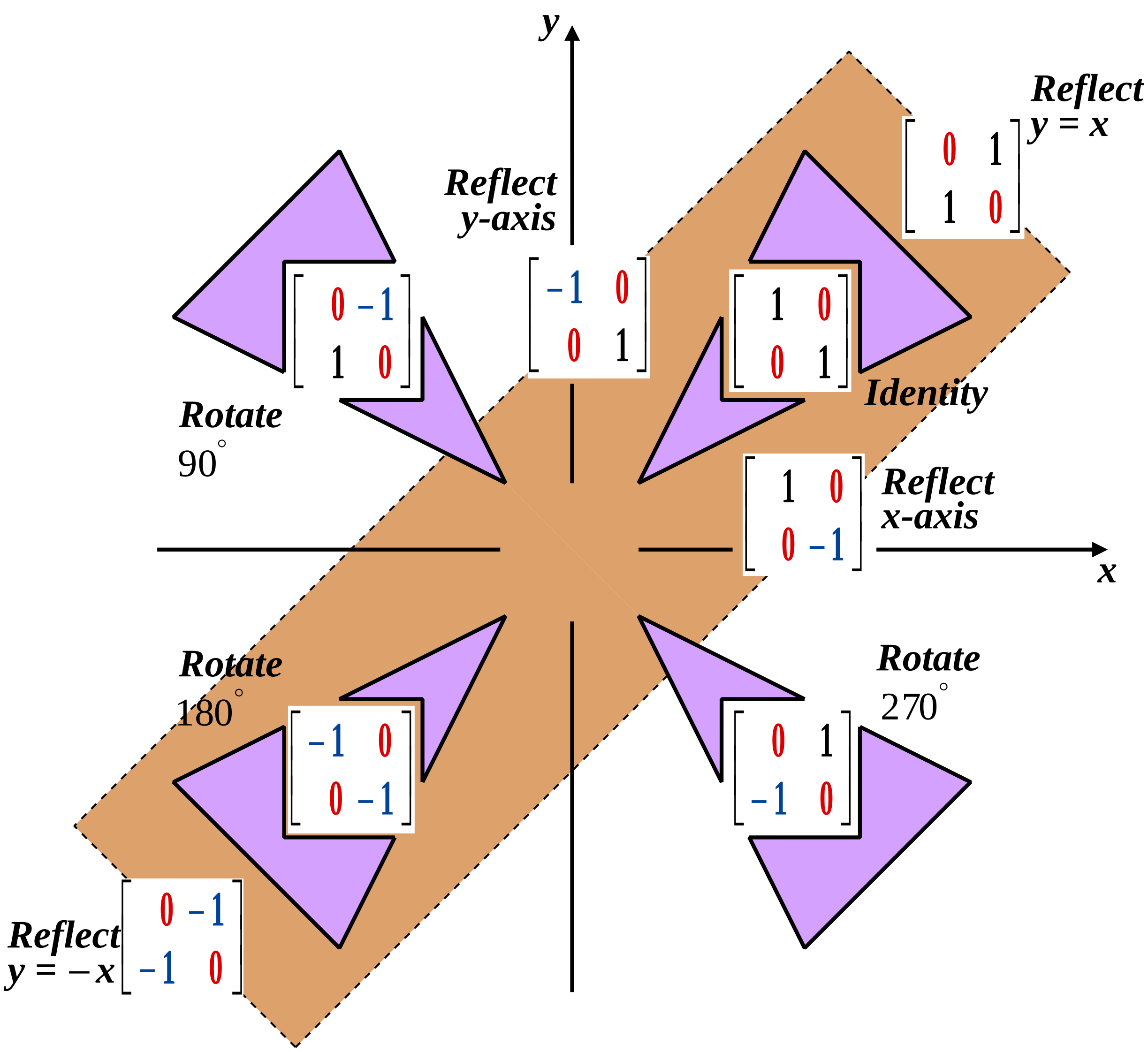
Further Pure A-Level Mathematics
Compulsory Course Component
Core 1

MATRIX TRANSFORMATIONS



M A T R I X T R A N S F O R M A T I O N S

Lesson 1

1.1 Introduction

A matrix is an array of elements (typically numbers) arranged in rows and columns. Here, for example, is a 3×4 matrix, **A**,

$$\mathbf{A} = \begin{pmatrix} 3 & 2 & 4 & 6 \\ -6 & 1 & 12 & 4 \\ 5 & 0 & -7 & 9 \end{pmatrix}$$

The 3×4 references that **A** has three rows and 4 columns.

An individual element within a matrix can be referenced by by specifying which row and which column it is in.
For matrix **A**,

$$\mathbf{A} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{23} & a_{33} & a_{34} \end{pmatrix} \end{array}$$

Thus, for example,

$$a_{23} = 12$$

If the matrix is small, the individual entries may instead be described with a single lower case letter, bold upper case letters usually being reserved for the name of an entire matrix.
For example, a generalised 2×2 square matrix, **B**, could be described as,

$$\mathbf{B} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

One major use of matrices is in the description and analysis of networks; in other words, “connectivity”. This is a vast and thriving area of mathematics called Graph Theory. Applications include modelling how a virus might spread, for example.

Most computer languages can do arithmetic with matrices. This is because matrices are used extensively in computer graphics. They are used to move points about a computer screen and to transform shapes; rotations, reflections, enlargements and sheers can all be handled by matrices. These manipulations can be in both two or three dimensions. Matrices can also handle the mathematics of projecting a three dimensional shape onto a two dimensional screen.

1.2 Matrix Arithmetic

1.2.1 Matrix Addition

To add two matrices together, add corresponding elements.
For example,

$$\begin{pmatrix} 4 & -6 \\ 7 & 3 \\ -1 & -4 \end{pmatrix} + \begin{pmatrix} 5 & -2 \\ 2 & -9 \\ 1 & 8 \end{pmatrix} = \begin{pmatrix} 9 & -8 \\ 9 & -6 \\ 0 & 4 \end{pmatrix}$$

Only matrices of the same size can be added.

1.2.2 Matrix Subtraction

Subtraction problems can be turned into addition problems.
The idea is to reverse all signs in the second matrix.
For example,

$$\begin{pmatrix} 5 & -8 & -9 \\ -7 & 4 & 8 \end{pmatrix} - \begin{pmatrix} -6 & 3 & -9 \\ 2 & -1 & -8 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & -8 & -9 \\ -7 & 4 & 8 \end{pmatrix} + \begin{pmatrix} 6 & -3 & 9 \\ -2 & 1 & 8 \end{pmatrix}$$

$$= \begin{pmatrix} 11 & -11 & 0 \\ -9 & 5 & 16 \end{pmatrix}$$

Only matrices of the same size can be subtracted.

1.2.3 Scalar Multiplication

To multiply a matrix by a scalar, multiply every element by the scalar.
For example,

$$4 \begin{pmatrix} 2 & -1 & -7 \\ 0 & 0.5 & -3 \\ -8 & 6 & 9 \end{pmatrix} = \begin{pmatrix} 8 & -4 & -28 \\ 0 & 2 & -12 \\ -32 & 24 & 36 \end{pmatrix}$$

1.2.4 Scalar Division

Scalar division problems can be turned into scalar multiplication problems.

To divide by s, multiply by $\frac{1}{s}$

For example,

to divide the matrix $\begin{pmatrix} 18 & -9 \\ 0 & 42 \end{pmatrix}$ by 3, instead multiply by $\frac{1}{3}$

$$\frac{1}{3} \begin{pmatrix} 18 & -9 \\ 0 & 42 \end{pmatrix} = \begin{pmatrix} 6 & -3 \\ 0 & 14 \end{pmatrix}$$

1.3 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable

Marks Available : 30

Question 1

The matrices **A** and **B** are defined as,

$$\mathbf{A} = \begin{pmatrix} 4 & -8 \\ 2 & 0 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 9 & 6 \\ 0 & -3 \end{pmatrix}$$

Find,

(i) $\mathbf{A} + \mathbf{B}$ **(ii)** $\mathbf{A} - \mathbf{B}$

(iii) $4\mathbf{A} - 3\mathbf{B}$ **(iv)** $\frac{1}{2}\mathbf{A} + \frac{2}{3}\mathbf{B}$

[8 marks]

Question 2

Given that,

$$\begin{pmatrix} a & 2 \\ -1 & b \end{pmatrix} - \begin{pmatrix} 2 & c \\ d & -2 \end{pmatrix} = 5\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

find the values of the constants a, b, c and d

[4 marks]

Question 3

Given that,

$$\begin{pmatrix} 4 & b \\ a & c \end{pmatrix} - \begin{pmatrix} a & 6 \\ a & d \end{pmatrix} = \begin{pmatrix} 11 & a \\ c & b \end{pmatrix}$$

find the values of the constants a, b, c and d

[4 marks]

Question 4

Find the value of k such that,

$$2\,k^2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} - 3\begin{pmatrix} 1 \\ 4k \end{pmatrix} = \begin{pmatrix} k \\ -9 \end{pmatrix}$$

[4 marks]

Question 5

Given that,

$$\begin{pmatrix} p & 0 & 0 \\ 0 & q^2 & r \\ 0 & 0 & 5 \end{pmatrix} - k\begin{pmatrix} 2q & 0 & 0 \\ 0 & 4 & 6 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

find the value of k and the positive constants, p, q and r

[4 marks]

Did you know ?

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{I}$ and is called the 3×3 Identity matrix.

It's the matrix equivalent of the number 1. The 2×2 version is $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Definition of the Determinant

Given the generalised 2×2 square matrix, **M**, where,

$$\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

the determinant **M** is given by,

$$|\mathbf{M}| = ad - bc$$

This can also be written *det* **M** or Δ**M**

- If $|\mathbf{M}| = 0$ then **M** is a singular matrix
- If $|\mathbf{M}| \neq 0$ then **M** is a non-singular matrix

Question 6

Given that $\mathbf{A} = \begin{pmatrix} 7 & 4 \\ 5 & 3 \end{pmatrix}$, find *det* **A**

[1 mark]

Question 7

The matrix $\mathbf{A} = \begin{pmatrix} -2 & k \\ 5 & 3 \end{pmatrix}$ has $|\mathbf{A}| = 24$. Find the value of *k*.

[2 marks]

Question 8

Given that the 2×2 matrix $\mathbf{A} = \begin{pmatrix} 3 & (k+3) \\ (k-2) & 8 \end{pmatrix}$ is singular, find the two possible values of the real number *k*.

[3 marks]

1.4 Answers

Answer 1

(i) $\mathbf{A} + \mathbf{B} = \begin{pmatrix} 4 & -8 \\ 2 & 0 \end{pmatrix} + \begin{pmatrix} 9 & 6 \\ 0 & -3 \end{pmatrix}$

$= \begin{pmatrix} 13 & -2 \\ 2 & -3 \end{pmatrix}$

[2 marks]

(ii) $\mathbf{A} - \mathbf{B} = \begin{pmatrix} 4 & -8 \\ 2 & 0 \end{pmatrix} - \begin{pmatrix} 9 & 6 \\ 0 & -3 \end{pmatrix}$

$= \begin{pmatrix} 4 & -8 \\ 2 & 0 \end{pmatrix} + \begin{pmatrix} -9 & -6 \\ 0 & 3 \end{pmatrix}$

$= \begin{pmatrix} -5 & -14 \\ 2 & 3 \end{pmatrix}$

[2 marks]

(iii) $4\mathbf{A} - 3\mathbf{B} = 4\begin{pmatrix} 4 & -8 \\ 2 & 0 \end{pmatrix} - 3\begin{pmatrix} 9 & 6 \\ 0 & -3 \end{pmatrix}$

$= \begin{pmatrix} 16 & -32 \\ 8 & 0 \end{pmatrix} + \begin{pmatrix} -27 & -18 \\ 0 & 9 \end{pmatrix}$

$= \begin{pmatrix} -11 & -50 \\ 8 & 9 \end{pmatrix}$

[2 marks]

(iv) $\frac{1}{2}\mathbf{A} + \frac{2}{3}\mathbf{B} = \frac{1}{2}\begin{pmatrix} 4 & -8 \\ 2 & 0 \end{pmatrix} + \frac{2}{3}\begin{pmatrix} 9 & 6 \\ 0 & -3 \end{pmatrix}$

$= \begin{pmatrix} 2 & -4 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 6 & 4 \\ 0 & -2 \end{pmatrix}$

$= \begin{pmatrix} 8 & 0 \\ 1 & -2 \end{pmatrix}$

[2 marks]

Answer 2

$$\begin{pmatrix} a & 2 \\ -1 & b \end{pmatrix} - \begin{pmatrix} 2 & c \\ d & -2 \end{pmatrix} = 5\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} (a-2) & (2-c) \\ (-1-d) & (b+2) \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$$

$$\therefore\ a = 7.\quad b = 3,\quad c = 2 \text{ and } d = -1$$

[4 marks]

Answer 3

$$\begin{pmatrix} 4 & b \\ a & c \end{pmatrix} - \begin{pmatrix} a & 6 \\ a & d \end{pmatrix} = \begin{pmatrix} 11 & a \\ c & b \end{pmatrix}$$

$$\begin{pmatrix} (4 - a) & (b - 6) \\ 0 & (c - d) \end{pmatrix} = \begin{pmatrix} 11 & a \\ c & b \end{pmatrix}$$

$$\therefore\ a = -7, \quad b = -1, \quad c = 0 \text{ and } d = 1$$

[4 marks]

Answer 4

$$2k^2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} - 3\begin{pmatrix} 1 \\ 4k \end{pmatrix} = \begin{pmatrix} k \\ -9 \end{pmatrix}$$

$$\begin{pmatrix} 2k^2 - 3 \\ 4k^2 - 12k \end{pmatrix} = \begin{pmatrix} k \\ -9 \end{pmatrix}$$

From the upper row, $2k^2 - k - 3 = 0$

$$(2k - 3)(k + 1) = 0$$

Either $k = \dfrac{3}{2}$ or $k = -1$

From the lower row, $4k^2 - 12k + 9 = 0$

$$(2k - 3)(2k - 3) = 0$$

$$k = \frac{3}{2}$$

The only common solution for the system of equations is $k = \dfrac{3}{2}$

[4 marks]

Answer 5

$$\begin{pmatrix} p & 0 & 0 \\ 0 & q^2 & r \\ 0 & 0 & 5 \end{pmatrix} - k\begin{pmatrix} 2q & 0 & 0 \\ 0 & 4 & 6 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Lower right : $5 - 2k = 1 \quad \Rightarrow k = 2$

Centre right : $r - 6k = 0 \quad \Rightarrow r = 12$

Centre : $q^2 - 4k = 1 \quad \Rightarrow q = 3$

Upper left : $p - 2qk = 1 \quad \Rightarrow p = 13$

[4 marks]

Answer 6

$$\begin{aligned} |\mathbf{A}| &= \left| \begin{pmatrix} 7 & 4 \\ 5 & 3 \end{pmatrix} \right| \\ &= 7 \times 3 - 4 \times 5 \\ &= 1 \end{aligned}$$

[1 mark]

Answer 7

$$\begin{aligned} |\mathbf{A}| &= 24 \\ \left| \begin{pmatrix} -2 & k \\ 5 & 3 \end{pmatrix} \right| &= 24 \\ -2 \times 3 - k \times 5 &= 24 \\ -5k &= 30 \\ k &= -6 \end{aligned}$$

[2 marks]

Answer 8

$$\begin{aligned} |\mathbf{A}| &= 0 \\ \left| \begin{pmatrix} 3 & (k + 3) \\ (k - 2) & 8 \end{pmatrix} \right| &= 0 \\ 3 \times 8 - (k + 3)(k - 2) &= 0 \\ 24 - \left(k^2 + k - 6\right) &= 0 \\ 24 - k^2 - k + 6 &= 0 \\ k^2 + k - 30 &= 0 \\ (k + 6)(k - 5) &= 0 \end{aligned}$$

Either $k = -6$ or $k = 5$

[3 marks]

2.1 Point Transformations

A square matrix of size 2×2 can be thought of as a transformation that moves a point on a two dimensional surface to a different location on that surface. This is an application of what is termed "matrix multiplication". This is a completely new type of mathematical operation, and the word "multiplication" is given a new meaning in the context of manipulating matrices. The rule for performing matrix multiplication with a 2×2 matrix on a point is now given;

Point Transformation by a Matrix (Two Dimensions)

The point $(x,\ y)$ is written $\begin{pmatrix} x \\ y \end{pmatrix}$ and placed right of the transforming matrix.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

The point $(x,\ y)$ has been transformed to the point $(ax + by,\ cx + dy)$

2.2 Four Practice Questions

Use the above rule to transform the given point.
Once done, check your answers with those over the page

(i) $\begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$(1, 3) \rightarrow (\quad ,\quad)$

(ii) $\begin{pmatrix} 4 & -1 \\ 6 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ 5 \end{pmatrix}$

$(3, 5) \rightarrow (\quad ,\quad)$

(iii) $\begin{pmatrix} 7 & 4 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ -3 \end{pmatrix}$

$(2, -3) \rightarrow (\quad ,\quad)$

(iv) $\begin{pmatrix} 4 & -1 \\ -8 & 5 \end{pmatrix}\begin{pmatrix} -3 \\ 5 \end{pmatrix}$

$(-3, 5) \rightarrow (\quad ,\quad)$

[8 marks]

2.3 Practice Question's Answers

(i) $$\begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \times 1 + 3 \times 3 \\ 4 \times 1 + 1 \times 3 \end{pmatrix} = \begin{pmatrix} 11 \\ 7 \end{pmatrix}$$

$\therefore \ (1,\ 3) \ \rightarrow \ (11,\ 7)$

(ii) $$\begin{pmatrix} 4 & -1 \\ 6 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 4 \times 3 + (-1) \times 5 \\ 6 \times 3 + 2 \times 5 \end{pmatrix} = \begin{pmatrix} 7 \\ 28 \end{pmatrix}$$

$\therefore \ (3,\ 5) \ \rightarrow \ (7,\ 28)$

(iii) $$\begin{pmatrix} 7 & 4 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 7 \times 2 + 4 \times (-3) \\ 5 \times 2 + 3 \times (-3) \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$\therefore \ (2, -3) \ \rightarrow \ (2.\ 1)$

(iv) $$\begin{pmatrix} 4 & -1 \\ -8 & 5 \end{pmatrix}\begin{pmatrix} -3 \\ 5 \end{pmatrix} = \begin{pmatrix} 4 \times (-3) + (-1) \times 5 \\ (-8) \times (-3) + 5 \times 5 \end{pmatrix} = \begin{pmatrix} -17 \\ 49 \end{pmatrix}$$

$\therefore \ (-3,\ 5) \ \rightarrow \ (-17,\ 49)$

[8 marks]

2.4 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 30

Question 1

Determine where each of the following points are moved to by the given matrix.

