2.5 Answers

Reminder :

Point Transformation by a Matrix (Two Dimensions)

The point $(x,\ y)$ is written $\begin{pmatrix} x \\ y \end{pmatrix}$ and placed right of the transforming matrix.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

The point $(x,\ y)$ has been transformed to the point $(ax + by,\ cx + dy)$

Answer 1

(i) $$\begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} 7 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \times 7 + 1 \times 4 \\ 5 \times 7 + 2 \times 4 \end{pmatrix} = \begin{pmatrix} 25 \\ 43 \end{pmatrix}$$

$\therefore\ (7,\ 4) \rightarrow (25,\ 43)$

[2 marks]

(ii) $$\begin{pmatrix} 4 & 3 \\ 5 & -2 \end{pmatrix}\begin{pmatrix} 6 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \times 6 + 3 \times 1 \\ 5 \times 6 + (-2) \times 1 \end{pmatrix} = \begin{pmatrix} 27 \\ 28 \end{pmatrix}$$

$\therefore\ (6,\ 1) \rightarrow (27,\ 28)$

[2 marks]

(iii) $$\begin{pmatrix} -7 & 3 \\ 6 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ -5 \end{pmatrix} = \begin{pmatrix} (-7) \times 2 + 3 \times (-5) \\ 6 \times 2 + (-1) \times (-5) \end{pmatrix} = \begin{pmatrix} -29 \\ 17 \end{pmatrix}$$

$\therefore\ (2, -5) \rightarrow (-29. 17)$

[2 marks]

(iv) $$\begin{pmatrix} 5 & -4 \\ -6 & 2 \end{pmatrix}\begin{pmatrix} -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \times (-3) + (-4) \times 1 \\ (-6) \times (-3) + 2 \times 1 \end{pmatrix} = \begin{pmatrix} -19 \\ 20 \end{pmatrix}$$

$\therefore\ (-3,\ 1) \rightarrow (-19,\ 20)$

[2 marks]

Answer 2

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + 4y \\ 3x - y \end{pmatrix}$$

$$\begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} = \begin{pmatrix} 2x + 4y \\ 3x - y \end{pmatrix}$$

$$\therefore\ a = 2,\ \ b = 4,\ \ c = 3 \text{ and } d = -1$$

[4 marks]

Answer 3

$$\begin{pmatrix} 5 & p \\ -6 & p \end{pmatrix}\begin{pmatrix} -3 \\ p \end{pmatrix} = \begin{pmatrix} 2p \\ -9p \end{pmatrix}$$

$$\begin{pmatrix} 5 \times (-3) + p \times p \\ (-6) \times (-3) + p \times p \end{pmatrix} = \begin{pmatrix} 2p \\ -9p \end{pmatrix}$$

$$\begin{pmatrix} -15 + p^2 \\ 18 + p^2 \end{pmatrix} - \begin{pmatrix} 2p \\ -9p \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} p^2 - 2p - 15 \\ p^2 + 9p + 18 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} (p - 5)\ (p + 3) \\ (p + 6)\ (p + 3) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\therefore$ The only solution is $p = -3$

[4 marks]

Answer 4

Let a generalised point on $y = x$ be (p, p)
Then,

$$\begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} p \\ p \end{pmatrix} = \begin{pmatrix} 2 \times p + 3 \times p \\ 4 \times p + 1 \times p \end{pmatrix}$$

$$= \begin{pmatrix} 5p \\ 5p \end{pmatrix}$$

As the point ($5p$, $5p$) is on the line $y = x$ it has been proved that any point on the line $y = x$ stays on the line.

