\frac{y}{2} \right) dy$$

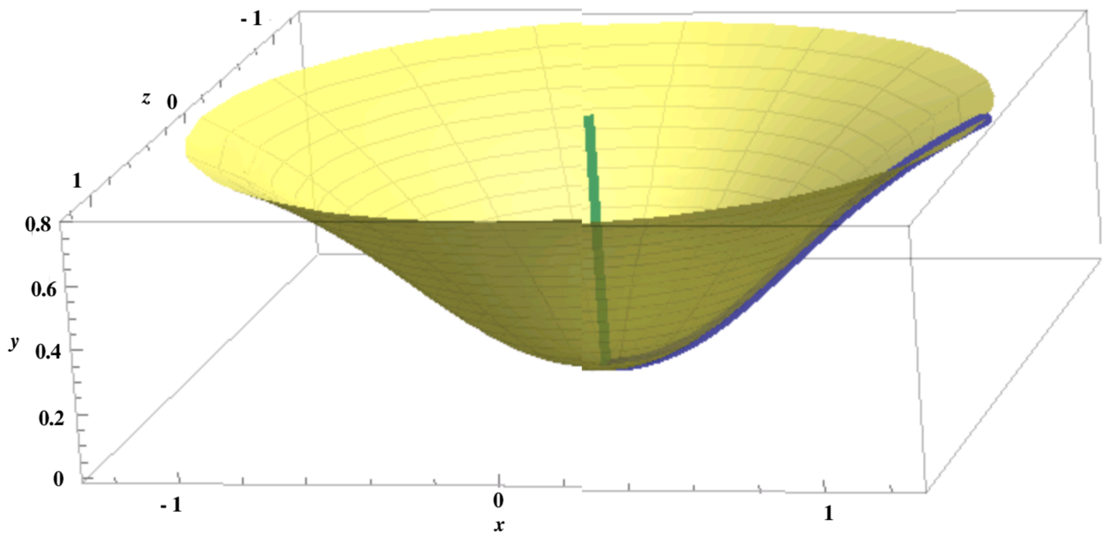
$$= \frac{\pi}{9} \left[-2 \, cos \left(\frac{y}{2} \right) \right]_0^{\frac{\pi}{6}}$$

$$= \frac{\pi}{9} \left\{ \left(-2 \, cos \left(\frac{\pi}{12} \right) \right) - \left(-2 \, cos \, 0 \right) \right\}$$

$$= 0.0238 \qquad \text{(3 significant figures asked for)}$$

[6 marks]

4.1 Revolving Integration By Parts



Find the exact volume swept out by the part of the following profile curve between the bounding lines given when it is rotated by $2\pi^c$ about the y-axis.

$$x = \frac{\sqrt{y}}{\cos y}, \qquad y = 0, \qquad y = \frac{\pi}{4}$$

Teaching Video : http://www.NumberWonder.co.uk/v9087/4.mp4

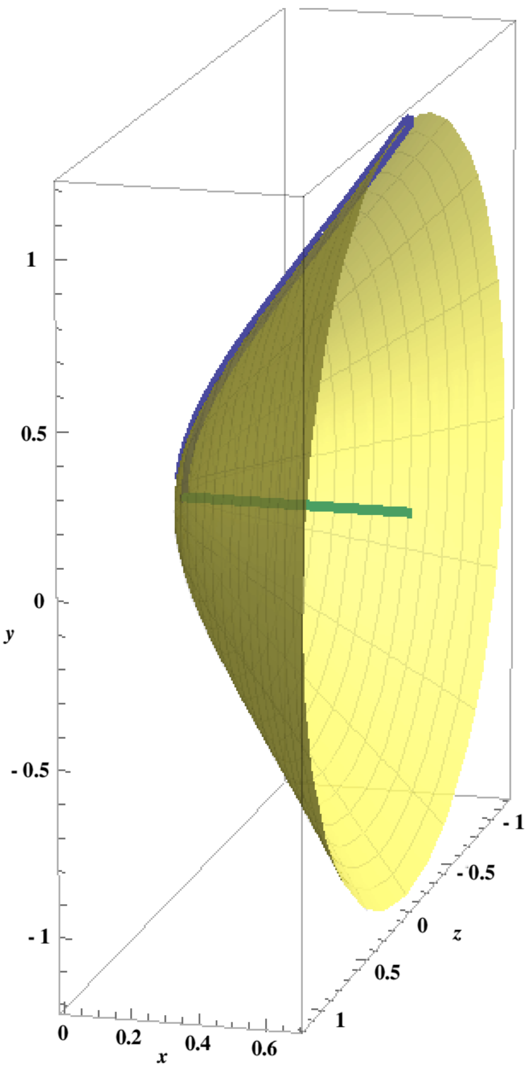
[10 marks]