(i) $\begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} 7 \\ 4 \end{pmatrix}$

$(7, 4) \rightarrow (\quad , \quad)$

(ii) $\begin{pmatrix} 4 & 3 \\ 5 & -2 \end{pmatrix}\begin{pmatrix} 6 \\ 1 \end{pmatrix}$

$(6, 1) \rightarrow (\quad , \quad)$

(iii) $\begin{pmatrix} -7 & 3 \\ 6 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ -5 \end{pmatrix}$

$(2, -5) \rightarrow (\quad , \quad)$

(iv) $\begin{pmatrix} 5 & -4 \\ -6 & 2 \end{pmatrix}\begin{pmatrix} -3 \\ 1 \end{pmatrix}$

$(-3, 1) \rightarrow (\quad , \quad)$

[8 marks]

Question 2
Given that,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + 4y \\ 3x - y \end{pmatrix}$$

State the values of a, b, c and d

[4 marks]

Question 3
Given that,

$$\begin{pmatrix} 5 & p \\ -6 & p \end{pmatrix}\begin{pmatrix} -3 \\ p \end{pmatrix} = \begin{pmatrix} 2p \\ -9p \end{pmatrix}$$

Find the value of p

[4 marks]

Question 4
Prove that any point on the line $y = x$ remains on the line $y = x$ when it is transformed by the matrix $\mathbf{M} = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}$

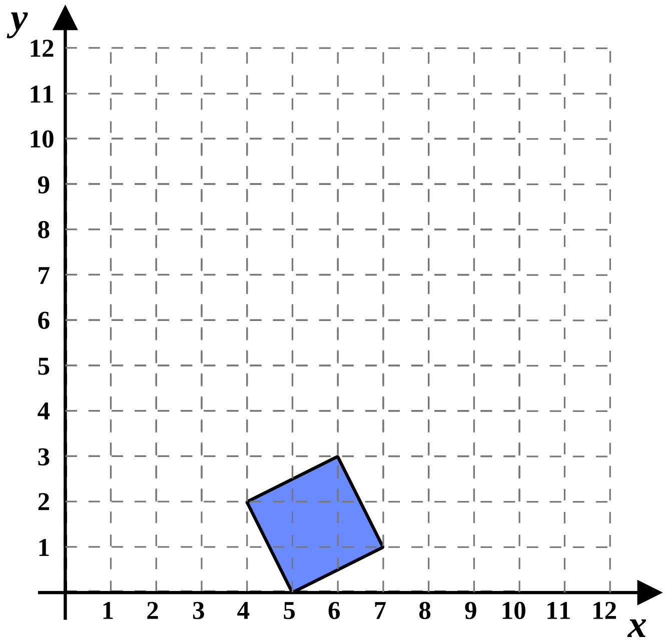
HINT : Let a generalised point on $y = x$ be (p, p) and work out $\begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} p \\ p \end{pmatrix}$

[4 marks]

Question 5

A square has vertices (5, 0), (7, 1), (6, 3) and (4, 2)

It is to be transformed by the matrix **M** $= \begin{pmatrix} 2 & -3 \\ 1 & 2 \end{pmatrix}$



(i) Move each point by **M** and plot the resulting shape on the graph.

[5 marks]

(ii) Work out the area of the original square.

[1 marks]

(iii) Work out the area of the transformed shape.

[2 marks]

(iv) There is a connecting between $|\mathbf{M}|$ and the areas of the two shapes.
Guess what this might be !

[2 marks]

2.5 Answers

Reminder :

Point Transformation by a Matrix (Two Dimensions)

The point (x, y) is written $\begin{pmatrix} x \\ y \end{pmatrix}$ and placed right of the transforming matrix.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

The point (x, y) has been transformed to the point $(ax + by, cx + dy)$

Answer 1

(i) $\begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} 7 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \times 7 + 1 \times 4 \\ 5 \times 7 + 2 \times 4 \end{pmatrix} = \begin{pmatrix} 25 \\ 43 \end{pmatrix}$

$\therefore\ (7, 4) \rightarrow (25, 43)$

[2 marks]

(ii) $\begin{pmatrix} 4 & 3 \\ 5 & -2 \end{pmatrix}\begin{pmatrix} 6 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \times 6 + 3 \times 1 \\ 5 \times 6 + (-2) \times 1 \end{pmatrix} = \begin{pmatrix} 27 \\ 28 \end{pmatrix}$

$\therefore\ (6, 1) \rightarrow (27, 28)$

[2 marks]

(iii) $\begin{pmatrix} -7 & 3 \\ 6 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ -5 \end{pmatrix} = \begin{pmatrix} (-7) \times 2 + 3 \times (-5) \\ 6 \times 2 + (-1) \times (-5) \end{pmatrix} = \begin{pmatrix} -29 \\ 17 \end{pmatrix}$

$\therefore\ (2, -5) \rightarrow (-29. 17)$

[2 marks]

(iv) $\begin{pmatrix} 5 & -4 \\ -6 & 2 \end{pmatrix}\begin{pmatrix} -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \times (-3) + (-4) \times 1 \\ (-6) \times (-3) + 2 \times 1 \end{pmatrix} = \begin{pmatrix} -19 \\ 20 \end{pmatrix}$

$\therefore\ (-3, 1) \rightarrow (-19, 20)$

[2 marks]

Answer 2

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + 4y \\ 3x - y \end{pmatrix}$$

$$\begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} = \begin{pmatrix} 2x + 4y \\ 3x - y \end{pmatrix}$$

$$\therefore\ a = 2,\ b = 4,\ c = 3 \text{ and } d = -1$$

[4 marks]

Answer 3

$$\begin{pmatrix} 5 & p \\ -6 & p \end{pmatrix}\begin{pmatrix} -3 \\ p \end{pmatrix} = \begin{pmatrix} 2p \\ -9p \end{pmatrix}$$

$$\begin{pmatrix} 5 \times (-3) + p \times p \\ (-6) \times (-3) + p \times p \end{pmatrix} = \begin{pmatrix} 2p \\ -9p \end{pmatrix}$$

$$\begin{pmatrix} -15 + p^2 \\ 18 + p^2 \end{pmatrix} - \begin{pmatrix} 2p \\ -9p \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} p^2 - 2p - 15 \\ p^2 + 9p + 18 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} (p - 5)\ (p + 3) \\ (p + 6)\ (p + 3) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\therefore$ The only solution is $p = -3$

[4 marks]

Answer 4

Let a generalised point on $y = x$ be (p, p)
Then,

$$\begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} p \\ p \end{pmatrix} = \begin{pmatrix} 2 \times p + 3 \times p \\ 4 \times p + 1 \times p \end{pmatrix}$$

$$= \begin{pmatrix} 5p \\ 5p \end{pmatrix}$$

As the point ($5p$, $5p$) is on the line $y = x$ it has been proved that any point on the line $y = x$ stays on the line.

(See the notes on eigenvalues and eigenvectors at the end of this set of answers)

[4 marks]

Answer 5

(i)

$$\begin{pmatrix} 2 & -3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 5 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \times 5 + (-3) \times 0 \\ 1 \times 5 + 2 \times 0 \end{pmatrix} = \begin{pmatrix} 10 \\ 5 \end{pmatrix}$$

$\therefore \; (5, \; 0) \; \rightarrow \; (10, \; 5)$

[1 mark]

$$\begin{pmatrix} 2 & -3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 7 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \times 7 + (-3) \times 1 \\ 1 \times 7 + 2 \times 1 \end{pmatrix} = \begin{pmatrix} 11 \\ 9 \end{pmatrix}$$

$\therefore \; (7, \; 1) \; \rightarrow \; (11, \; 9)$

[1 mark]

$$\begin{pmatrix} 2 & -3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 6 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \times 6 + (-3) \times 3 \\ 1 \times 6 + 2 \times 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 12 \end{pmatrix}$$

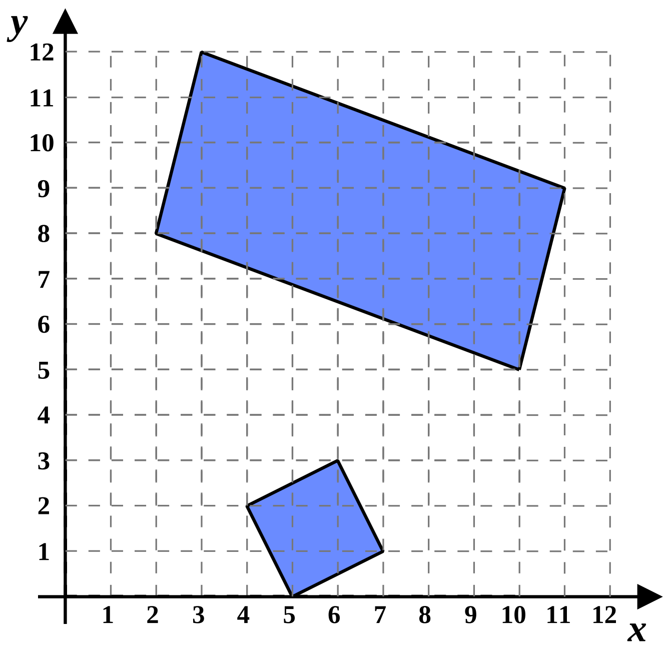
$\therefore \; (6, \; 3) \rightarrow (3. \, 12)$

[1 mark]

$$\begin{pmatrix} 2 & -3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \times 4 + (-3) \times 2 \\ 1 \times 4 + 2 \times 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \end{pmatrix}$$

$\therefore \; (4, \; 2) \rightarrow (2, \; 8)$

[1 mark]



[1 mark]

(ii) Area square is 5 units^2

[1 mark]

(iii) Area of transformed shape is 35 units^2

[2 marks]

(iv) $|\mathbf{M}| = 7$ which is the area scale factor of the transformation

[2 marks]

Extra Notes for Answer 4

To find the eigenvalues λ_1 and λ_2 :

$$\left| \mathbf{M} - \lambda \mathbf{I} \right| = 0$$

$$\left| \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = 0$$

$$\left| \begin{pmatrix} 2 - \lambda & 3 \\ 4 & 1 - \lambda \end{pmatrix} \right| = 0$$

$$(2 - \lambda)(1 - \lambda) - 3 \times 4 = 0$$

$$2 - 2\lambda - \lambda + \lambda^2 - 12 = 0$$

$$\lambda^2 - 3\lambda - 10 = 0 \quad \text{The characteristic equation}$$

$$(\lambda - 5)(\lambda + 2) = 0$$

$$\lambda_1 = 5 \quad \text{or} \quad \lambda_2 = -2$$

To find the eigenvectors v_1 and v_2 :

$$(\mathbf{M} - \lambda \mathbf{I})v = 0$$

$$\left(\begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\begin{pmatrix} 2 - \lambda & 3 \\ 4 & 1 - \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

When $\lambda = 5$,
$$-3x + 3y = 0 \Rightarrow y = x$$
$$4x - 4y = 0 \Rightarrow y = x$$
$$\therefore v_1 = k \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad k \in \mathbb{R}$$

When $\lambda = -2$
$$4x + 3y = 0 \Rightarrow y = \frac{4}{3} x$$
$$4x + 3y = 0 \Rightarrow y = \frac{4}{3} x$$
$$\therefore v_1 = k \begin{pmatrix} 3 \\ 4 \end{pmatrix} \qquad k \in \mathbb{R}$$

3.1 The Multipoint Matrix

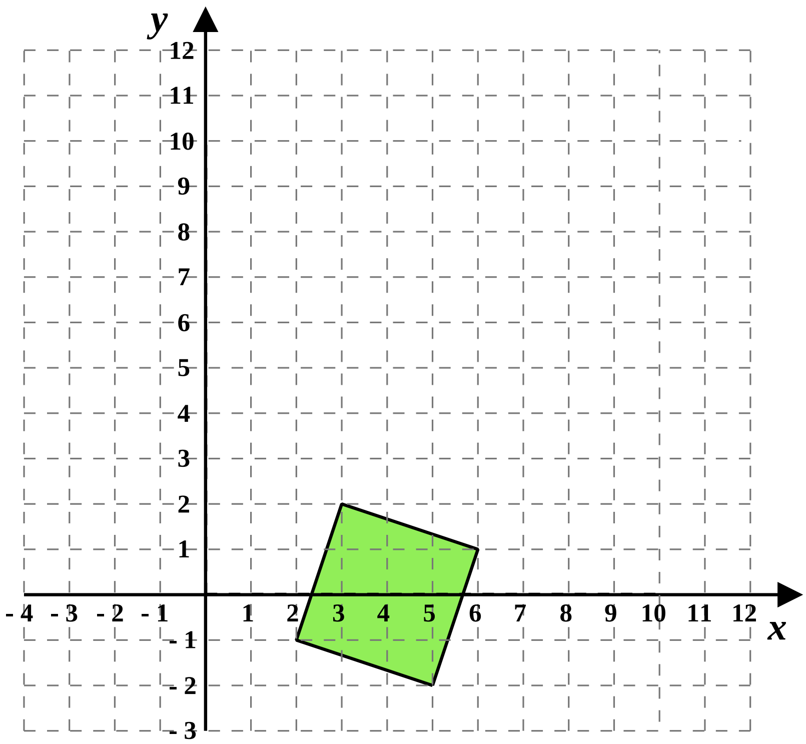
A square has vertices (3, 2), (6, 1), (5, – 2) and (2, – 1)

(i) Write the square's vertices as a multipoint matrix. and transform it with **M**

(ii) Transform the square's vertices using the matrix $\mathbf{M} = \begin{pmatrix} 1 & 4 \\ 2 & -1 \end{pmatrix}$

(iii) Add a plot of the transformed shape to the graph below.

Teaching Video : http://www.NumberWonder.co.uk/v9090/3.mp4



[1, 4, 1 marks]

Note : The determinant of **M** is a negative number, – 9

The magnitude of *det* **M** is 9 : the Area Scale Factor of the transformation.

3.2 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 30

Question 1

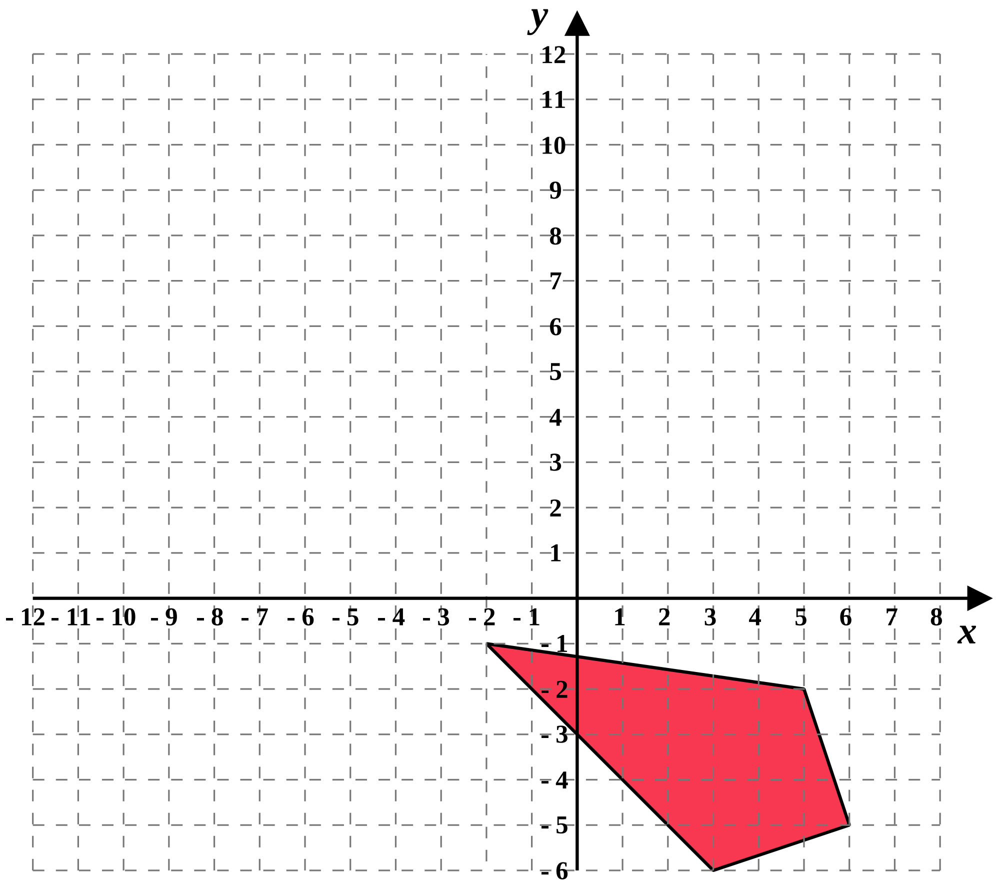
A kite has vertices (5, – 2), (6, – 5), (3, – 6) and (– 2, – 1)

(i) Write the kite's vertices as a multipoint matrix.

(ii) Transform the kite's vertices using the matrix **M** $= \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$

(iii) Add a plot of the transformed shape to the graph below.

(iv) What is the area scale factor of the transformation ?



[1, 4, 1, 1 marks]

Question 2

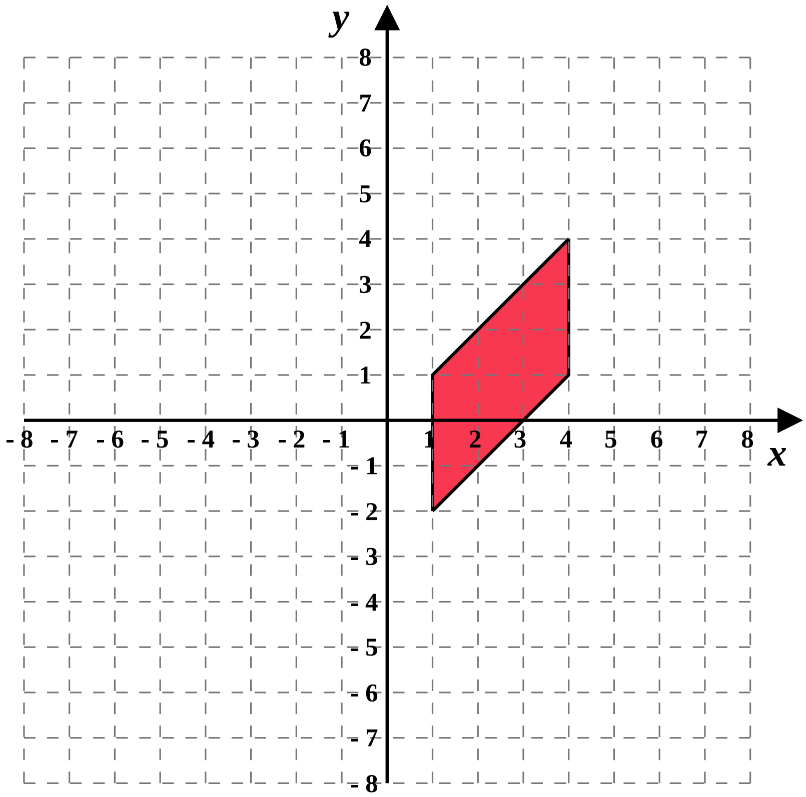
A parallelogram has vertices (1, – 2), (4, 1), (4, 4) and (1, 1)

(i) Write the parallelogram's vertices as a multipoint matrix.

(ii) Transform the parallelogram's vertices using the matrix **M** $= \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}$

(iii) Add a plot of the transformed shape to the graph below.