(See the notes on eigenvalues and eigenvectors at the end of this set of answers)

[4 marks]

Answer 5

(i)

$$\begin{pmatrix} 2 & -3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 5 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \times 5 + (-3) \times 0 \\ 1 \times 5 + 2 \times 0 \end{pmatrix} = \begin{pmatrix} 10 \\ 5 \end{pmatrix}$$

$\therefore$ (5, 0) $\rightarrow$ (10, 5)

[1 mark]

$$\begin{pmatrix} 2 & -3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 7 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \times 7 + (-3) \times 1 \\ 1 \times 7 + 2 \times 1 \end{pmatrix} = \begin{pmatrix} 11 \\ 9 \end{pmatrix}$$

$\therefore$ (7, 1) $\rightarrow$ (11, 9)

[1 mark]

$$\begin{pmatrix} 2 & -3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 6 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \times 6 + (-3) \times 3 \\ 1 \times 6 + 2 \times 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 12 \end{pmatrix}$$

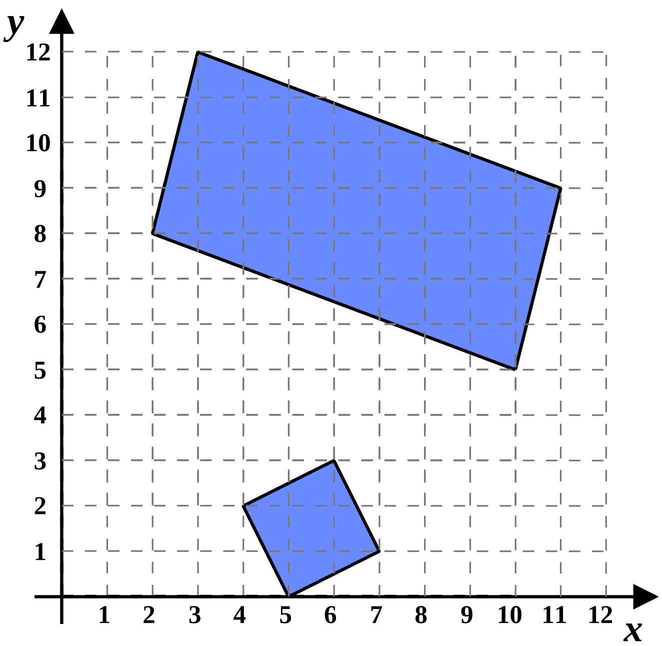
$\therefore$ (6, 3) $\rightarrow$ (3. 12)

[1 mark]

$$\begin{pmatrix} 2 & -3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \times 4 + (-3) \times 2 \\ 1 \times 4 + 2 \times 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \end{pmatrix}$$

$\therefore$ (4, 2) $\rightarrow$ (2, 8)

[1 mark]



[1 mark]

(ii) Area square is 5 units^2

[1 mark]

(iii) Area of transformed shape is 35 units^2

[2 marks]

(iv) $|\mathbf{M}| = 7$ which is the area scale factor of the transformation

[2 marks]

Extra Notes for Answer 4

To find the eigenvalues λ_1 and λ_2 :

$$\left| \mathbf{M} - \lambda \mathbf{I} \right| = 0$$

$$\left| \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = 0$$

$$\left| \begin{pmatrix} 2 - \lambda & 3 \\ 4 & 1 - \lambda \end{pmatrix} \right| = 0$$

$$(2 - \lambda)(1 - \lambda) - 3 \times 4 = 0$$

$$2 - 2\lambda - \lambda + \lambda^2 - 12 = 0$$

$$\lambda^2 - 3\lambda - 10 = 0 \quad \text{The characteristic equation}$$

$$(\lambda - 5)(\lambda + 2) = 0$$

$$\lambda_1 = 5 \quad \text{or} \quad \lambda_2 = -2$$

To find the eigenvectors v_1 and v_2 :

$$(\mathbf{M} - \lambda \mathbf{I}) v = 0$$

$$\left(\begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\begin{pmatrix} 2 - \lambda & 3 \\ 4 & 1 - \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

When $\lambda = 5$, $\quad - 3x + 3y = 0 \Rightarrow y = x$

$$4x - 4y = 0 \Rightarrow y = x$$

$$\therefore \ v_1 = k \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad k \in \mathbb{R}$$

When $\lambda = - 2$ $\quad 4x + 3y = 0 \Rightarrow y = \frac{4}{3} x$

$$4x + 3y = 0 \Rightarrow y = \frac{4}{3} x$$

$$\therefore \ v_1 = k \begin{pmatrix} 3 \\ 4 \end{pmatrix} \qquad k \in \mathbb{R}$$