4.2 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 30

Question 1

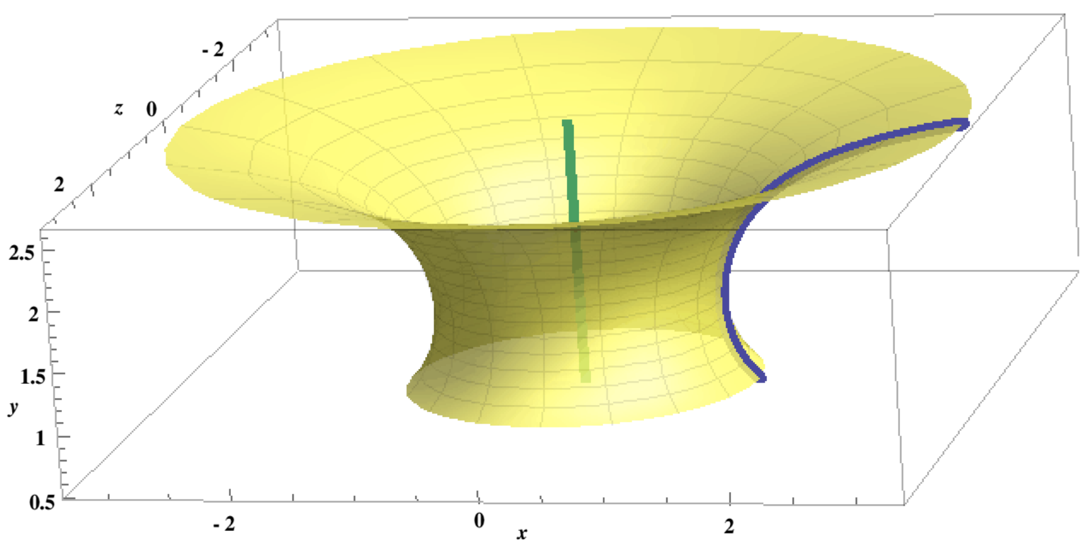
Show that the volume swept out by the curve $y = \sqrt{x}\, e^{\frac{x}{2}}$ between $x = 0$ and $x = \ln 2$ when it is rotated by $2\pi^c$ about the x-axis is exactly, $\pi\,(\,2\ln(2) - 1\,)$



[10 marks]

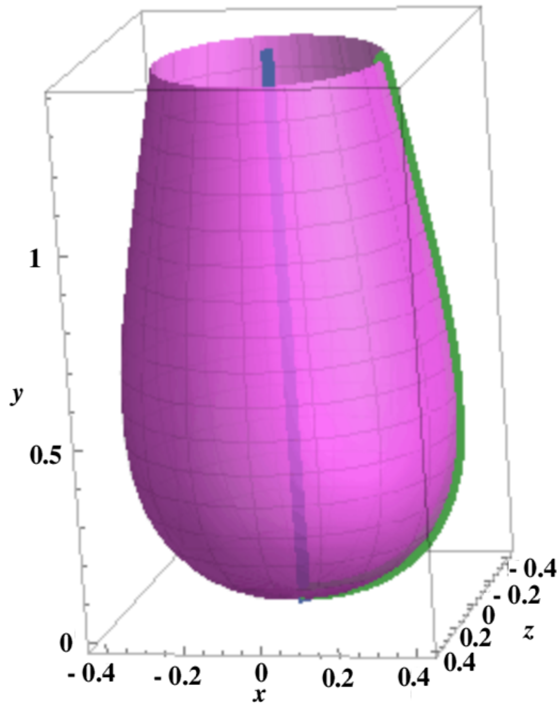
Question 2

Show that the volume swept out by the curve $x = \dfrac{\sqrt{y}}{\sin y}$ between $y = \dfrac{\pi}{6}$ and $y = \dfrac{5\pi}{6}$ when it is rotated by $2\pi^c$ about the y-axis is exactly, $\pi^2\sqrt{3}$