(iv) Find the magnitude of *det* **M** and explain what it tells you about the transformation.



[1, 4, 1, 1 marks]

Question 3

This question is about working out the following matrix multiplication,

$$\begin{pmatrix} 5 & -1 & 6 \\ 8 & 3 & -4 \end{pmatrix} \times \begin{pmatrix} 2 & 9 & 1 & -6 \\ -3 & 12 & -5 & 7 \\ 4 & -2 & 0 & 11 \end{pmatrix}$$

Here is the multiplication grid already set up,

$$\begin{array}{c|c} & \begin{pmatrix} 2 & 9 & 1 & -6 \\ -3 & 12 & -5 & 7 \\ 4 & -2 & 0 & 11 \end{pmatrix} \\ \hline \begin{pmatrix} 5 & -1 & 6 \\ 8 & 3 & -4 \end{pmatrix} & \begin{pmatrix} & 21 & & \\ & & & -71 \end{pmatrix} \end{array}$$

And here is how the 21 and the (-71) were found:

- In the product matrix, the 21 came from, $5 \times 9 + (-1) \times 12 + 6 \times (-2)$
- In the product matrix, the (-71) came from, $8 \times (-6) + 3 \times 7 + (-4) \times 11$

Complete the matrix multiplication grid.

[5 marks]

Question 4

Further A-Level Examination Question from January 2015, IAL, F1, Q6 (ii) (Edexcel)

$$\mathbf{M} = \begin{pmatrix} 2k+5 & -4 \\ 1 & k \end{pmatrix} \text{ where } k \text{ is a real number}$$

Show that $\det \mathbf{M} \neq 0$ for all values of k

[4 marks]

Question 5

Further A-Level Examination Question from January 2014, IAL, F1, Q2 (Edexcel)

(i) $\mathbf{A} = \begin{pmatrix} -4 & 10 \\ -3 & k \end{pmatrix}$ where k is a constant.

The triangle T is transformed to the triangle T' by the transformation represented by **A**

Given that the area of triangle T' is twice the area of triangle T, find the possible values of k

[4 marks]

(ii) Given that,

$$\mathbf{B} = \begin{pmatrix} 1 & -2 & 3 \\ -2 & 5 & 1 \end{pmatrix} \qquad \mathbf{C} = \begin{pmatrix} 2 & 8 \\ 0 & 2 \\ 1 & -2 \end{pmatrix}$$

find **BC**

[3 marks]

3.3 Answers

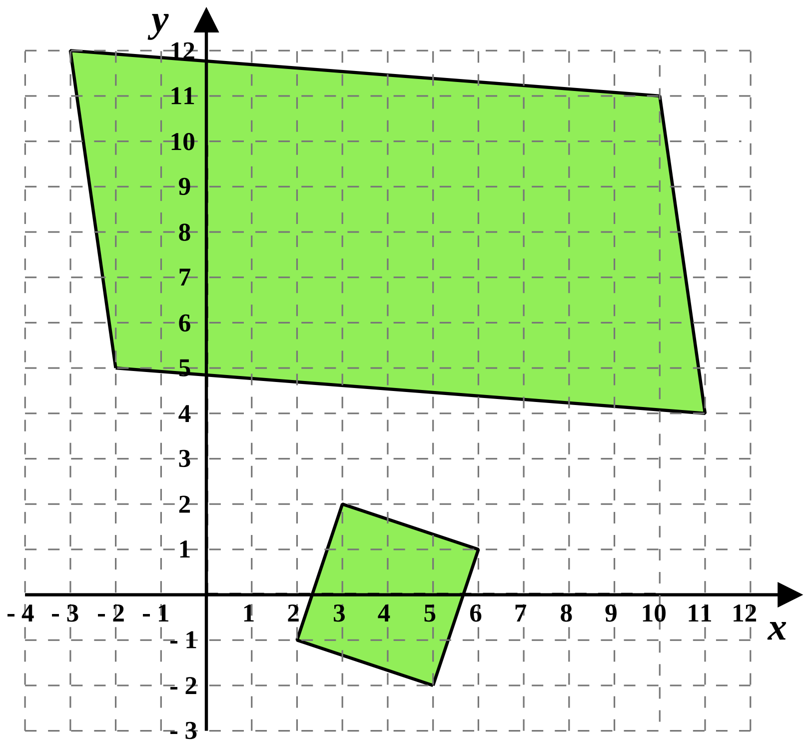
3.3.1 Solution to 3.1 Teaching Video Example

(i) and **(ii)**

	$\begin{pmatrix} 3 & 6 & 5 & 2 \\ 2 & 1 & -2 & -1 \end{pmatrix}$
$\begin{pmatrix} 1 & 4 \\ 2 & -1 \end{pmatrix}$	$\begin{pmatrix} 11 & 10 & -3 & -2 \\ 4 & 11 & 12 & 5 \end{pmatrix}$

[1, 4 marks]

(iii)



[1 mark]

Note that there is an assumption made that the straight lines between the vertices of the original shape transform to straight lines between the vertices of the transformed shape.
This is indeed what happens - a challenge question would be to prove this !

3.3.2 Solutions to 3.2 Exercise

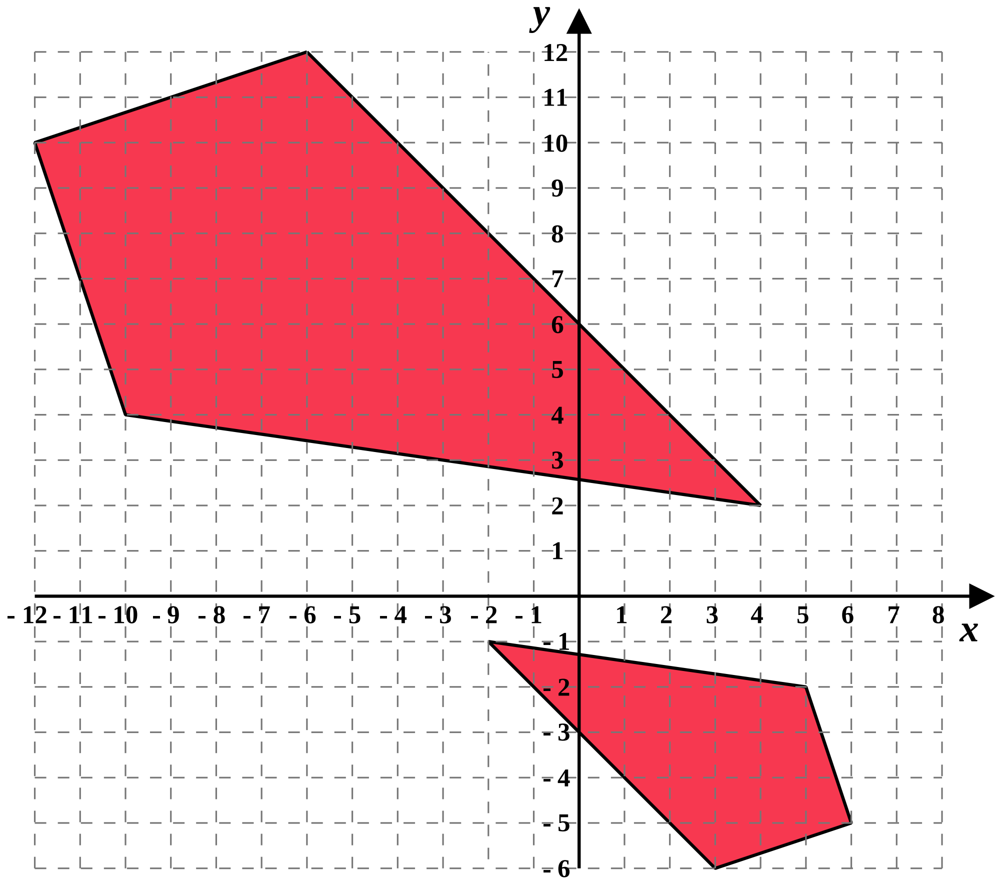
Answer 1

(i) and **(ii)**

$$\begin{array}{c|c} & \begin{pmatrix} 5 & 6 & 3 & -2 \\ -2 & -5 & -6 & -1 \end{pmatrix} \\ \hline \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} & \begin{pmatrix} -10 & -12 & -6 & 4 \\ 4 & 10 & 12 & 2 \end{pmatrix} \end{array}$$

[1, 4 marks]

(iii)



[1 mark]

(iv)

$$\text{Area scale factor} = |\mathbf{M}|$$

$$= \left| \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} \right|$$

$$= (-2) \times (-2) - 0 \times 0$$

$$= 4$$

[1 mark]

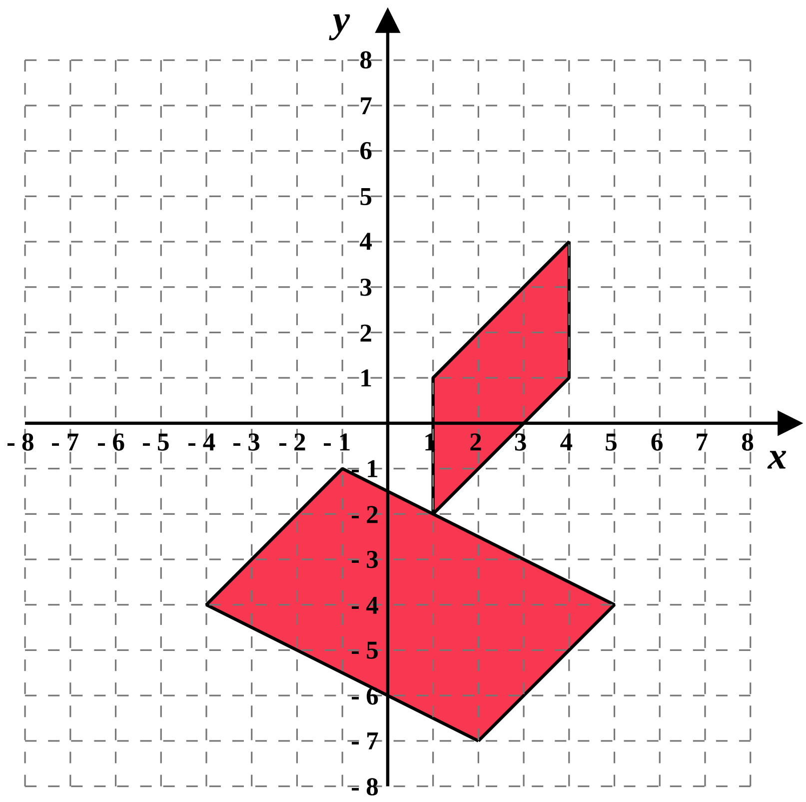
Answer 2

(i) and **(ii)**

	$\begin{pmatrix} 1 & 4 & 4 & 1 \\ -2 & 1 & 4 & 1 \end{pmatrix}$
$\begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}$	$\begin{pmatrix} 5 & 2 & -4 & -1 \\ -4 & -7 & -4 & -1 \end{pmatrix}$

[1, 4 marks]

(iii)



[1 mark]

(iv)

$$\mathit{det}\ \mathbf{M} = \left| \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix} \right|$$

$$= 1 \times 1 - (-2) \times (-2)$$

$$= -3$$

The magnitude of the determinant gives the area scale factor, greater than one, of the transformation.

(The minus sign means that the orientation of the shape has reversed)

[1 mark]

Answer 3

$$\begin{array}{c|c} & \begin{pmatrix} 2 & 9 & 1 & -6 \\ -3 & 12 & -5 & 7 \\ 4 & -2 & 0 & 11 \end{pmatrix} \\ \hline \begin{pmatrix} 5 & -1 & 6 \\ 8 & 3 & -4 \end{pmatrix} & \begin{pmatrix} 37 & 21 & 10 & 29 \\ -9 & 116 & -7 & -71 \end{pmatrix} \end{array}$$

Thus,

$$\begin{pmatrix} 5 & -1 & 6 \\ 8 & 3 & -4 \end{pmatrix} \times \begin{pmatrix} 2 & 9 & 1 & -6 \\ -3 & 12 & -5 & 7 \\ 4 & -2 & 0 & 11 \end{pmatrix} = \begin{pmatrix} 37 & 21 & 10 & 29 \\ -9 & 116 & -7 & -71 \end{pmatrix}$$

[5 marks]

Answer 4

$$\begin{aligned} det\,\mathbf{M} &= |\mathbf{M}| \\ &= \left| \begin{pmatrix} 2k+5 & -4 \\ 1 & k \end{pmatrix} \right| \\ &= (2k+5) \times k - (-4) \times 1 \\ &= 2k^2 + 5k + 4 \end{aligned}$$

$\therefore$ For $det\,\mathbf{M}$ to equal zero requires that $2k^2 + 5k + 4 = 0$

METHOD 1

Consider the discriminant, $D = 5^2 - 4 \times 2 \times 4$
$= -7$

As $D < 0$ there are no solutions to the equation $2k^2 + 5k + 4 = 0$

Thus, $det\,\mathbf{M} \neq 0$ for all values of k $\square$

[4 marks]

METHOD 2

By completing the square,

$$2k^2 + 5k + 4 = 2\left(k + \frac{5}{4}\right)^2 + \frac{7}{8}$$

Anything squared has a minimum value of zero and, even then seven eighths is added on, so the quadratic in k is always positive.

Thus, $det\,\mathbf{M} \neq 0$ for all values of k $\square$

[4 marks]

Answer 5

(i)

$$\begin{aligned} det\,\mathbf{M} &= \pm 2 \\ det\begin{pmatrix} -4 & 10 \\ -3 & k \end{pmatrix} &= \pm 2 \\ (-4) \times k - 10 \times (-3) &= \pm 2 \\ -4k + 30 &= \pm 2 \\ 4k - 30 &= \mp 2 \\ 4k &= \mp 2 + 30 \\ k &= \frac{30 \mp 2}{4} \\ k &= 7 \text{ or } 8 \end{aligned}$$

[4 marks]

(ii)

$$\begin{array}{c|c} & \begin{pmatrix} 2 & 8 \\ 0 & 2 \\ 1 & -2 \end{pmatrix} \\ \hline \begin{pmatrix} 1 & -2 & 3 \\ -2 & 5 & 1 \end{pmatrix} & \begin{pmatrix} 5 & -2 \\ -3 & -8 \end{pmatrix} \\ \hline \end{array}$$

$$\therefore\ \mathbf{BC} = \begin{pmatrix} 5 & -2 \\ -3 & -8 \end{pmatrix}$$

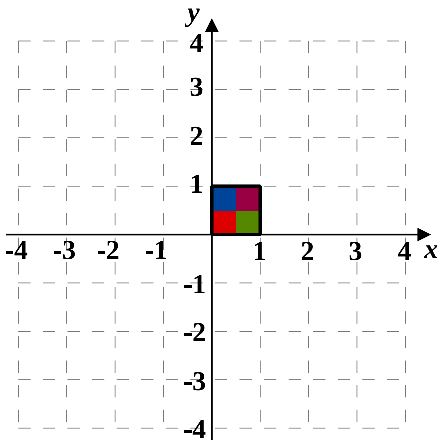
[3 marks]

4.1 The Unit Square

Faced with an unfamiliar 2×2 transformation matrix, one way to investigate its properties is to apply it to a unit square.

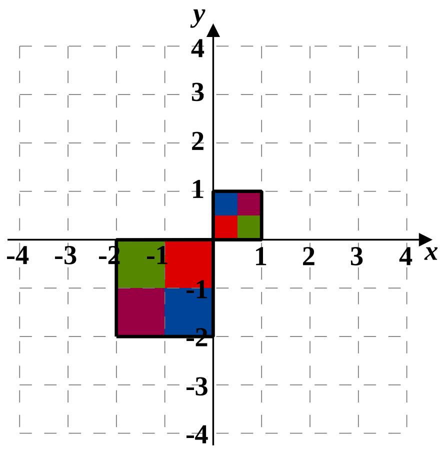
Written as a multipoint matrix the unit square to use is, $\mathbf{U} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

which is visualised as,



Consider the matrix, $\mathbf{M} = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$ used in Exercise 3.2, Question 1.

MU	$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$
$\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$	$\begin{pmatrix} 0 & -2 & -2 & 0 \\ 0 & 0 & -2 & -2 \end{pmatrix}$



The transformation can be seen to be a rotation of 180° about the origin and an enlargement of Length Scale Factor 2.

The Area Scale Factor is 4, which is also given by *det* $\mathbf{M} = +4$.

The + sign indicates that the orientation is unchanged. In both the original shape and the image the colours go red, green, purple, blue in anticlockwise order.

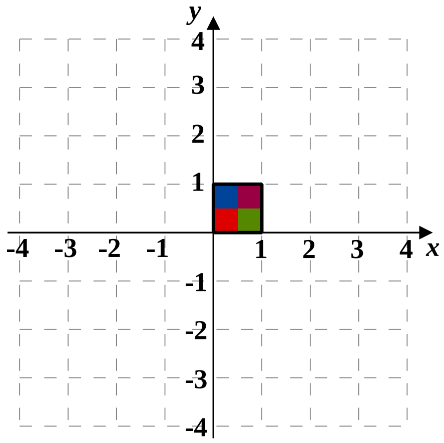
4.2 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 50

Question 1

A matrix, **N**, is used as a transformation, where $\mathbf{N} = \begin{pmatrix} -2 & -2 \\ -2 & 2 \end{pmatrix}$

(i) Apply **N** to the unit square and plot the resulting shape on the graph below.



[3 marks]

(ii) Calculate the determinant of **N**

[1 mark]

(iii) Explain carefully what the sign of the determinant tells you about the transformation, **N**

[2 marks]

(iv) What is the Length Scale Factor of the transformation ?

[1 mark]

(v) What is the Area Scale Factor of the transformation ?

[1 mark]

Question 2

Complete the following,

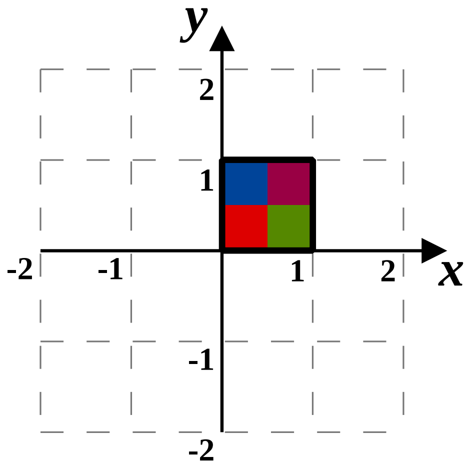
Ask your teacher to check your answers as it's important to get these correct !

