Answer 1

(a) For the matrix to be singular,

$$\begin{aligned} det\,\mathbf{X} &= 0 \\ det\begin{pmatrix} 1 & a \\ 3 & 2 \end{pmatrix} &= 0 \\ 1 \times 2 - a \times 3 &= 0 \\ 3a &= 2 \\ a &= \frac{2}{3} \end{aligned}$$

[2 marks]

(b)

$$\begin{aligned} \mathbf{Y}^{-1} &= \begin{pmatrix} 1 & -1 \\ 3 & 2 \end{pmatrix}^{-1} \\ &= \frac{1}{1 \times 2 - (-1) \times 3}\begin{pmatrix} 2 & 1 \\ -3 & 1 \end{pmatrix} \\ &= \frac{1}{5}\begin{pmatrix} 2 & 1 \\ -3 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0.4 & 0.2 \\ -0.6 & 0.2 \end{pmatrix} \end{aligned}$$

[2 marks]

(c)

$$\mathbf{Y}A = B$$

$$\mathbf{Y}^{-1}\mathbf{Y}A = \mathbf{Y}^{-1}B \qquad \text{pre-multiply both sides by } \mathbf{Y}^{-1}$$

$$\begin{aligned} A &= \mathbf{Y}^{-1}B \\ A &= \frac{1}{5}\begin{pmatrix} 2 & 1 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 1 - \lambda \\ 7\lambda - 2 \end{pmatrix} \\ &= \frac{1}{5}\begin{pmatrix} 2 - 2\lambda + 7\lambda - 2 \\ -3 + 3\lambda + 7\lambda - 2 \end{pmatrix} \\ &= \frac{1}{5}\begin{pmatrix} 5\lambda \\ -5 + 10\lambda \end{pmatrix} \\ &= \begin{pmatrix} \lambda \\ 2\lambda - 1 \end{pmatrix} \end{aligned}$$

So A has coordinates $(\lambda,\ 2\lambda - 1)$

[4 marks]

Answer 2

(a)

$$\begin{aligned} det\,\mathbf{A} &= det\begin{pmatrix} 2 & -2 \\ -1 & 3 \end{pmatrix} \\ &= 2 \times 3 - (-2) \times (-1) \\ &= 4 \end{aligned}$$

[1 mark]

(b)

$$\begin{aligned} \mathbf{A}^{-1} &= \begin{pmatrix} 2 & -2 \\ -1 & 3 \end{pmatrix}^{-1} \\ &= \frac{1}{4}\begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 0.75 & 0.5 \\ 0.25 & 0.5 \end{pmatrix} \end{aligned}$$

[2 marks]

(c) Area of R is 18 square units

[2 marks]

(d)

$$\begin{aligned} \mathbf{A}\,R &= S \\ \mathbf{A}^{-1}\mathbf{A}\,R &= \mathbf{A}^{-1}S \qquad \text{pre-multiply both sides by } \mathbf{A}^{-1} \\ R &= \frac{1}{4}\begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 0 & 8 & 12 \\ 4 & 16 & 4 \end{pmatrix} \\ &= \frac{1}{4}\begin{pmatrix} 8 & 56 & 44 \\ 8 & 40 & 20 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 14 & 11 \\ 2 & 10 & 5 \end{pmatrix} \end{aligned}$$

That is, the vertices of R are $(2,\ 2)$, $(14,\ 10)$ and $(11,\ 5)$

[4 marks]

Answer 3

(a)

$$det\,\mathbf{A} = det\begin{pmatrix} a & -5 \\ 2 & a+4 \end{pmatrix}$$

$$= a \times (a+4) - (-5) \times 2$$

$$= a^2 + 4a + 10$$

[2 marks]

(b)

$$det\,\mathbf{A} = \left[a^2 + 4a \right] + 10$$

$$= \left[(a+2)^2 - 4 \right] + 10$$

$$= (a+2)^2 + 6$$

As the square is $\geqslant 0$, and then 6 is added on, $det\,\mathbf{A} > 0$ for all real $\mathbf{A}$

$\therefore$ **A** is non-singular for all values of a ☐

[3 marks]

(c)

$$\mathbf{A}^{-1} = \begin{pmatrix} a & -5 \\ 2 & a+4 \end{pmatrix}$$

$$= \frac{1}{a^2 + 4a + 10}\begin{pmatrix} a+4 & 5 \\ -2 & a \end{pmatrix}$$

Told that $a = 0$,

$$\therefore\ \mathbf{A}^{-1} = \frac{1}{10}\begin{pmatrix} 4 & 5 \\ -2 & 0 \end{pmatrix}$$

[3 marks]

Answer 4

(i) As **A** is a singular matrix,

$$det\,\mathbf{A} = 0$$

$$det\begin{pmatrix} 5k & 3k-1 \\ -3 & k+1 \end{pmatrix} = 0$$

$$(5k) \times (k+1) - (3k-1) \times (-3) = 0$$

$$5k^2 + 5k + 9k - 3 = 0$$

$$5k^2 + 14k - 3 = 0$$

$$(5k-1)(k+3) = 0$$

Either $5k - 1 = 0$ or $k + 3 = 0$

$$k = \frac{1}{5} \quad \text{or} \quad k = -3$$

[4 marks]

(ii) **(a)**

$$\mathbf{B}^{-1} = \begin{pmatrix} 10 & 5 \\ -3 & 3 \end{pmatrix}^{-1}$$

$$= \frac{1}{10 \times 3 - 5 \times (-3)} \begin{pmatrix} 3 & -5 \\ 3 & 10 \end{pmatrix}$$

$$= \frac{1}{45} \begin{pmatrix} 3 & -5 \\ 3 & 10 \end{pmatrix}$$

[2 marks]

(b)

$$\mathbf{B}\,T = T'$$

$$\mathbf{B}^{-1}\,\mathbf{B}\,T = \mathbf{B}^{-1}\,T' \qquad \text{pre-multiply both sides by } \mathbf{A}^{-1}$$

$$T = \mathbf{B}^{-1}\,T'$$

$$T = \frac{1}{45} \begin{pmatrix} 3 & -5 \\ 3 & 10 \end{pmatrix} \begin{pmatrix} 0 & -20 & 10c \\ 0 & 6 & 6c \end{pmatrix}$$

$$= \frac{1}{45} \begin{pmatrix} 0 & -90 & 0 \\ 0 & 0 & 90c \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 2c \end{pmatrix}$$

That is, the vertices of T are $(0,\ 0)$, $(-2,\ 0)$ and $(0,\ 2c)$

[3 marks]

(c)

T has an area of $135 \div 45 = 3$

$$\text{Area } \Delta = \frac{1}{2} \times b \times h$$

$$3 = \frac{1}{2} \times 2 \times 2c$$

$$c = 1.5$$

[3 marks]