[10 marks]

Question 3



Show that the volume swept out by the curve $x = \sqrt{y}\, e^{-y}$ between $y = 0$ and $y = \ln 4$ when it is rotated by $2\pi^c$ about the y-axis is exactly, $\frac{\pi}{64}(15 - 2\ln 4)$

[10 marks]

4.3 Answers

4.3.1 Solution to 4.1 Teaching Video Example

$$Volume = \pi \int x^2 \, dy$$

$$= \pi \int_0^{\frac{\pi}{4}} \left(\frac{\sqrt{y}}{\cos y} \right)^2 dy$$

$$= \pi \int_0^{\frac{\pi}{4}} y \sec^2 y \, dy$$

L I D I

$$= \pi \left[y \tan y \right]_0^{\frac{\pi}{4}} - \pi \int_0^{\frac{\pi}{4}} 1 \ \tan y \ dy$$

$$= \pi \left[\frac{\pi}{4} \tan\left(\frac{\pi}{4} \right) - 0 \right] - \pi \left[\ln |\sec x| \right]_0^{\frac{\pi}{4}}$$

$$= \frac{\pi^2}{4} - \pi \left[\ln \left| \frac{1}{\cos\left(\frac{\pi}{4}\right)} \right| - \ln \left| \frac{1}{\cos(0)} \right| \right]$$

$$= \frac{\pi^2}{4} - \pi \left[\ln \sqrt{2} - \ln 1 \right]$$

$$= \frac{\pi^2}{4} - \pi \ln 2^{\frac{1}{2}}$$

$$= \frac{\pi^2}{4} - \frac{\pi}{2} \ln 2$$

(As a decimal this is about 1.3786)

[10 marks]

4.3.2 Solution to 4.2 Exercise

Answer 1

$$Volume = \pi \int y^2\, dx = \pi \int_0^{ln\,2} \left(\sqrt{x}\ \ e^{\frac{x}{2}} \right)^2 dx$$

$$= \pi \int_0^{ln\,2} x\, e^x\, dx$$

$$\textbf{L} \quad \textbf{I} \quad \textbf{D} \quad \textbf{I}$$

$$= \pi \left[x\, e^x \right]_0^{ln\,2} - \pi \int_0^{ln\,2} e^x\, dx$$

$$= \pi \left[ln\,2\ \ e^{ln\,2} - 0 \right] - \pi \left[e^x \right]_0^{ln\,2}$$

$$= 2\pi\, ln\,2 - \pi \left[e^{ln\,2} - e^0 \right]$$

$$= 2\pi\, ln\,2 - \pi \left[2 - 1 \right]$$

$$= 2\pi\, ln\,2 - \pi$$

$$= \pi \,(\, 2\, ln\,2 - 1\,) \qquad \square$$

[10 marks]

Answer 2

$$Volume = \pi \int x^2\, dy$$

$$\pi \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(\frac{\sqrt{y}}{sin\,y} \right)^2 dy$$

$$= \pi \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} y\, csc^2 y \ \ dy$$

$$\textbf{L} \quad \textbf{I} \quad \textbf{D} \quad \textbf{I}$$

$$= \pi \left[y\ (-1)\, cot\, y \right]_{\frac{\pi}{6}}^{\frac{5\pi}{6}} - \pi \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (-1)\ cot\, y\ \ dy$$

$$= -\pi \left[\frac{5\pi}{6\, tan\left(\frac{5\pi}{6}\right)} - \frac{\pi}{6\, tan\left(\frac{\pi}{6}\right)} \right] + \pi \left[ln\, \left| sin\, y \right| \right]_{\frac{\pi}{6}}^{\frac{5\pi}{6}}$$

$$= -\pi \left[-\frac{5\pi\sqrt{3}}{6} - \frac{\pi\sqrt{3}}{6} \right] + \pi \left[ln \left| sin\left(\frac{5\pi}{6}\right) \right| - ln \left| sin\left(\frac{\pi}{6}\right) \right| \right]$$

$$= \pi^2 \sqrt{3} + \pi \left[ln\left(\frac{1}{2}\right) - ln\left(\frac{1}{2}\right) \right]$$

$$= \pi^2 \sqrt{3} \qquad \square$$

[10 marks]