A Catalogue of Two-Dimensional Transformations

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{array}{c|c} & \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \\ \hline \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & & & \\ 0 & & & \end{pmatrix} \end{array}$$

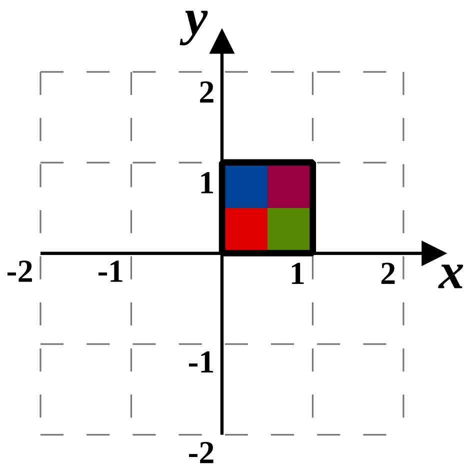
Description :



$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{array}{c|c} & \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \\ \hline \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & & & \\ 0 & & & \end{pmatrix} \end{array}$$

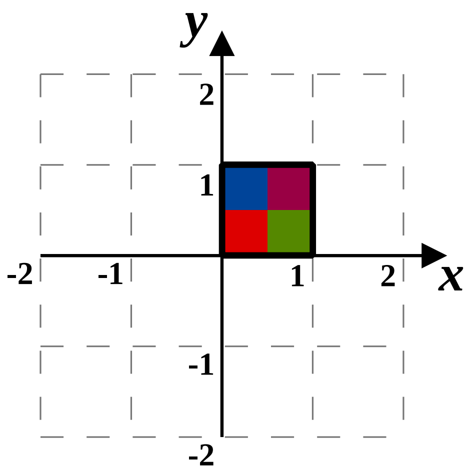
Description :



$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{array}{c|c} & \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \\ \hline \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} & \begin{pmatrix} 0 & & & \\ 0 & & & \end{pmatrix} \end{array}$$

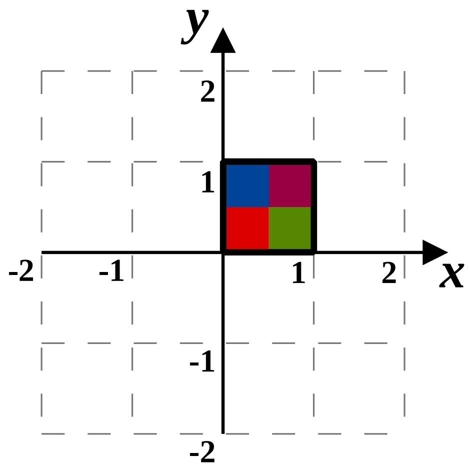
Description :



$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\begin{array}{c|c} & \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \\ \hline \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & & & \\ 0 & & & \end{pmatrix} \end{array}$$

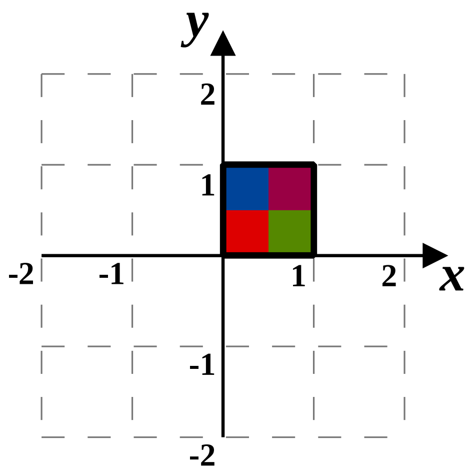
Description :



$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{array}{c|c} & \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \\ \hline \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & \begin{pmatrix} 0 & & & \\ 0 & & & \end{pmatrix} \end{array}$$

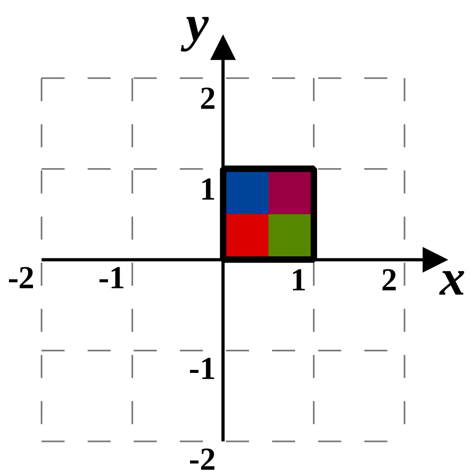
Description :



$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{array}{c|c} & \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \\ \hline \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & & & \\ 0 & & & \end{pmatrix} \end{array}$$

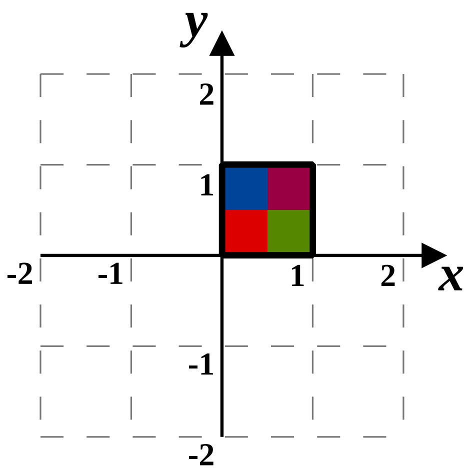
Description :



$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

	$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$
$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & & & \\ 0 & & & \end{pmatrix}$

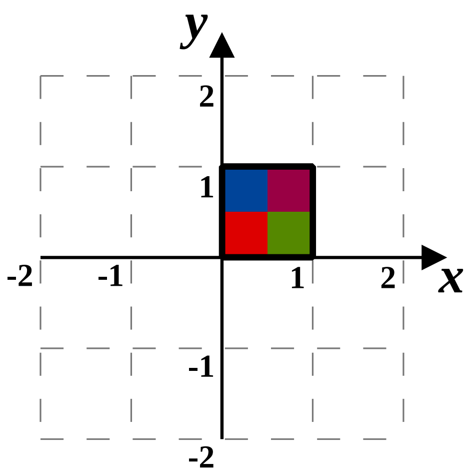
Description :



$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

	$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$
$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & & & \\ 0 & & & \end{pmatrix}$

Description :



[3 marks each = 24 marks total]

Question 3

Further A-Level Examination Question from May 2016, FP1, Q1 (Edexcel)

Given that k is a real number and that,

$$\mathbf{A} = \begin{pmatrix} 1 + k & k \\ k & 1 - k \end{pmatrix}$$

find the exact values of k for which $\mathbf{A}$ is a singular matrix.

Give your answers in their simplest form.

[3 marks]

Question 4

Further A-Level Examination Question from January 2011, FP1, Q2 (Edexcel)

$$\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 5 & 3 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} -3 & -1 \\ 5 & 2 \end{pmatrix}$$

(a) Find **AB**

[3 marks]

Given that,

$$\mathbf{C} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

(b) describe fully the geometrical transformation represented by **C**

[2 marks]

(c) write down $\mathbf{C}^{100}$

[1 mark]

Question 5

Further A-Level Examination Question from June 2011, FP1, Q3 (Edexcel)

(a) Given that $\mathbf{A} = \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix}$

(i) find $\mathbf{A}^2$

[3 marks]

(ii) describe fully the geometrical transformation represented by $\mathbf{A}^2$

[1 mark]

(b) Given that $\mathbf{B} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

describe fully the geometrical transformation represented by **B**

[2 marks]

(c) Given that $\mathbf{C} = \begin{pmatrix} k+1 & 12 \\ k & 9 \end{pmatrix}$

where k is a constant, find the value of k for which the matrix **C** is singular

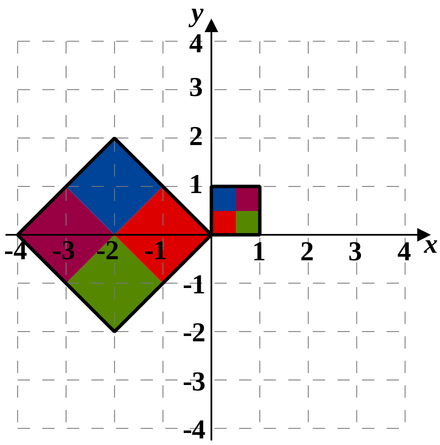
[3 marks]

4.3 Answers

Answer 1

(i)

	NU
	$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$
$\begin{pmatrix} -2 & -2 \\ -2 & 2 \end{pmatrix}$	$\begin{pmatrix} 0 & -2 & -4 & -2 \\ 0 & -2 & 0 & 2 \end{pmatrix}$



[3 marks]

(ii)

$$\begin{aligned} det\ \mathbf{N} &= |\mathbf{N}| \\ &= \left| \begin{pmatrix} -2 & -2 \\ -2 & 2 \end{pmatrix} \right| \\ &= (-2) \times 2 - (-2) \times (-2) \\ &= -8 \end{aligned}$$

[1 mark]

(iii) The minus sign of the determinant informs that the orientation of the shape has been reversed.
Going anticlockwise about the centre the colours used to go red, green purple, blue but in the image go red, blue, purple, green.

[2 marks]

(iv) $lsf = 2\sqrt{2}$

[1 mark]

(v) $asf = 8$

[1 mark]

Answer 2

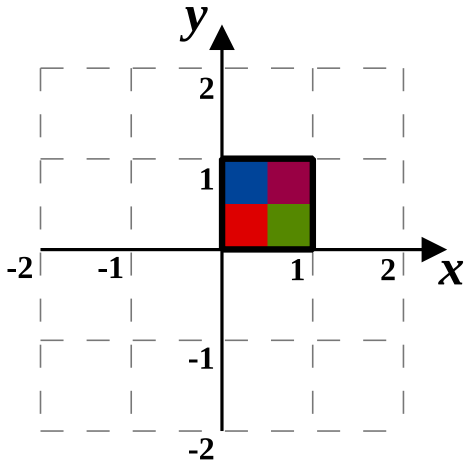
A Catalogue of Two-Dimensional Transformations

$\begin{pmatrix}1&0\\0&1\end{pmatrix}$

	$\begin{pmatrix}0&1&1&0\\0&0&1&1\end{pmatrix}$
$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}0&1&1&0\\0&0&1&1\end{pmatrix}$

Description : Identity matrix, **I**

Leaves points as is

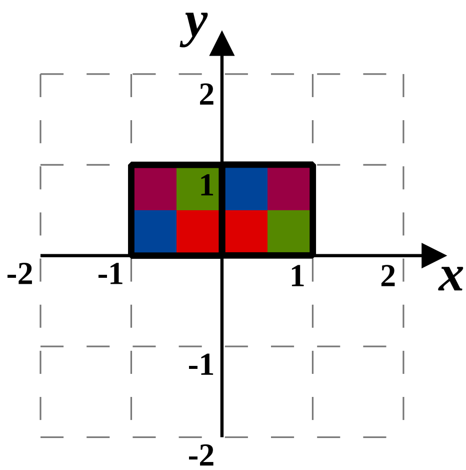


$\begin{pmatrix}0&-1\\1&0\end{pmatrix}$

	$\begin{pmatrix}0&1&1&0\\0&0&1&1\end{pmatrix}$
$\begin{pmatrix}0&-1\\1&0\end{pmatrix}$	$\begin{pmatrix}0&0&-1&-1\\0&1&1&0\end{pmatrix}$

Description : Rotation of 90°

about (0, 0)

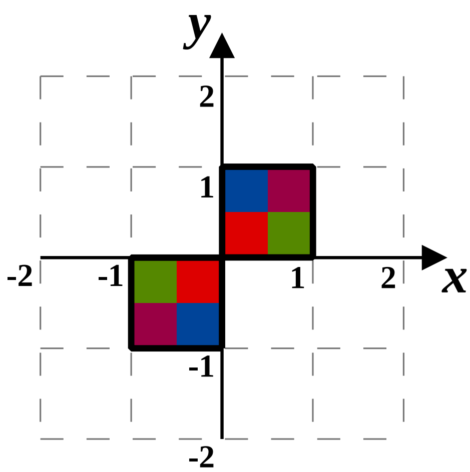


$\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$

	$\begin{pmatrix}0&1&1&0\\0&0&1&1\end{pmatrix}$
$\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}0&-1&-1&0\\0&0&-1&-1\end{pmatrix}$

Description : Rotation of 180°

about (0, 0)

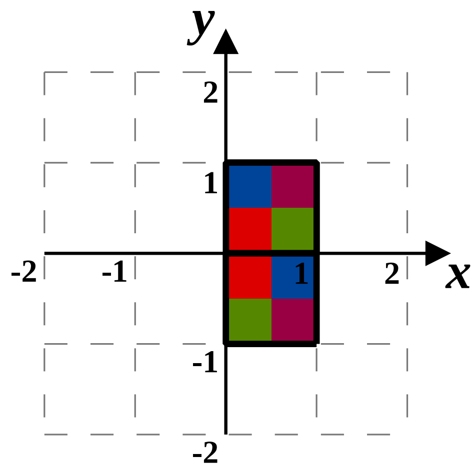


$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\begin{array}{c|c} & \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \\ \hline \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & -1 & -1 & 0 \end{pmatrix} \end{array}$$

Description : Rotation of 270°

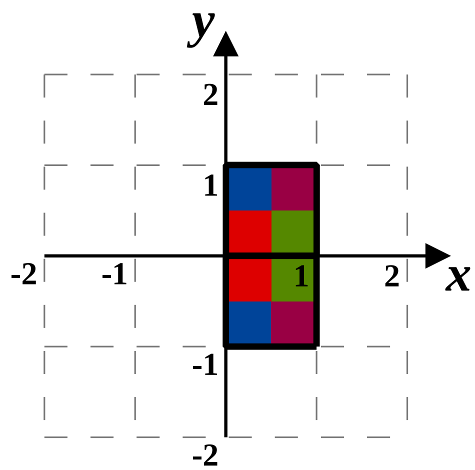
about (0, 0)



$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{array}{c|c} & \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \\ \hline \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & -1 \end{pmatrix} \end{array}$$

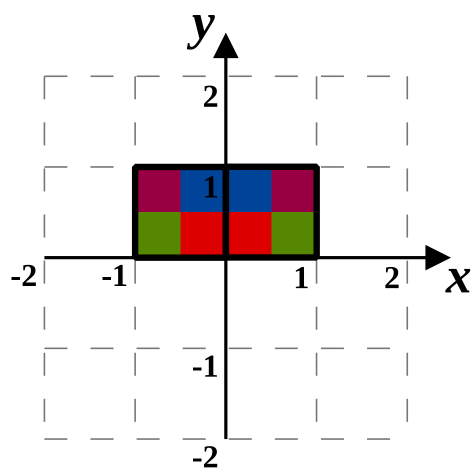
Description : Reflection in x-axis



$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{array}{c|c} & \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \\ \hline \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

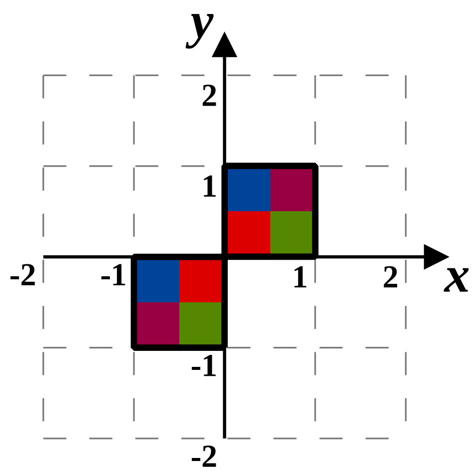
Description : Reflection in y-axis



$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

	$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$
$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & -1 & -1 \\ 0 & -1 & -1 & 0 \end{pmatrix}$

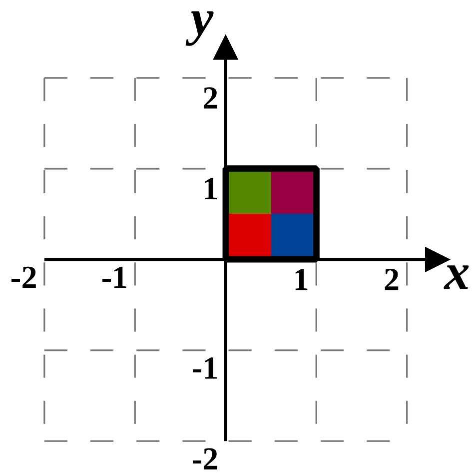
Description : Reflection in $y = -x$



$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

	$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$
$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$

Description : Reflection in $y = x$



[3 marks each = 24 marks total]

Answer 3

When a matrix is singular, $\quad det\ \mathbf{M} = 0$

$$\left| \begin{pmatrix} 1 + k & k \\ k & 1 - k \end{pmatrix} \right| = 0$$

$$(1 + k) \times (1 - k) - k \times k = 0$$

$$1 - k^2 - k^2 = 0$$

$$2k^2 - 1 = 0$$

$$k^2 = \frac{1}{2}$$

$$k = \pm \frac{1}{\sqrt{2}}$$

$$\pm \frac{\sqrt{2}}{2}$$

[3 marks]

Answer 4

(a)

$\mathbf{AB}$	$\begin{pmatrix} -3 & -1 \\ 5 & 2 \end{pmatrix}$
$\begin{pmatrix} 2 & 0 \\ 5 & 3 \end{pmatrix}$	$\begin{pmatrix} -6 & -2 \\ 0 & 1 \end{pmatrix}$

$$\therefore\ \mathbf{AB} = \begin{pmatrix} -6 & -2 \\ 0 & 1 \end{pmatrix}$$

[3 marks]

(b) **C** is Reflection in the y-axis

[2 marks]

(c) $\mathbf{C}^{100} = \mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

[1 mark]

Answer 5

(a) **(i)**

$\mathbf{A}^2$	$\begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix}$
$\begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix}$	$\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$

[3 marks]

(ii) Enlargement, scale factor 3, centre (0, 0)

[1 mark]

(b) Reflection in the line $y = -x$

[2 marks]

(c) When a matrix is singular, $det\ \mathbf{C} = 0$

$$det\ \mathbf{C} = 0$$

$$\left| \begin{pmatrix} k+1 & 12 \\ k & 9 \end{pmatrix} \right| = 0$$

$$(k+1) \times 9 - 12 \times k = 0$$

$$9 - 3k = 0$$

$$k = 3$$

[3 marks]

5.1 Composite Transformations

Let $\mathbf{X} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ This is the matrix for reflection in the x-axis

Let $\mathbf{D} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ This is the matrix, reflection in the diagonal line $y = -x$

Next is determined **DX** and, separately, **XD.**

XD	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$
$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

DX	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

The Commutative Law

Notice that **XD** $\neq$ **DX** : These two matrices do not obey The Commutative Law. In general, matrices do not commute, although there are special cases where they do.

Also notice that the answers to the multiplications are recognizable.

XD $= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ This is the matrix for rotation of 90° about the origin.

DX $= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ This is the matrix for rotation of 270° about the origin.

Getting The Order of Operations Correct

By way of working towards a concrete example, suppose now that a parallelogram, **P**, is defined by the points (5, 0), (5, 2), (3, 3) and (3, 1).

$$\mathbf{P} = \begin{pmatrix} 5 & 5 & 3 & 3 \\ 0 & 2 & 3 & 1 \end{pmatrix}$$

From what is already know of composite functions, it should not be a surprise that **XDP** means to **P**, first do **D**, then do **X**.

This is like $fg(x)$ meaning put x into function g first, then that result into function f.

The Associative Law

With matrices, The Associative Law holds; it does not matter how the matrices in **XDP** are grouped. In other words, (**XD**)**P** = **X**(**DP**)

Advantage

The advantage of working out **XD** first rather than **DP** is that a general result was obtained; that reflection in the line $y = -x$ followed by reflection in the x-axis is equivalent to a rotation of 90° about the origin.

Disadvantage

The disadvantage of working out **XD** first rather than **DP** is that the interim position of the parallelogram is not found. So, next, this alternative method will be worked through for both **XDP** and **DXP** and diagrams obtained showing, step-by-step, the composite transformations.

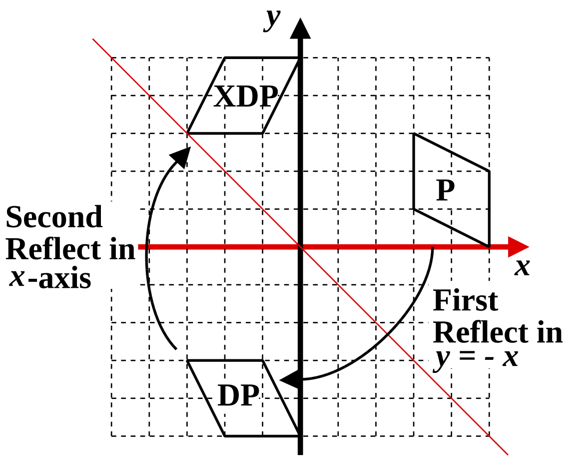
X(DP)

First : Reflect in $y = -x$

DP	$\begin{pmatrix} 5 & 5 & 3 & 3 \\ 0 & 2 & 3 & 1 \end{pmatrix}$
$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -2 & -3 & -1 \\ -5 & -5 & -3 & -3 \end{pmatrix}$

Second : Reflect in the x-axis

X(DP)	$\begin{pmatrix} 0 & -2 & -3 & -1 \\ -5 & -5 & -3 & -3 \end{pmatrix}$
$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & -2 & -3 & -1 \\ 5 & 5 & 3 & 3 \end{pmatrix}$



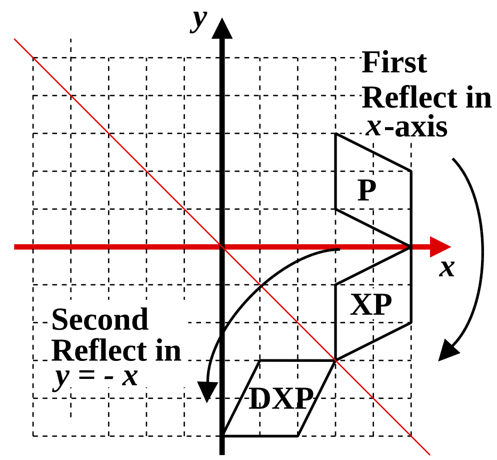
D(XP)

First : Reflect in the x-axis

XP	$\begin{pmatrix} 5 & 5 & 3 & 3 \\ 0 & 2 & 3 & 1 \end{pmatrix}$
$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 5 & 5 & 3 & 3 \\ 0 & -2 & -3 & -1 \end{pmatrix}$

Second : Reflect in $y = -x$

D(XP)	$\begin{pmatrix} 5 & 5 & 3 & 3 \\ 0 & -2 & -3 & -1 \end{pmatrix}$
$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 2 & 3 & 1 \\ -5 & -5 & -3 & -3 \end{pmatrix}$



5.2 Exercise

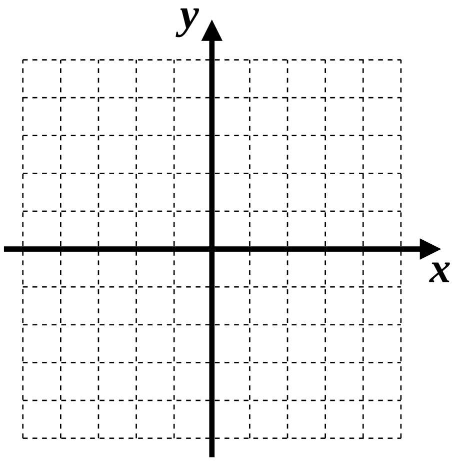
Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 50

Question 1

(i) Plot the arrow shape **A** described by the multipoint matrix,

$$\mathbf{A} = \begin{pmatrix} -2 & 0 & -2 & -4 \\ 0 & -4 & -3 & -4 \end{pmatrix}$$



[2 marks]

(ii) Transform **A** using the matrix $\mathbf{R} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and plot the result, **RA**.

[3 marks]

(iii) Transform **RA** using the matrix $\mathbf{M} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and plot the result, **MRA**.

[3 marks]

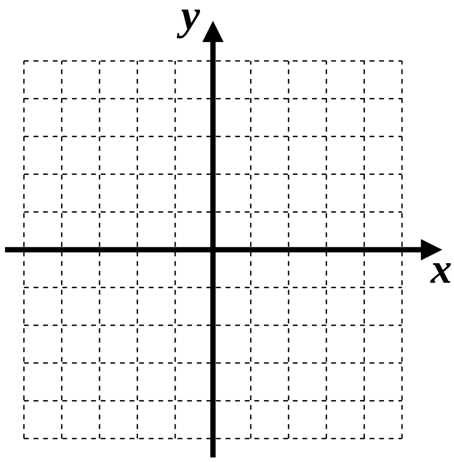
(iv) Calculate **MR** and hence or otherwise state what single transformation would have transformed **A** to **MRA**.

[4 marks]

Question 2

(i) Plot the shape described by the multipoint matrix,

$$\mathbf{P} = \begin{pmatrix} 0 & 2 & 0 & -2 \\ 0 & 0 & -4 & -4 \end{pmatrix}$$



[2 marks]

(ii) Transform **P** using the matrix $\mathbf{S} = \begin{pmatrix} -1 & 1 \\ 1 & -0.5 \end{pmatrix}$ and plot the result, **SP**

[3 marks]

(iii) Transform **SP** using the matrix $\mathbf{T} = \begin{pmatrix} 1 & 2 \\ -2 & -2 \end{pmatrix}$ and plot the result, **TSP**

[3 marks]

(iv) Calculate **TS**, and hence or otherwise state what single transformation would have transformed **P** to **TSP**.

[4 marks]

Question 3

Further A-Level Examination Question from January 2013, FP1, Q4 (Edexcel)

The transformation U, represented by the 2×2 matrix $\mathbf{P}$, is a rotation of $90°$ anticlockwise about the origin.

(a) Write down the matrix **P**

[1 mark]

The transformation V, represented by the 2×2 matrix $\mathbf{Q}$, is a reflection in the line $y = -x$.

(b) Write down the matrix **Q**

[1 mark]

Given that U followed by V is transformation T, which is represented by the matrix **R**,

(c) express **R** in terms of **P** and **Q**

[1 mark]

(d) find the matrix **R**

[2 marks]

(e) give a full geometric description of T as a single transformation

[2 marks]

Question 4

Further A-Level Examination Question from January 2012, FP1, Q4 (Edexcel)

A right angled triangle T has vertices $A(1, 1)$, $B(2, 1)$ and $C(2, 4)$.

When T is transformed by the matrix $\mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ the image is T'

(a) Find the coordinates of the vertices of T'

[2 marks]

(b) Describe fully the transformation represented by **P**

[2 marks]

The matrices $\mathbf{Q} = \begin{pmatrix} 4 & -2 \\ 3 & -1 \end{pmatrix}$ and $\mathbf{R} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ represent two transformations

When T is transformed by the matrix **QR**, the image is T''

(c) Find **QR**

[2 marks]

(d) Find the determinant of **QR**

[2 marks]

(e) Using your answer to part (d), find the area of T''

[3 marks]

Question 5

Further A-Level Examination Question from June 2011, FP1, Q5 (Edexcel)

$$\mathbf{A} = \begin{pmatrix} -4 & a \\ b & -2 \end{pmatrix} \text{ where } a \text{ and } b \text{ are constants}$$

Given that the matrix **A** maps the point with coordinates (4, 6) onto the point with coordinates (2, – 8),

(a) find the value of a and the value of b

[4 marks]

A quadrilateral R has area 30 square units

It is transformed into another quadrilateral S by the matrix **A**

Using your values of a and b

(b) find the area of quadrilateral S

[4 marks]

5.3 Answers to 5.2 Exercise

Answer 1

(i) Plotted as **A**

(ii)

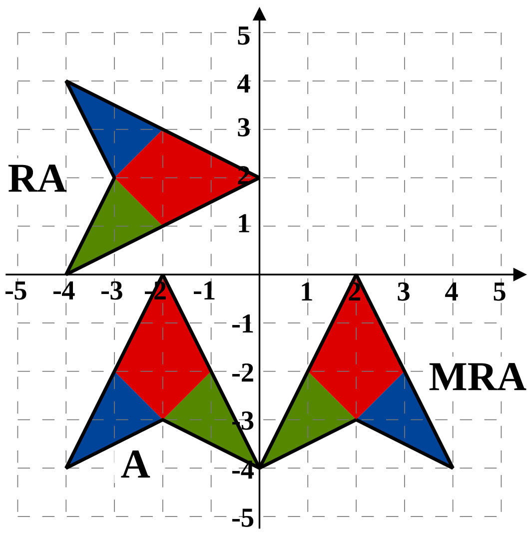
RA	$\begin{pmatrix} -2 & 0 & -2 & -4 \\ 0 & -4 & -3 & -4 \end{pmatrix}$
$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -4 & -3 & -4 \\ 2 & 0 & 2 & 4 \end{pmatrix}$

Plotted as **RA**

(iii)

MRA	$\begin{pmatrix} 0 & -4 & -3 & -4 \\ 2 & 0 & 2 & 4 \end{pmatrix}$
$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 2 & 0 & 2 & 4 \\ 0 & -4 & -3 & -4 \end{pmatrix}$

Plotted as **MRA**



(iv)

MR	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

This is recognisable as the matrix for reflection in the y-axis

The single matrix that would have transformed **A** to **MRA** is $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

[2, 3, 3, 4 marks]

Answer 2

(i) Plotted as **P**

[2 marks]

(ii)

SP	$\begin{pmatrix} 0 & 2 & 0 & -2 \\ 0 & 0 & -4 & -4 \end{pmatrix}$
$\begin{pmatrix} -1 & 1 \\ 1 & -0.5 \end{pmatrix}$	$\begin{pmatrix} 0 & -2 & -4 & -2 \\ 0 & 2 & 2 & 0 \end{pmatrix}$

Plotted as **SP**

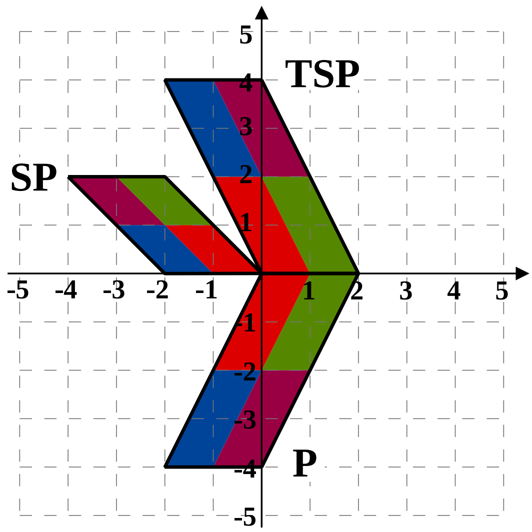
[3 marks]

(iii)

TSP	$\begin{pmatrix} 0 & -2 & -4 & -2 \\ 0 & 2 & 2 & 0 \end{pmatrix}$
$\begin{pmatrix} 1 & 2 \\ -2 & -2 \end{pmatrix}$	$\begin{pmatrix} 0 & 2 & 0 & -2 \\ 0 & 0 & 4 & 4 \end{pmatrix}$

Plotted as **TSP**

[3 marks]



(iv)

TS	$\begin{pmatrix} -1 & 1 \\ 1 & -0.5 \end{pmatrix}$
$\begin{pmatrix} 1 & 2 \\ -2 & -2 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

This is recognisable as the matrix for reflection in the *x*-axis

The single matrix that would have transformed **P** to **TSP** is $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

[4 marks]

Answer 3

(a) $\mathbf{P} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

(b) $\mathbf{Q} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

(c) **R** = **QP**

(d)

QP	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\mathbf{R} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

(e) **R** is a reflection in the y-axis

This question is showing that a rotation 90° anticlockwise about the origin followed by a reflection in the line $y = -x$ is equivalent to a reflection in the in y-axis.

[1, 1, 1, 2, 2 marks]

Answer 4

(a)

PA	$\begin{pmatrix} 1 & 2 & 2 \\ 1 & 1 & 4 \end{pmatrix}$
$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 4 \\ 1 & 2 & 2 \end{pmatrix}$

So T' has coordinates $A'(1, 1)$, $B'(1, 2)$ and $C'(4, 2)$

(b) **P** represents reflection in the line $y = x$

(c)

QR	$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$
$\begin{pmatrix} 4 & -2 \\ 3 & -1 \end{pmatrix}$	$\begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$

(d) $|\mathbf{QR}| = (-2) \times 2 - 0 \times 0 = -4$

(e) T has area $0.5 \times 1 \times 3 = 1.5$

$\therefore\ T''$ has area $1.5 \times 4 = 6$ units^2

[2, 2, 2, 2, 3 marks]

Answer 5

(a) A "matrix methods solution"...

$$\begin{pmatrix} -4 & a \\ b & -2 \end{pmatrix}\begin{pmatrix} 4 \\ 6 \end{pmatrix} = \begin{pmatrix} 2 \\ -8 \end{pmatrix}$$

$$\begin{pmatrix} (-4 \times 4 + a \times 6) \\ b \times 4 + (-2) \times 6 \end{pmatrix} = \begin{pmatrix} 2 \\ -8 \end{pmatrix}$$

$$\begin{pmatrix} -16 + 6a \\ 4b - 12 \end{pmatrix} = \begin{pmatrix} 2 \\ -8 \end{pmatrix}$$

$$\begin{pmatrix} 6a \\ 4b \end{pmatrix} - \begin{pmatrix} 16 \\ 12 \end{pmatrix} = \begin{pmatrix} 2 \\ -8 \end{pmatrix}$$

$$\begin{pmatrix} 6a \\ 4b \end{pmatrix} = \begin{pmatrix} 2 \\ -8 \end{pmatrix} + \begin{pmatrix} 16 \\ 12 \end{pmatrix}$$

$$\begin{pmatrix} 6a \\ 4b \end{pmatrix} = \begin{pmatrix} 18 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

[4 marks]

(b)

$$\begin{aligned} det\,|\mathbf{A}| &= \left|\begin{pmatrix} -4 & a \\ b & -2 \end{pmatrix}\right| \\ &= \left|\begin{pmatrix} -4 & 3 \\ 1 & -2 \end{pmatrix}\right| \\ &= (-4) \times (-2) - 3 \times 1 \\ &= 8 - 3 \\ &= 5 \end{aligned}$$

$\therefore$ Area of S will be $30 \times 5 = 150$ square units

[4 marks]

6.1 Generalised Rotation

The rotation matrices looked at previously are special cases of a more general result,

---

Rotation through angle θ about (0, 0)

$$\mathbf{R}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

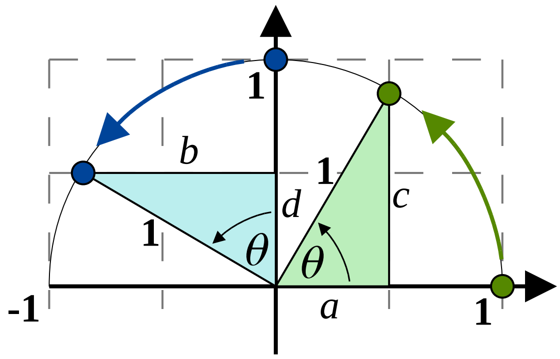
Proof

Let the matrix that causes rotation of $\theta°$ about the origin be $\mathbf{R}_\theta = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Consider what this matrix does to the unit square, $\mathbf{U} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

$\mathbf{R}_\theta\mathbf{U}$	$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$
$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$	$\begin{pmatrix} 0 & a & a+b & b \\ 0 & c & c+d & d \end{pmatrix}$

Thus, under the rotation of $\theta°$ $(1, 0) \rightarrow (a, c)$ and $(0, 1) \rightarrow (b, d)$



From the diagram, the green triangle has hypotenuse 1, opposite c and adjacent a
Applying SOH CAH TOA in this green triangle gives $a = \cos\theta$ and $c = \sin\theta$

The displacement vector to the rotated green point is thus $\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$.

The green point has coordinates ($\cos\theta, \sin\theta$).
The blue and green triangles are congruent, with lengths a and c equal to lengths d and b respectively; the displacement vector to the rotated blue point is $\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$.

The blue point has coordinates ($-\sin\theta$, $\cos\theta$).

Thus $\mathbf{R}_\theta = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ □

Exercise 6.2

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 50

Question 1

(i) By comparing the matrix $\mathbf{Y} = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$ to $\mathbf{R}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

state the transformation represented by matrix **Y**.

[2 marks]

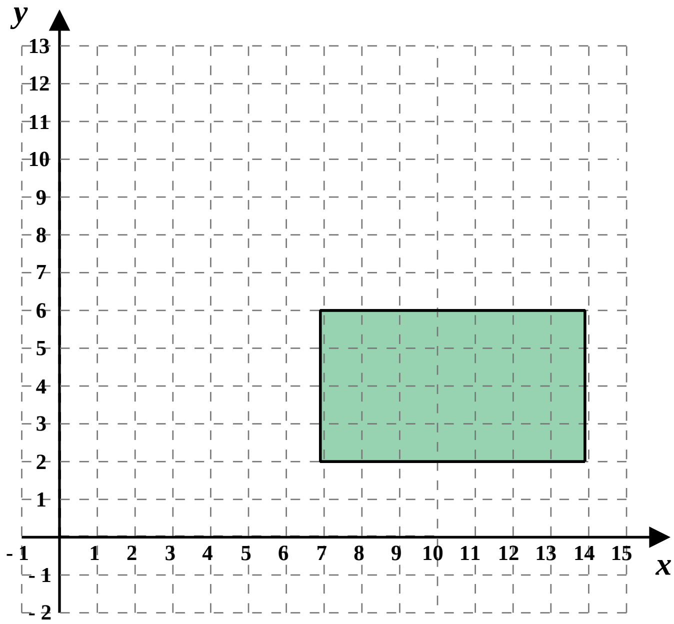
(ii) A rectangle is represented by the multipoint matrix,

$$\mathbf{M} = \begin{pmatrix} 4\sqrt{3} & 8\sqrt{3} & 8\sqrt{3} & 4\sqrt{3} \\ 2 & 2 & 6 & 6 \end{pmatrix}$$

Apply the transformation represented by matrix **Y** to the rectangle.
Give the exact coordinates of the transformed rectangle.

[5 marks]

(iii) Plot the transformed rectangle on the graph below.



[2 marks]

Question 2

Further A-Level Examination Question from February 2010, FP1, Q9 (Edexcel)

$$\mathbf{M} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

(a) Describe fully the geometrical transformation represented by the matrix **M**

[2 marks]

The transformation represented by **M** maps the point A with coordinates (p, q) onto the point B with coordinates $(3\sqrt{2}, \ 4\sqrt{2})$

(b) Find the value of p and the value of q

[4 marks]

(c) Find, in its simplest surd form, the length OA, where O is the origin.