Answer 3

$$
\begin{aligned}
Volume &= \pi \int x^2 \, dy \\
&= \pi \int_0^{ln\,4} \left(\sqrt{y}\ e^{-y} \right)^2 dy \\
&= \pi \int_0^{ln\,4} y\, e^{-2y}\, dy \\
& \quad \textbf{\textit{L}} \qquad \textbf{\textit{I}} \qquad\quad \textbf{\textit{D}} \quad \textbf{\textit{I}} \\
&= \pi \left[\frac{y\, e^{-2y}}{-2} \right]_0^{ln\,4} - \pi \int_0^{ln\,4} \left(\frac{e^{-2y}}{-2} \right) dy \\
&= -\frac{\pi}{2} \left[ln\,4\ e^{-2\,ln\,4} - 0 \right] + \frac{\pi}{2} \int_0^{ln\,4} e^{-2y}\, dy \\
&= -\frac{\pi}{2} \left[ln\,4\, e^{ln\,4^{-2}} \right] + \frac{\pi}{2} \left[\frac{e^{-2y}}{-2} \right]_0^{ln\,4} \\
&= -\frac{\pi}{2}\, ln\,4\ \times\ 4^{-2} - \frac{\pi}{4} \left[e^{-2\,ln\,4} - e^0 \right] \\
&= -\frac{\pi\, ln\,4}{32} - \frac{\pi}{4} \left[e^{ln\,4^{-2}} - 1 \right] \\
&= -\frac{\pi\, ln\,4}{32} - \frac{\pi}{4} \left[4^{-2} - 1 \right] \\
&= -\frac{\pi\, ln\,4}{32} - \frac{\pi}{4} \left[\frac{1}{16} - \frac{16}{16} \right] \\
&= -\frac{\pi\, ln\,4}{32} - \frac{\pi}{4} \left[-\frac{15}{16} \right] \\
&= -\frac{\pi\, ln\,4}{32} + \frac{15\,\pi}{64} \\
&= \frac{\pi}{64} \left(15 - 2\, ln\,4 \right) \qquad \square
\end{aligned}
$$

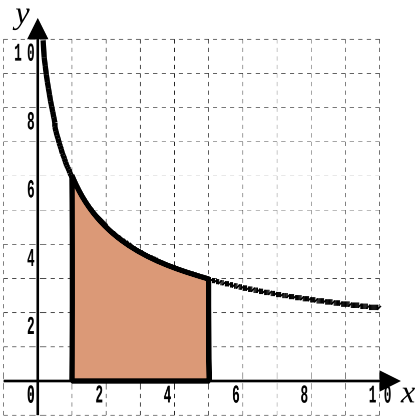
[10 marks]

5.1 Test

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 40

Question 1

The graph is of the function, $f(x) = \dfrac{12}{\sqrt{3x + 1}}, \quad x \in \mathbb{R}, \; x > -\dfrac{1}{3}$



(i) Find the area shown shaded in the graph.

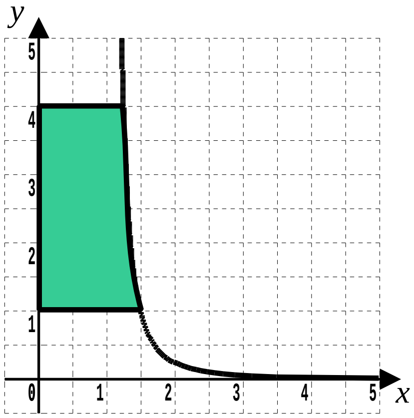
[3 marks]

(ii) The shaded area is now spun 360° about the x-axis.
Show that the volume of the solid formed is $96\pi \ln 2$

[6 marks]

Question 2

The graph is of the function, $f(y) = 1 + \dfrac{1}{2\sqrt{y}}$, $\quad y \in \mathbb{R},\ y \neq 0$