[2 marks]

(d) Find $\mathbf{M}^2$

[2 marks]

The point B is mapped onto the point C by the transformation represented by $\mathbf{M}^2$

(e) Find the coordinates of C

[2 marks]

Question 3

Further A-Level Examination Question from May 2016, FP1, Q6 (Edexcel)

$$\mathbf{P} = \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix}$$

(a) Describe fully the single geometrical transformation U represented by the matrix $\mathbf{P}$

[2 marks]

The transformation U maps the point A with coordinates (p, q) onto the point B, with coordinates $(6\sqrt{2},\ 3\sqrt{2})$

(b) Find the value of p and the value of q

[3 marks]

The transformation V, represented by the 2×2 matrix $\mathbf{Q}$, is a reflection in the line with equation $y = x$.

(c) Write down the matrix $\mathbf{Q}$

[1 mark]

The transformation U followed by the transformation V is the transformation T. The transformation T is represented by the matrix $\mathbf{R}$

(d) Find the matrix $\mathbf{R}$

[3 marks]

(e) Deduce that the transformation T is self-inverse

[2 marks]

Question 4

Further A-Level Examination Question from June 2014, FP1, Q7 (Edexcel)

(i) In each of the following cases, find a 2×2 matrix that represents

(a) a reflection in the line $y = -x$

(b) a rotation of 135° anticlockwise about (0, 0)

(c) a reflection in the line $y = -x$ followed by a rotation of 135° anticlockwise about (0, 0)

[4 marks]

(ii) The triangle T has vertices at the points (1, k) (3, 0) and (11, 0) where k is a constant,

The triangle T is transformed onto the triangle T' by the matrix

$$\begin{pmatrix} 6 & -2 \\ 1 & 2 \end{pmatrix}$$

Given that the area of triangle T' is 364 square units, find the value of k

[6 marks]

Question 5

(i) Given that $\mathbf{P} = \begin{pmatrix} cos\,\theta & -sin\,\theta \\ sin\,\theta & cos\,\theta \end{pmatrix}$

show algebraically that $\mathbf{P}^2 = \begin{pmatrix} cos\,2\theta & -sin\,2\theta \\ sin\,2\theta & cos\,2\theta \end{pmatrix}$

[6 marks]

(ii) Interpret this result geometrically

[2 marks]

Answer 1

(i) As $\cos\theta$ and $\sin\theta$ are both positive, $0 < \theta < 90$

$\therefore\ \theta = 30°$

Y represents a rotation of 30° about the origin.

[2 marks]

(ii)

$$\mathbf{YM} = \frac{1}{2}\begin{pmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{pmatrix} \times 2 \times \begin{pmatrix} 2\sqrt{3} & 4\sqrt{3} & 4\sqrt{3} & 2\sqrt{3} \\ 1 & 1 & 3 & 3 \end{pmatrix}$$

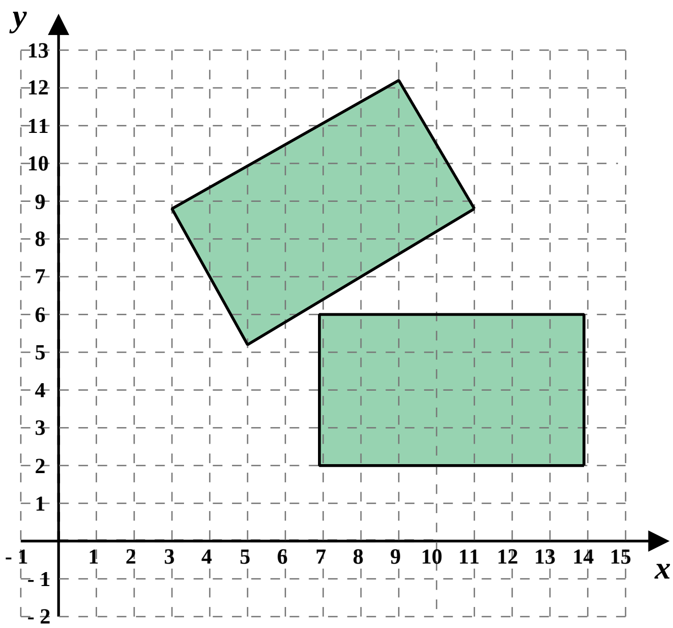
$$= \begin{pmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{pmatrix}\begin{pmatrix} 2\sqrt{3} & 4\sqrt{3} & 4\sqrt{3} & 2\sqrt{3} \\ 1 & 1 & 3 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 11 & 9 & 3 \\ 3\sqrt{3} & 5\sqrt{3} & 7\sqrt{3} & 5\sqrt{3} \end{pmatrix}$$

YM	$\begin{pmatrix} 2\sqrt{3} & 4\sqrt{3} & 4\sqrt{3} & 2\sqrt{3} \\ 1 & 1 & 3 & 3 \end{pmatrix}$
$\begin{pmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{pmatrix}$	$\begin{pmatrix} 5 & 11 & 9 & 3 \\ 3\sqrt{3} & 5\sqrt{3} & 7\sqrt{3} & 5\sqrt{3} \end{pmatrix}$

[5 marks]

(iii)



$$\text{YM} = \begin{pmatrix} 5 & 11 & 9 & 3 \\ 5.20 & 8.66 & 12.12 & 8.66 \end{pmatrix}$$

[2 marks]

Answer 2

(a) As *cos* θ and *sin* θ are both positive, $0 < \theta < 90$

$\therefore\ \theta = 45^\circ$

M represents a rotation of 45° about the origin.

[2 marks]

(b)

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 3\sqrt{2} \\ 4\sqrt{2} \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \sqrt{2} \times \begin{pmatrix} 3\sqrt{2} \\ 4\sqrt{2} \end{pmatrix}$$

$$\begin{pmatrix} p - q \\ p + q \end{pmatrix} = \begin{pmatrix} 6 \\ 8 \end{pmatrix}$$

Simultaneous Equations

$$2p = 14$$

$$p = 7 \qquad \Rightarrow\ q = 1$$

[4 marks]

(c)

$$|OA| = \sqrt{p^2 + q^2}$$

$$= \sqrt{7^2 + 1^2}$$

$$= \sqrt{50}$$

$$= 5\sqrt{2}$$

[2 marks]

(d)

$$\mathbf{M}^2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \times \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$= \frac{1}{2} \times \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$ Rotation 90° about (0, 0)

[2 marks]

(e)

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 3\sqrt{2} \\ 4\sqrt{2} \end{pmatrix} = \begin{pmatrix} -4\sqrt{2} \\ 3\sqrt{2} \end{pmatrix}$$

$$\therefore\ C\left(-4\sqrt{2},\ 3\sqrt{2}\right)$$

[2 marks]

Answer 3

(a) As *cos* θ is negative and *sin* θ is positive positive, $90 < \theta < 180$

$\therefore\ \theta = 135°$

P represents a rotation of 135° about the origin.

[2 marks]

(b)

$$\frac{1}{\sqrt{2}}\begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 6\sqrt{2} \\ 3\sqrt{2} \end{pmatrix}$$

$$\begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \sqrt{2} \times \begin{pmatrix} 6\sqrt{2} \\ 3\sqrt{2} \end{pmatrix}$$

$$\begin{pmatrix} -p - q \\ p - q \end{pmatrix} = \begin{pmatrix} 12 \\ 6 \end{pmatrix}$$

Simultaneous Equations

$$-2q = 18$$

$$q = -9 \qquad \Rightarrow\ p = -3$$

[3 marks]

(c)

$$\mathbf{Q} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

[1 mark]

(d)

$$\mathbf{R} = \mathbf{QP}$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \times \frac{1}{\sqrt{2}} \times \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}$$

[3 marks]

(e) If **R** is self inverse then **RR** = **I**, the identity matrix

$$\mathbf{RR} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} \times \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \mathbf{I} \qquad \therefore\ \mathbf{R} \text{ is self inverse}$$

It can also be thought of as a reflection.
I think the mirror line is $y = -\left(1 + \sqrt{2}\right)x$
Any reflection twice in a line gets back to the original position.

[2 marks]

Answer 4

(i) **(a)** Reflection in the line $y = -x$: $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

(b) Put 135° into the general rotation matrix : $\begin{pmatrix} cos\,135° & -\,sin\,135° \\ sin\,135° & cos\,135° \end{pmatrix}$

$$\begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$$

(c) Get the order correct...

$$\frac{1}{\sqrt{2}}\begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

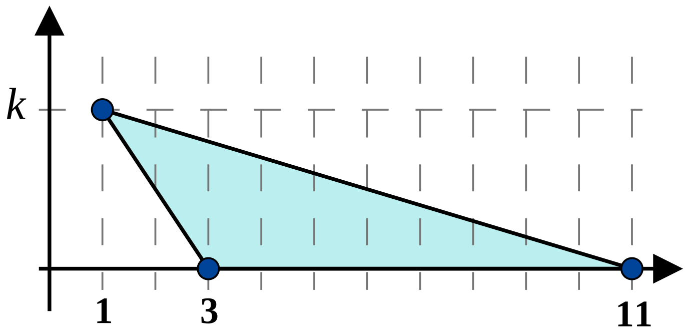
$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix}$$

[4 marks]

(ii) $det\begin{pmatrix} 6 & -2 \\ 1 & 2 \end{pmatrix} = 6 \times 2 - (-2) \times 1 = 14$

So the area of triangle T must be $\frac{364}{14} = 26$ square units

A quick sketch...



$$\text{Area } \Delta = \frac{1}{2}\,b\,h$$

$$26 = \frac{1}{2} \times 8 \times k$$

$$k = 6.5 \text{ units}$$

[6 marks]

Answer 5

(i)

$$\mathbf{P}^2 = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \times \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\theta - \sin^2\theta & -\cos\theta\sin\theta - \sin\theta\cos\theta \\ \sin\theta\cos\theta + \sin\theta\cos\theta & -\sin^2\theta + \cos^2\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\theta - \sin^2\theta & -2\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}$$

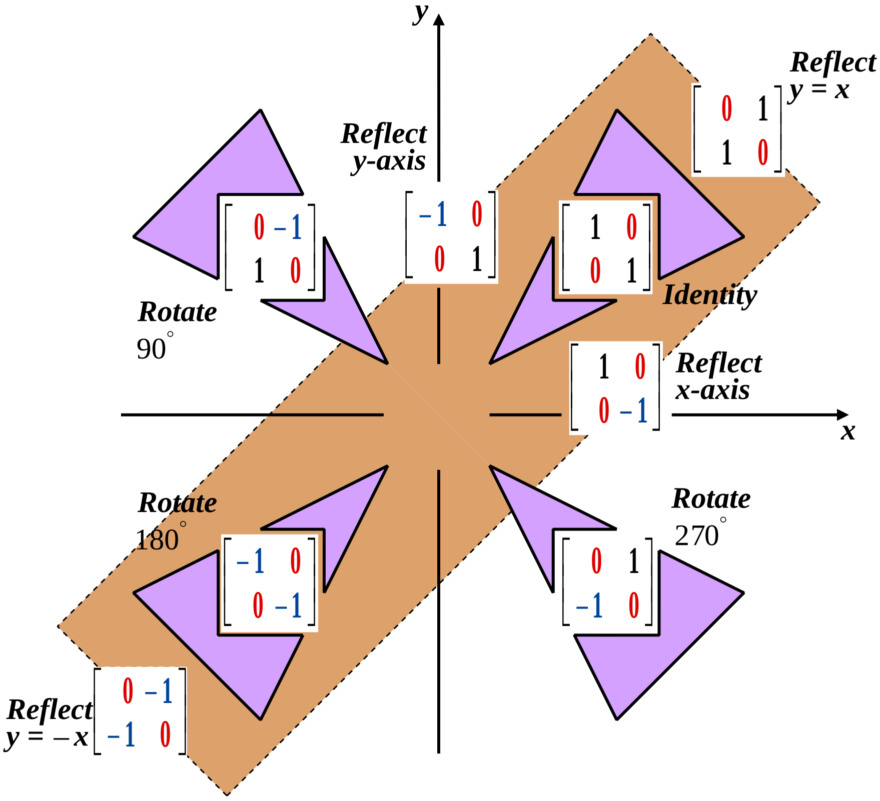
[6 marks]

(ii) Two successive anticlockwise rotations about the origin by an angle θ are equivalent to a single anticlockwise rotation by an angle 2θ

[2 marks]

7.1 Congruent Shapes

In general a matrix can represent a transformation that moves points around. Of more interest is the movement of a set of connected points; a shape. Sometimes a shape is savagely distorted by the transformation represented by the matrix. Thus far, of particular interest are matrices that give rise to transformations that preserve both size and shape. In other words, transformations in which the image is congruent with the pre-image.



A useful summary of some matrices which represent transformations that preserve both the size and shape of a pre-image and its image. The idea is that the zeros of the matrices lie along zero-lines and so allow the diagram to be memorised.

7.2 A Calculation Technique

The rotation matrices above, $\mathbf{R}_0$, $\mathbf{R}_{90}$, $\mathbf{R}_{180}$ and $\mathbf{R}_{270}$ are special cases of,

$$\mathbf{R}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

With an angle such as 45° the $\sqrt{2}$ involved can be factorised out to give,

$$\mathrm{R}_{45} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

which may help to keep subsequent multiplications compact and easier to follow. The angle θ can, and often is, expressed in radians.

7.3 Stretches and Enlargements

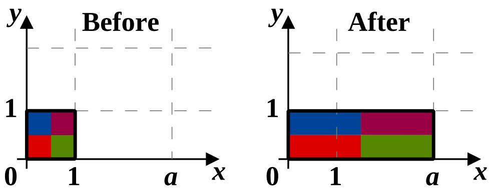
In starting to probe the distortions that a linear transformation can do to a shape the world of the stretch is entered.
The image is no longer congruent with the pre-image.

Parallel to *x*-axis

A stretch, (length) scale factor a parallel to the x-axis (only) will have matrix $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$

Applying this to the unit square, **U**, gives,

$$\begin{array}{c|c} & \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \\ \hline \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & a & a & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

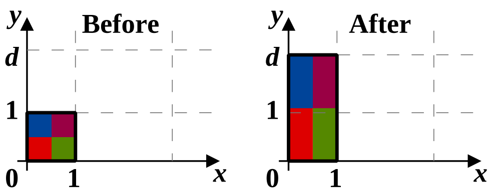


Parallel to *y*-axis

A stretch. (length) scale factor d parallel to the y-axis (only) will have matrix $\begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}$

Applying this to the unit square, **U**, gives,

$$\begin{array}{c|c} & \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \\ \hline \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} & \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & d & d \end{pmatrix} \end{array}$$

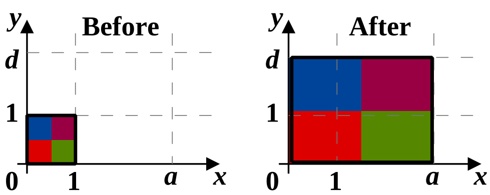


Parallel to both *x*-axis and *y*-axis

A stretch, (length scale factor a parallel to the x-axis, and (length) scale factor d parallel to the y-axis will have matrix $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$

Applying this to the unit square, **U**, gives,

$$\begin{array}{c|c} & \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \\ \hline \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} & \begin{pmatrix} 0 & a & a & 0 \\ 0 & 0 & d & d \end{pmatrix} \end{array}$$



Enlargement : A Special Case of Parallel to both *x*-axis and *y*-axis

In the special case where $a = d = k$, $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ becomes $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$ which represents an enlargement with (length) scale factor k centre (0, 0)

7.4 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 50

Question 1

(i) Describe fully the transformation represented by $\mathbf{M} = \begin{pmatrix} 5 & 0 \\ 0 & 7 \end{pmatrix}$

[3 marks]

(ii) A triangle *T* has vertices at (2, 0), (0, 3) and (7, 0)
Find the area of the triangle.

[3 marks]

(iii) The triangle *T* is transformed using the matrix **M**.
Show how to use the determinant of **M** to find the area of the image of *T*.

[3 marks]

Question 2

(i) Describe fully the transformation represented by $\mathbf{A} = \begin{pmatrix} 2\sqrt{5} & 0 \\ 0 & 2\sqrt{5} \end{pmatrix}$

[3 marks]

(ii) A triangle *T* has coordinates (*a*, 1), (4, 1) and (4, 3).
Given that *T* is transformed using matrix A, and the area of the resulting triangle is 60, find the two possible values of *a*

[4 marks]

Question 3

Further A-Level Examination Question from June 2011, FP1, Q3 (Edexcel)

(a) Given that $\mathbf{A} = \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix}$

(i) find $\mathbf{A}^2$

(ii) describe fully the geometrical transformation represented by $\mathbf{A}^2$

[4 marks]

(b) Given that $\mathbf{B} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ describe fully the geometrical transformation represented by $\mathbf{B}$

[2 marks]

(c) Given that $\mathbf{C} = \begin{pmatrix} k+1 & 12 \\ k & 9 \end{pmatrix}$ where k is a constant,

find the value of k for which the matrix $\mathbf{C}$ is singular.

[3 marks]

Question 4

Further A-Level Examination Question from May 2018, F1, Q2 (Edexcel)

The transformation represented by the 2×2 matrix $\mathbf{P}$ is an anticlockwise rotation about the origin through 45 degrees.

(a) Write down the matrix $\mathbf{P}$, giving the exact numerical value of each element.

[1 mark]

$$\mathbf{Q} = \begin{pmatrix} k\sqrt{2} & 0 \\ 0 & k\sqrt{2} \end{pmatrix} \text{ where } k \text{ is a constant and } k > 0$$

(b) Describe fully the single geometrical transformation represented by the matrix $\mathbf{Q}$

[2 marks]

The combined transformation represented by the matrix $\mathbf{PQ}$ transforms the rhombus R_1 onto the rhombus R_2

The area of the rhombus R_1 is 6 and the area of the rhombus R_2 is 147

(c) Find the value of the constant k

[4 marks]

Question 5

Further A-Level Examination Question from June 2010, FP1, Q6 (Edexcel)

Write down the 2×2 matrix that represents,

(a) an enlargement with centre (0, 0) and scale factor 8

[1 mark]

(b) a reflection in the x-axis

[1 mark]

Hence, or otherwise,

(c) find the matrix **T** that represents an enlargement with centre (0, 0) and scale factor 8 followed by reflection in the x-axis

[2 marks]

$\mathbf{A} = \begin{pmatrix} 6 & 1 \\ 4 & 2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} k & 1 \\ c & -6 \end{pmatrix}$ where k and c are constants

(d) Find **AB**

[3 marks]

Given that **AB** represents the same transformation as **T**

(e) find the value of k and the value of c

[2 marks]

Question 6

Further A-Level Examination Question from January 2019, F1, Q7 (Edexcel)

$$\mathbf{P} = \begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$$

(a) Show that $\mathbf{P}^3 = 8\mathbf{I}$, where $\mathbf{I}$ is the 2×2 identity matrix

[3 marks]

(b) Describe fully the transformation represented by the matrix $\mathbf{P}$ as a combination of two simple geometrical transformations,

[4 marks]

(c) Find the matrix $\mathbf{P}^{35}$, giving your answer in the form

$$\mathbf{P}^{35} = 2^k \begin{pmatrix} -1 & a \\ b & -1 \end{pmatrix}$$

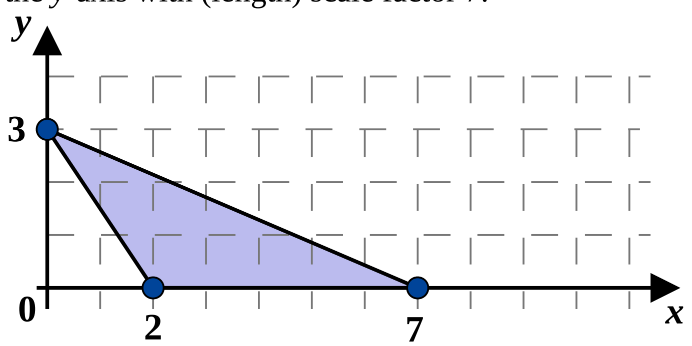
where k is an integer and a and b are surds to be found.