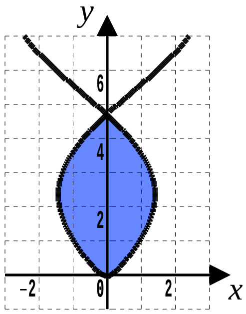


The shaded region bounded by the curve, the y-axis and the lines $y = 1$ and $y = 4$, is rotated through 360° about the y-axis. Using integration, show that the volume of the solid generated is $\pi\left(5 + \dfrac{1}{2}\ln 2\right)$

[9 marks]

Question 3

The curve shown in the diagram has equation $2x^2 = y\sin y + y$



(i) Show that the coordinates of the positive y-axis crossing are $\left(0, \dfrac{3\pi}{2}\right)$

[1 mark]

The shaded region is rotated about the y-axis to generate a solid of revolution.

(ii) Find the exact volume of the solid generated.

[8 marks]

Question 4

Find the exact volume of the solid generated when the curve $y = \dfrac{2\sqrt{2}}{\sqrt{x}\,(5x - 2)}$

between $x = \dfrac{1}{2}$ and $x = 1$ is rotated 2π radians about the x-axis.

Simplify your answer.

[13 marks]

5.2 Answers

Answer 1

(i)

$$\int_1^5 \frac{12}{\sqrt{3x+1}}\,dx = 4\int_1^5 3\,(3x+1)^{-\frac{1}{2}}\,dx$$

$$= 4\left[\,2\,(3x+1)^{\frac{1}{2}}\,\right]_1^5$$

$$= 8\left[\,(3x+1)^{\frac{1}{2}}\,\right]_1^5$$

$$= 8\{\,4 - 2\,\}$$

$$= 16$$

[3 marks]

(ii)

$$Volume = \pi\int y^2\,dx$$

$$= \pi\int_1^5 \left(\frac{12}{\sqrt{3x+1}}\right)^2 dx$$

$$= \pi\int_1^5 \frac{144}{3x+1}\,dx$$

$$= 48\pi\int_1^5 3\,(3x+1)^{-1}\,dx$$

$$= 48\pi\left[\,ln\,|\,3x+1\,|\,\right]_1^5$$

$$= 48\pi\{\,ln\,16 - ln\,4\,\}$$

$$= 48\pi\{\,ln\,2^4 - ln\,2^2\,\}$$

$$= 48\pi\{\,4\,ln\,2 - 2\,ln\,2\,\}$$

$$= 48\pi\{\,2\,ln\,2\,\}$$

$$= 96\,\pi\,ln\,2 \qquad \square$$

[6 marks]

Answer 2

$$Volume = \pi \int x^2 \, dy$$

$$= \pi \int_1^4 \left(1 + \frac{1}{2\sqrt{y}} \right)^2 dy$$

$$= \pi \int_1^4 \left(1 + y^{-\frac{1}{2}} + \frac{1}{4y} \right) dy$$

$$= \pi \left\{ y + 2y^{\frac{1}{2}} + \frac{1}{4} \ln y \right\}_1^4$$

$$= \pi \left\{ \left[4 + 4 + \frac{1}{4} \ln 4 \right] - \left[1 + 2 + \frac{1}{4} \ln 1 \right] \right\}$$

$$= \pi \left\{ \left[8 + \frac{1}{4} \ln 2^2 \right] - [3] \right\}$$

$$= \pi \left\{ 5 + \frac{1}{2} \ln 2 \right\} \qquad \square$$

[9 marks]

Answer 3

(i) When $x = 0$,

$$y \sin y + y = 0$$

$$y (\sin y + 1) = 0$$

Positive solution sought will come from, $\sin y + 1 = 0$

$$\therefore \sin y = -1$$

$$y = \arcsin (-1)$$

$$= \frac{3\pi}{2} \text{ as the first positive solution}$$

That is, coordinates are $\left(0, \frac{3\pi}{2} \right)$ $\square$

[1 mark]