[2 marks]

Answer 1

(i) The matrix is of the form $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ and so $\begin{pmatrix} 5 & 0 \\ 0 & 7 \end{pmatrix}$ represents a stretch parallel to the x-axis, with (length) scale factor 5 and a stretch parallel to the y-axis with (length) scale factor 7.

(ii)



$$\text{Area } \Delta = \frac{1}{2}bh = \frac{1}{2} \times 5 \times 3 = 7.5 \text{ units}^2$$

(iii) $det \begin{pmatrix} 5 & 0 \\ 0 & 7 \end{pmatrix} = 5 \times 7 - 0 \times 0 = 35$ (which is the transformation's *asf*)

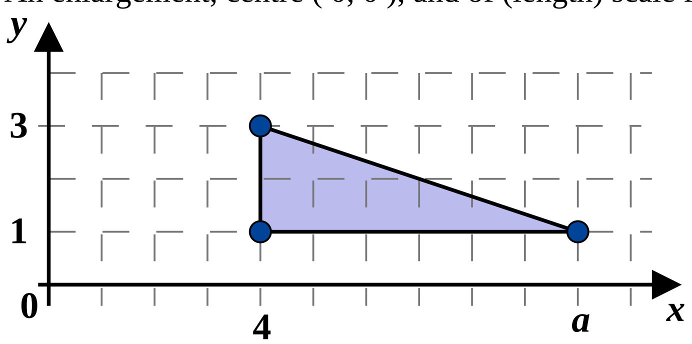
$\therefore$ Area of image of $T = 35 \times 7.5 = 262.5$ units^2

[3, 3, 3 marks]

Answer 2

(i) An enlargement, centre (0, 0), and of (length) scale factor $2\sqrt{5}$

(ii)



$$det \begin{pmatrix} 2\sqrt{5} & 0 \\ 0 & 2\sqrt{5} \end{pmatrix} = 2\sqrt{5} \times 2\sqrt{5} - 0 \times 0 = 20$$

This is the Area Scale Factor of the transformation

$\therefore$ Area of of $T = 3$ units^2

$$\text{Area } \Delta = \frac{1}{2}bh$$

$$3 = \frac{1}{2} \times base \times 2$$

$$base = 3 \text{ units} \qquad \Rightarrow a = 1 \text{ or } 7$$

[3, 4 marks]

Answer 3

(a) **(i)**

	$\begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix}$
$\mathbf{A}^2$	
$\begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix}$	$\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$

(ii) Enlargement, centre (0, 0), (length) scale factor 3

[4 marks]

(b) Reflection in the line $y = -x$

[2 marks]

(c) A singular matrix has a determinant of zero,

$$det\,\mathbf{C} = det\begin{pmatrix} k+1 & 12 \\ k & 9 \end{pmatrix}$$

$$0 = (k+1) \times 9 - 12k$$

$$0 = 9k + 9 - 12k$$

$$3k = 9 \Leftrightarrow k = 3$$

[3 marks]

Answer 4

(a) $\begin{pmatrix} \dfrac{1}{\sqrt{2}} & \dfrac{-1}{\sqrt{2}} \\ \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} \end{pmatrix}$

(b) Enlargement, centre (0, 0), (length) scale factor $k\sqrt{2}$

(c) $\mathbf{PQ} = \dfrac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} k\sqrt{2} & 0 \\ 0 & k\sqrt{2} \end{pmatrix}$

$$= \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$$

$$= \begin{pmatrix} k & -k \\ k & k \end{pmatrix} \quad \text{or} \quad k\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

The Area Scale Factor of the transformation, also $det\,(\mathbf{PQ})$ will be,

$$\frac{\text{Area } R_2}{\text{Area } R_1} = \frac{147}{6} = \frac{49}{2}$$

$$det\,(\mathbf{PQ}) = k \times k - (-k) \times k = 2k^2$$

$$\therefore\ k = \sqrt{\frac{49}{4}} = \frac{7}{2} = 3.5$$

[1, 2, 4 marks]

Answer 5

(a) $\begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix} = \mathbf{E}$

[1 mark]

(b) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \mathbf{R}$

[1 mark]

(c) **T** = **RE**

RE	$\begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix}$
$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 8 & 0 \\ 0 & -8 \end{pmatrix}$

$$\therefore \ \mathbf{T} = \begin{pmatrix} 8 & 0 \\ 0 & -8 \end{pmatrix}$$

[2 marks]

(d)

AB	$\begin{pmatrix} k & 1 \\ c & -6 \end{pmatrix}$
$\begin{pmatrix} 6 & 1 \\ 4 & 2 \end{pmatrix}$	$\begin{pmatrix} 6k + c & 0 \\ 4k + 2c & -8 \end{pmatrix}$

$$\therefore \ \mathbf{AB} = \begin{pmatrix} 6k + c & 0 \\ 4k + 2c & -8 \end{pmatrix}$$

[3 marks]

(e)

$$\begin{pmatrix} 8 & 0 \\ 0 & -8 \end{pmatrix} = \begin{pmatrix} 6k + c & 0 \\ 4k + 2c & -8 \end{pmatrix}$$

$$6k + c = 8$$
$$4k + 2c = 0$$

$$12k + 2c = 16$$
$$4k + 2c = 0 \qquad \text{SUBTRACT}$$

$$8k = 16$$
$$k = 2 \quad \Rightarrow \quad c = -4$$

[2 marks]

Answer 6

(a)

$$\begin{aligned}
\text{LHS} &= \mathbf{P}^3 \\
&= \begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} \\
&= \begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}\begin{pmatrix} -2 & 2\sqrt{3} \\ -2\sqrt{3} & -2 \end{pmatrix} \\
&= \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix} \\
&= 8\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= 8\mathbf{I} \\
&= \text{RHS} \qquad \square
\end{aligned}$$

[3 marks]

(b) It's a spiral similarity which can be broken down into,

- An enlargement, centre (0, 0), and (length) scale factor 2
- A rotation of 120° about (0, 0)

In this case the order will not matter.

[4 marks]

(c)

$$\begin{aligned}
\mathbf{P}^{35} &= \left(\mathbf{P}^3\right)^{11} \times \mathbf{P}^2 \\
&= (8\mathbf{I})^{11} \times \begin{pmatrix} -2 & 2\sqrt{3} \\ -2\sqrt{3} & -2 \end{pmatrix} \\
&= 8^{11} \times \mathbf{I}^{11} \times 2 \times \begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix} \\
&= \left(2^3\right)^{11} \times 1^{11} \times 2 \times \begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix} \\
&= 2^{34}\begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}
\end{aligned}$$

$$\therefore \quad k = 34, \quad a = \sqrt{3} \text{ and } b = -\sqrt{3}$$

[2 marks]

8.1 Singularities: The Implications

In Lesson 1, the definition of the determinant for a 2×2 matrix was given,

---

Definition of the Determinant

Given the generalised 2×2 square matrix, **M**, where,

$$\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

the determinant **M** is given by,

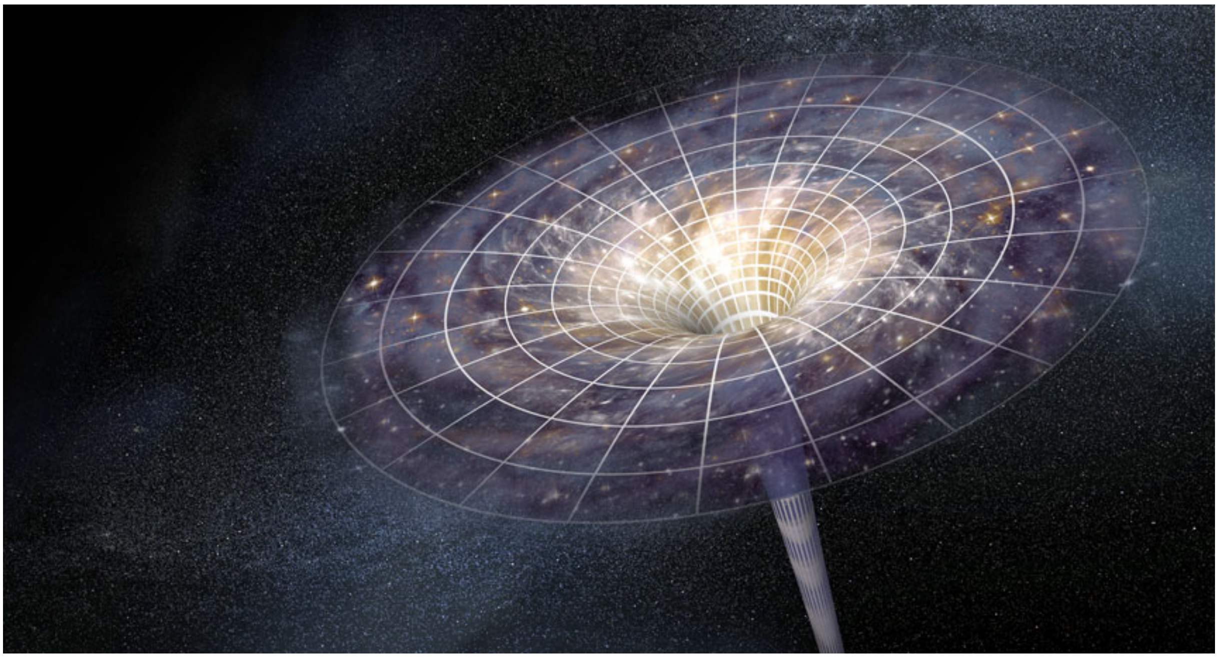
$$|\mathbf{M}| = ad - bc$$

This can also be written *det* **M** or $\Delta\mathbf{M}$

- If $|\mathbf{M}| = 0$ then **M** is a singular matrix
- If $|\mathbf{M}| \neq 0$ then **M** is a non-singular matrix

---

The magnitude of a matrix's determinant gives the Area Scale Factor of the transformation represented by the matrix.



A singularity is generally seen as a place to avoid. Mathematically they typically correspond to a division by zero. As a singularity is approached savage distortions occur and the singularity itself corresponds to a collapse of rules and structure, a loss of information, and an explosion of contradictions.
Singularities occur in nature; a black hole being the obvious example. The extreme curvature of time and space around a black hole mean that the standard rules of physics cease to apply.

8.2 Exploring A Singularity

Consider the matrix $\mathbf{M} = \begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix}$

$$\begin{aligned} det\,\mathbf{M} &= 1 \times 2 - (-1) \times (-2) \\ &= 0 \end{aligned}$$

This shows that **M** is a singular matrix.

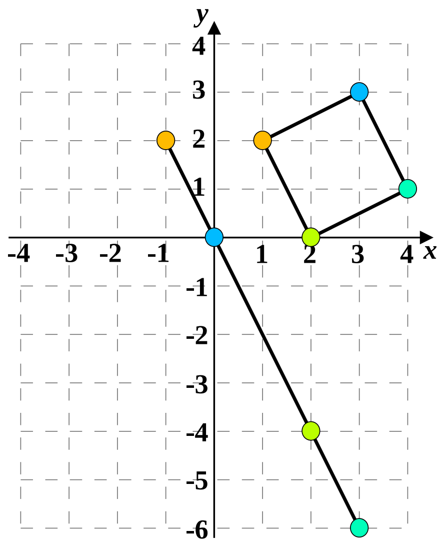
To explore what this means geometrically it will be applied to a test shape, the square represented by the multipoint matrix,

$$\mathbf{S} = \begin{pmatrix} 2 & 4 & 3 & 1 \\ 0 & 1 & 3 & 2 \end{pmatrix}$$

$$\begin{array}{c|c} \mathbf{MS} & \begin{pmatrix} 2 & 4 & 3 & 1 \\ 0 & 1 & 3 & 2 \end{pmatrix} \\ \hline \begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix} & \begin{pmatrix} 2 & 3 & 0 & -1 \\ -4 & -6 & 0 & 2 \end{pmatrix} \end{array}$$

$$\therefore\ \mathbf{MS} = \begin{pmatrix} 2 & 3 & 0 & -1 \\ -4 & -6 & 0 & 2 \end{pmatrix}$$

When the pre-image square is plotted and its image, it's clear that all the points of the image are on a straight line. In one respect this is not unexpected for the determinant has indeed given the Area Scale Factor of the transformation of zero.



Were we just unlucky in our choice of test shape ?
Will all points be moved onto this line ?
A more general argument is needed to answer these questions.

For a general pre-image point (u, v), and image point (x, y)

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

$$= \begin{pmatrix} u - v \\ -2u + 2v \end{pmatrix}$$

$$= \begin{pmatrix} u - v \\ -2(u - v) \end{pmatrix}$$

Let $p = u - v$ to get the result that,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p \\ -2p \end{pmatrix}$$

Which are points on the line $y = -2x$
All points under the transformation represented by this matrix, have images on the line $y = -2x$. The two dimensional surface collapses into a one dimensional line !

The situation is even worse than has so far been revealed.
Consider any point on the line $y = x$. Such a point could be written as (k, k).
Applying **M** to this point,

$$\begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} k \\ k \end{pmatrix} = \begin{pmatrix} k - k \\ -2k + 2k \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This is revealing that all of the points on the line $y = x$ map to the origin.

This in turn means that any hope of undoing the transformation has been extinguished for it could not be decided where on the line $y = x$ a point with image (0, 0) had come from. The singularity has caused information to be irretrievably lost.

Naked Singularity by Mark Tedin

8.3 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable

Marks Available : 40

Question 1

(i) Consider any point on the line $y = x + 1$

Such a generalised point could be written as (k, $k + 1$)

Apply matrix **M** to this generalised point where **M** is the singularity matrix investigated in 8.2 , namely,

$$\mathbf{M} = \begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix}$$

[2 marks]

(ii) To what single point are all points on the line $y = x + 1$ mapped under the transformation associated with matrix **M** ?

[1 mark]

(iii) Prove that all points on the line $y = x - 2$ are mapped to a single point under the transformation associated with matrix **M**.

Also, state the coordinates of that single point.

[3 marks]

Question 2

The two matrices, **M** and **N**, are,

$$\mathbf{M} = \begin{pmatrix} 1 & -3 \\ 2 & 1 \end{pmatrix} \qquad \mathbf{N} = \begin{pmatrix} -1 & k \\ 4 & 3 \end{pmatrix} \text{ where } k \text{ is a constant}$$

(a) Evaluate the determinant of **M**

[1 mark]

(b) Given that the determinant of **N** is 7, find the value of k

[1 mark]

(c) Using the value of k found in part (b), find **MN**

[1 mark]

(d) Verify that

$$\det \mathbf{MN} = \det \mathbf{M} \times \det \mathbf{N}$$

[1 mark]

(e) Does,

$$\det(\mathbf{M} + \mathbf{N}) = \det \mathbf{M} + \det \mathbf{N} \qquad ?$$

Justify your answer.

[2 marks]

Question 3

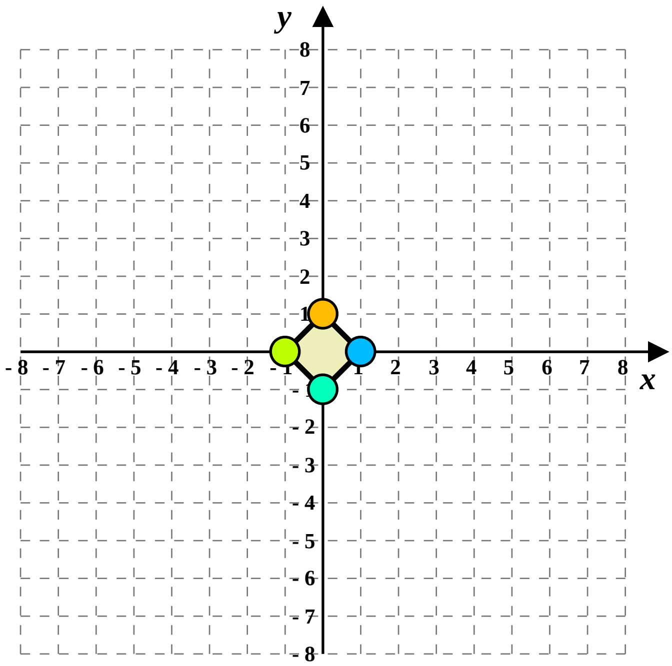
(i) Which one of the following matrices is singular ?

$$\mathbf{A} = \begin{pmatrix} 3 & -2 \\ 6 & 4 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 4 & 2 \\ 8 & 4 \end{pmatrix} \qquad \mathbf{C} = \begin{pmatrix} 3 & 4 \\ 7 & 8 \end{pmatrix}$$

Justify your answer.

[2 marks]

(ii) Apply the singular matrix from part **(i)** to transform the square below, and plot the resulting points and the line through them.



[3 marks]

(iii) Write down the equation of the line along which all image points lie.

[1 mark]

(iv) Prove that a general point in the plane, (u, v) when transformed using your part (i) matrix will lie on the straight line of your your part (ii) answer.

[3 marks]

(v) Prove that under the transformation represented by your part (i) matrix all points on the line $y = -2x + 3$ map onto the same point, and state the coordinates of that point.

[3 marks]

(vi) Write down the equation of another line along which all points will have the same image when transformed using the part (i) matrix.
For the example you give, state the coordinates of the image point to which all pre-images points will be mapped.

[3 marks]

Question 4

Using only 1s and 0s find as many 2×2 singular matrices as you can.

There are ten to be found.

[5 marks]

Question 5

Given that *k* is a real number and that,

$$\mathbf{M} = \begin{pmatrix} -2 & 1 - k \\ k - 1 & k \end{pmatrix}$$

find the exact values of *k* for which M is a singular matrix.