(ii)

$$Volume = \pi \int x^2 \, dy$$

$$= \frac{\pi}{2} \int_0^{\frac{3\pi}{2}} y \sin y + y \, dy$$

L I D I

$$= \frac{\pi}{2} \left[y (-\cos y) \right]_0^{\frac{3\pi}{2}} - \frac{\pi}{2} \int_0^{\frac{3\pi}{2}} (-\cos y) \, dy + \frac{\pi}{2} \left[\frac{y^2}{2} \right]_0^{\frac{3\pi}{2}}$$

$$= -\frac{\pi}{2} \left[\frac{3\pi}{2} \times 0 - 0 \right] + \frac{\pi}{2} \left[\sin y \right]_0^{\frac{3\pi}{2}} + \frac{\pi}{4} \left[\left(\frac{3\pi}{2} \right)^2 - 0 \right]$$

$$= \frac{\pi}{2} \left[\sin \left(\frac{3\pi}{2} \right) - \sin (0) \right] + \frac{9\pi^3}{16}$$

$$= \frac{\pi}{2} [-1] + \frac{9\pi^3}{16}$$

$$= \frac{\pi}{2} \left(\frac{9\pi^2}{8} - 1 \right) \quad \text{or} \quad \frac{9\pi^3}{16} - \frac{8\pi}{16} \quad \text{or} \quad \frac{\pi}{16} \left(9\pi^2 - 8 \right)$$

[8 marks]

Answer 4

$$Volume \;=\; \pi \int y^2 \, dx$$

$$= \; \pi \int_{\frac{1}{2}}^{2} \left(\frac{2\sqrt{2}}{\sqrt{x}\;(5x - 2)} \right)^2 dx$$

$$= \; \pi \int_{\frac{1}{2}}^{2} \frac{8}{x\,(5x - 2)^2} \, dx$$

$$\frac{8}{x\,(5x - 2)^2} \;=\; \frac{A}{x} + \frac{B}{5x - 2} + \frac{C}{(5x - 2)^2}$$

$$\frac{8 \times [\,\dots\dots\,]}{[\,x\,(5x - 2)^2\,]} \;=\; \frac{A \times [\,\dots\dots\,]}{x} + \frac{B \times [\,\dots\dots\,]}{5x - 2} + \frac{C \times [\,\dots\dots\,]}{(5x - 2)^2}$$

$$8 \;=\; A\,(5x - 2)^2 + Bx\,(5x - 2) + Cx$$

Let $x = 0 \;:\; 8 = 4A \qquad \Rightarrow \; A = 2$

Let $x = \dfrac{2}{5} \;:\; 8 = \dfrac{2C}{5} \qquad \Rightarrow \; C = 20$

Let $x = 1 \;:\; 8 = 9A + 3B + C \qquad \Rightarrow \; B = -10$

$$\frac{8}{x\,(5x - 2)^2} \;=\; \frac{2}{x} - \frac{10}{5x - 2} + \frac{20}{(5x - 2)^2}$$

$$\pi \int_{\frac{1}{2}}^{1} \frac{1}{x\,(5x - 2)^2} \, dx \;=\; 2\pi \int_{\frac{1}{2}}^{1} \frac{1}{x} \, dx - 2\pi \int_{\frac{1}{2}}^{1} \frac{5}{(5x - 2)} \, dx + 4\pi \int_{\frac{1}{2}}^{1} \frac{5}{(5x - 2)^2} \, dx$$

$$= \; 2\pi \Big[\, ln\,|x| \, \Big]_{\frac{1}{2}}^{1} - 2\pi \Big[\, ln\,|5x - 2| \, \Big]_{\frac{1}{2}}^{1} + 4\pi \left[\frac{(-1)}{5x - 2} \right]_{\frac{1}{2}}^{1}$$

$$= \; 2\pi \left[\, ln\,1 - ln \left(\frac{1}{2}\right) \right] - 2\pi \left[\, ln\,3 - ln \left(\frac{1}{2}\right) \right] - 4\pi \left[\frac{1}{3} - 2 \right]$$

$$= \; -2\pi\, ln \left(\frac{1}{2} \right) - 2\pi\, ln\,3 + 2\pi\, ln \left(\frac{1}{2} \right) + \frac{20\pi}{3}$$

$$= \; \frac{20\pi}{3} - 2\pi\, ln\,3$$

$$(= \; 14.04)$$

[13 marks]