[3 marks]

Question 6

By using the two general 2×2 matrices,

$$\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad \mathbf{N} = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

Prove that,

$$det\,\mathbf{MN} = det\,\mathbf{M} \times det\,\mathbf{N}$$

[5 marks]

8.4 Answers

Answer 1

(i)

$$\begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} k \\ k+1 \end{pmatrix} = \begin{pmatrix} k - (k+1) \\ -2k + 2(k+1) \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

[2 marks]

(ii) $(-1,\ 2)$

[1 mark]

(iii) A generalised point on the line $y = x - 2$ would $(k,\ k-2)$

$$\begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} k \\ k-2 \end{pmatrix} = \begin{pmatrix} k - (k-2) \\ -2k + 2(k-2) \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ -4 \end{pmatrix}$$

All points on the line $y = x - 2$ are mapped onto $(2, -4)$

[3 marks]

Answer 2

(a) $det\,\mathbf{M} = 1 \times 1 - (-3) \times 2 = 7$

[1 mark]

(b)

$$\begin{aligned} det\,\mathbf{N} &= 7 \\ (-1) \times 3 - 4k &= 7 \\ -4k &= 10 \\ k &= -2.5 \end{aligned}$$

[1 mark]

(c)

MN	$\begin{pmatrix} -1 & -2.5 \\ 4 & 3 \end{pmatrix}$
$\begin{pmatrix} 1 & -3 \\ 2 & 1 \end{pmatrix}$	$\begin{pmatrix} -13 & -11.5 \\ 2 & -2 \end{pmatrix}$

[1 mark]

(d)

$$\begin{aligned} \text{LHS} &= det\,\mathbf{MN} \\ &= (-13) \times (-2) - (-11.5) \times 2 \\ &= 26 + 23 \\ &= 49 \\ &= 7 \times 7 \\ &= det\,\mathbf{M} \times det\,\mathbf{N} \\ &= \text{RHS} \qquad \square \end{aligned}$$

[1 mark]

(e)

$$\begin{aligned} det\,(\mathbf{M} + \mathbf{N}) &= det\left(\begin{pmatrix} 1 & -3 \\ 2 & 1 \end{pmatrix} + \begin{pmatrix} -1 & -2.5 \\ 4 & 3 \end{pmatrix} \right) \\ &= det\begin{pmatrix} 0 & -5.5 \\ 6 & 4 \end{pmatrix} \\ &= 0 \times 4 - (-5.5) \times 6 \\ &= 33 \end{aligned}$$

$$\begin{aligned} det\,\mathbf{M} + det\,\mathbf{N} &= 7 + 7 \\ &= 14 \end{aligned}$$

$$\therefore \quad det\,(\mathbf{M} + \mathbf{N}) \neq det\,\mathbf{M} + det\,\mathbf{N}$$

[2 marks]

Answer 3

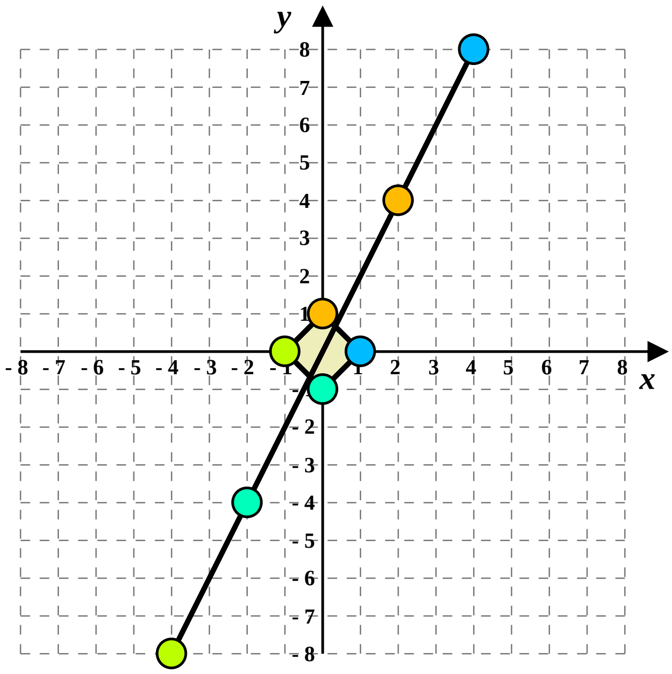
(i) $\quad det\ \mathbf{A} = 24 \qquad det\ \mathbf{B} = 0 \qquad det\ \mathbf{C} = -4$

$\therefore$ **B** is singular

[2 marks]

(ii)

M□	$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$
$\begin{pmatrix} 4 & 2 \\ 8 & 4 \end{pmatrix}$	$\begin{pmatrix} 4 & 2 & -4 & -2 \\ 8 & 4 & -8 & -4 \end{pmatrix}$



[3 marks]

(iii) $\quad y = 2x$

[1 mark]

(iv) For a general pre-image point (u, v), and image point (x, y)

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 8 & 4 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

$$= \begin{pmatrix} 4u + 2v \\ 8u + 4v \end{pmatrix}$$

$$= \begin{pmatrix} 4u + 2v\,) \\ 2\,(\,4u + 2v\,) \end{pmatrix}$$

Let $p = 4u + 2v$ to get the result that,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p \\ 2p \end{pmatrix}$$

Which are points on the line $y = 2\,x$

All points under the transformation represented by this matrix, have images on the line $y = 2\,x$.

The two dimensional surface is collapsing into a one dimensional line !

[3 marks]

(v) Let a general point on the line $y = -2x + 3$ be $(k, -2k + 3)$

$$\begin{pmatrix} 4 & 2 \\ 8 & 4 \end{pmatrix}\begin{pmatrix} k \\ -2k + 3 \end{pmatrix} = \begin{pmatrix} 4k + 2\,(-2k + 3\,) \\ 8k + 4\,(-2k + 3) \end{pmatrix}$$

$$= \begin{pmatrix} 6 \\ 12 \end{pmatrix}$$

So, all points of the line $y = -2x + 3$ map onto the point (6, 12)

[3 marks]

(vi) Let a general point on the line $y = -2x + c$ be $(k, -2k + c)$

$$\begin{pmatrix} 4 & 2 \\ 8 & 4 \end{pmatrix}\begin{pmatrix} k \\ -2k + c \end{pmatrix} = \begin{pmatrix} 4k + 2\,(-2k + c\,) \\ 8k + 4\,(-2k + c) \end{pmatrix}$$

$$= \begin{pmatrix} 2c \\ 4c \end{pmatrix}$$

So, all points of the line $y = -2x + c$ map onto the point $(2c, 4c)$

Question asks this be done for a value of c chosen by the candidate.

[3 marks]

Answer 4

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

[5 marks]

Answer 5

For **M** to be singular, $\quad det\,\mathbf{M} = 0$

$$(-2) \times k - (1 - k)(k - 1) = 0$$
$$-2k - \left(k - 1 - k^2 + k\right) = 0$$
$$-2k - k + 1 + k^2 - k = 0$$
$$k^2 - 4k + 1 = 0$$
$$\left[k^2 - 4k\right] + 1 = 0$$
$$\left[(k - 2)^2 - 4\right] + 1 = 0$$
$$(k - 2)^2 - 3 = 0$$
$$(k - 2)^2 = 3$$
$$k - 2 = \pm\sqrt{3}$$
$$k = 2 \pm \sqrt{3}$$

[3 marks]

Answer 6

$$\text{LHS} = det\,\mathbf{MN}$$
$$= det\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix}\right)$$
$$= det\begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}$$
$$= (ae + bg)(cf + dh) - (af + bh)(ce + dg)$$
$$= aecf + aedh + bgcf + bgdh - afce - afdg - bhce - bhdg$$
$$= (aecf - afce) + aedh + bgcf + (bgdh - bhdg) - afdg - bhce$$
$$= aedh + bgcf - afdg - bhce$$

$$\textbf{RHS} = det\,\mathbf{M} \times det\,\mathbf{N}$$
$$= det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times det\begin{pmatrix} e & f \\ g & h \end{pmatrix}$$
$$= (ad - bc)(eh - fg)$$
$$= adeh + bcfg - adfg - bceh$$
$$= \textbf{LHS} \qquad \square$$

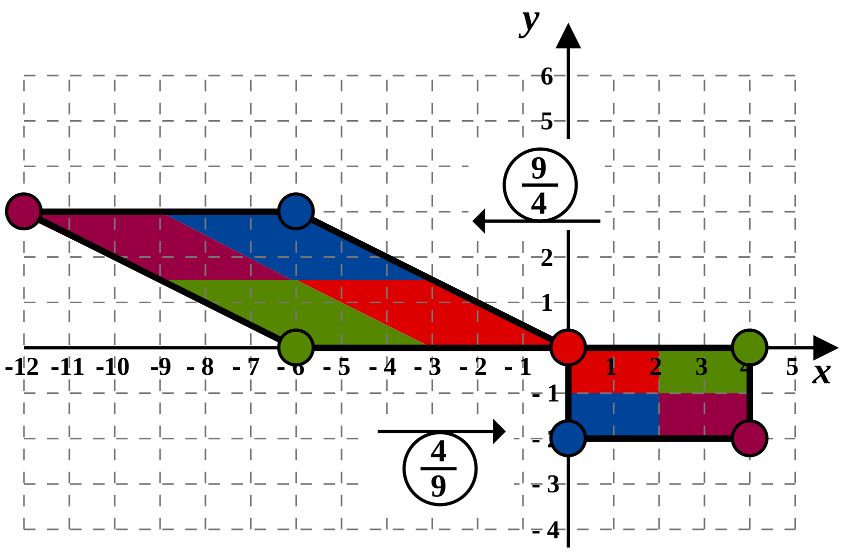
[5 marks]

9.1 The Undo Matrix

Let the matrix **T** be $\begin{pmatrix} -1.5 & 3 \\ 0 & -1.5 \end{pmatrix}$

The transformation represented by this matrix is applied to the rectangle described by $\mathbf{R} = \begin{pmatrix} 0 & 4 & 4 & 0 \\ 0 & 0 & -2 & -2 \end{pmatrix}$

TR	$\begin{pmatrix} 0 & 4 & 4 & 0 \\ 0 & 0 & -2 & -2 \end{pmatrix}$
$\begin{pmatrix} -1.5 & 3 \\ 0 & -1.5 \end{pmatrix}$	$\begin{pmatrix} 0 & -6 & -12 & -6 \\ 0 & 0 & 3 & 3 \end{pmatrix}$



The graph shows the original rectangle and its image, a parallelogram.

Also shown is the area scale factor of the transformation, the $\frac{9}{4}$.

This was calculated from,

$$\begin{aligned} \det \mathbf{T} &= (-1.5) \times (-1.5) - 3 \times 0 \\ &= 2.25 \end{aligned}$$

Writing this as $\frac{9}{4}$ is helpful as it's then obvious the inverse area scale factor is $\frac{4}{9}$.

In starting to think about the inverse transformation that will move the parallelogram back to the rectangle, one would expect $\frac{1}{\det \mathbf{T}}$ to be involved, given the connection between determinants and area scale factor. This insight also illuminates why a determinant of zero is a problem; it's the familiar issue of division by zero not being defined in mathematics emerging once again. Singular matrices can have no inverse.

9.2 Inverting a 2×2 Matrix

Given that we have been using matrices to move points, the inverse matrix will be that which moves the points back from whence they came.

The Inverse of a Matrix

In general, the inverse of a non-singular matrix $\mathbf{M}$ is the matrix $\mathbf{M}^{-1}$ such that

$$\mathbf{M}\,\mathbf{M}^{-1} = \mathbf{M}^{-1}\,\mathbf{M} = \mathbf{I}$$

In particular, if $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\mathbf{M}^{-1} = \dfrac{1}{det\,\mathbf{M}} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

For the matrix $\mathbf{T} = \begin{pmatrix} -1.5 & 3 \\ 0 & -1.5 \end{pmatrix}$ where $det\,\mathbf{T} = \dfrac{9}{4}$, we have $\dfrac{1}{det\,\mathbf{T}} = \dfrac{4}{9}$

Thus,

$$\mathbf{T}^{-1} = \frac{4}{9}\begin{pmatrix} -1.5 & -3 \\ 0 & -1.5 \end{pmatrix}$$

$$= -\frac{2}{3}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

Now to show that this inverse matrix moves the parallelogram back to the square.

$$-\frac{2}{3}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -6 & -12 & -6 \\ 0 & 0 & 3 & 3 \end{pmatrix} = -\frac{2}{3}\begin{pmatrix} 0 & -6 & -6 & 0 \\ 0 & 0 & -3 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 4 & 4 & 0 \\ 0 & 0 & 2 & -2 \end{pmatrix} \qquad \square$$

9.3 Spot Check

Find the inverse of each of the following matrices.

Having done so, check your answers with mine, to be found in 9.5 after the exercise.

For **C**, if you wish, you can deal with the third separately, inverting it to a 3.

$\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$ $\qquad$ $\mathbf{B} = \begin{pmatrix} 4 & 6 \\ 10 & 16 \end{pmatrix}$ $\qquad$ $\mathbf{C} = \dfrac{1}{3}\begin{pmatrix} 6 & -3 \\ -7 & 4 \end{pmatrix}$

[6 marks]

9.4 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 40

Question 1

Further A-Level Examination Question from January 2013, FP1, Q6 (Edexcel)

$$\mathbf{X} = \begin{pmatrix} 1 & a \\ 3 & 2 \end{pmatrix}$$

(a) Find the value of a for which the matrix X is singular

[2 marks]

$$\mathbf{Y} = \begin{pmatrix} 1 & -1 \\ 3 & 2 \end{pmatrix}$$

(b) Find $\mathbf{Y}^{-1}$

[2 marks]

The transformation represented by **Y** maps the point A onto the point B

Given that B has coordinates $(1 - \lambda,\ 7\lambda - 2)$ where λ is a constant,

(c) find, in terms of λ, the coordinates of point A

[4 marks]

Question 2

Further A-Level Examination Question from January 2011, FP1, Q8

$$\mathbf{A} = \begin{pmatrix} 2 & -2 \\ -1 & 3 \end{pmatrix}$$

(a) Find *det* **A**

[1 mark]

(b) Find $\mathbf{A}^{-1}$

[2 marks]

The triangle *R* is transformed to the triangle *S* by the matrix **A**
Given that the area of triangle *S* is 72 square units,

(c) find the area of triangle *R*

[2 marks]

The triangle *S* has vertices at the points (0, 4), (8, 16) and (12, 4).

(d) Find the coordinates of the vertices of *R*

[4 marks]

Question 3

Further A-Level Examination Question from January 2010, Q5 (Edexcel)

$$\mathbf{A} = \begin{pmatrix} a & -5 \\ 2 & a+4 \end{pmatrix}, \quad \text{where } a \text{ is real}$$

(a) Find *det* **A** in terms of *a*

[2 marks]

(b) Show that the matrix **A** is non-singular for all values of *a*

[3 marks]

Given that $a = 0$,

(c) find $\mathbf{A}^{-1}$

[3 marks]

Question 4

Further A-Level Examination Question from June 2015, Q7 (Edexcel)

(i) $\mathbf{A} = \begin{pmatrix} 5k & 3k - 1 \\ -3 & k + 1 \end{pmatrix}$, where k is a real constant.

Given that **A** is a singular matrix, find the possible values of k.

[4 marks]

(ii) $\mathbf{B} = \begin{pmatrix} 10 & 5 \\ -3 & 3 \end{pmatrix}$

A triangle T is transformed onto a triangle T' by the transformation represented by the matrix **B**. The vertices of triangle T' have coordinates $(0, 0)$, $(-20, 6)$ and $(10c, 6c)$, where c is a positive constant.

The area of triangle T' is 135 square units.

(a) Find the matrix $\mathbf{B}^{-1}$

[2 marks]

(b) Find the coordinates of the vertices of the triangle T in terms of c where necessary.

[3 marks]

(c) Find the value of c

[3 marks]

9.5 Answers to the 9.3 Spot Check

$$\mathbf{A}^{-1} = \frac{1}{3 \times 2 - 1 \times 5} \begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix}$$

$$\mathbf{B}^{-1} = \frac{1}{4 \times 16 - 6 \times 10} \begin{pmatrix} 16 & -6 \\ -10 & 4 \end{pmatrix}$$

$$= \frac{1}{4} \begin{pmatrix} 16 & -6 \\ -10 & 4 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 8 & -3 \\ -5 & 2 \end{pmatrix}$$

$$\mathbf{C}^{-1} = \frac{3}{6 \times 4 - (-3) \times (-7)} \begin{pmatrix} 4 & 3 \\ 7 & 6 \end{pmatrix}$$

$$= \frac{3}{3} \begin{pmatrix} 4 & 3 \\ 7 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 3 \\ 7 & 6 \end{pmatrix}$$

9.6 Answers

Answer 1

(a) For the matrix to be singular,

$$\begin{aligned} det\,\mathbf{X} &= 0 \\ det\begin{pmatrix} 1 & a \\ 3 & 2 \end{pmatrix} &= 0 \\ 1 \times 2 - a \times 3 &= 0 \\ 3a &= 2 \\ a &= \frac{2}{3} \end{aligned}$$

[2 marks]

(b)

$$\begin{aligned} \mathbf{Y}^{-1} &= \begin{pmatrix} 1 & -1 \\ 3 & 2 \end{pmatrix}^{-1} \\ &= \frac{1}{1 \times 2 - (-1) \times 3}\begin{pmatrix} 2 & 1 \\ -3 & 1 \end{pmatrix} \\ &= \frac{1}{5}\begin{pmatrix} 2 & 1 \\ -3 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0.4 & 0.2 \\ -0.6 & 0.2 \end{pmatrix} \end{aligned}$$

[2 marks]

(c)

$$\begin{aligned} \mathbf{Y}A &= B \\ \mathbf{Y}^{-1}\mathbf{Y}A &= \mathbf{Y}^{-1}B \qquad \text{pre-multiply both sides by } \mathbf{Y}^{-1} \\ A &= \mathbf{Y}^{-1}B \\ A &= \frac{1}{5}\begin{pmatrix} 2 & 1 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 1 - \lambda \\ 7\lambda - 2 \end{pmatrix} \\ &= \frac{1}{5}\begin{pmatrix} 2 - 2\lambda + 7\lambda - 2 \\ -3 + 3\lambda + 7\lambda - 2 \end{pmatrix} \\ &= \frac{1}{5}\begin{pmatrix} 5\lambda \\ -5 + 10\lambda \end{pmatrix} \\ &= \begin{pmatrix} \lambda \\ 2\lambda - 1 \end{pmatrix} \end{aligned}$$

So A has coordinates $(\lambda,\ 2\lambda - 1)$

[4 marks]

Answer 2

(a)

$$\begin{aligned} det\,\mathbf{A} &= det\begin{pmatrix} 2 & -2 \\ -1 & 3 \end{pmatrix} \\ &= 2 \times 3 - (-2) \times (-1) \\ &= 4 \end{aligned}$$

[1 mark]

(b)

$$\begin{aligned} \mathbf{A}^{-1} &= \begin{pmatrix} 2 & -2 \\ -1 & 3 \end{pmatrix}^{-1} \\ &= \frac{1}{4}\begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 0.75 & 0.5 \\ 0.25 & 0.5 \end{pmatrix} \end{aligned}$$

[2 marks]

(c) Area of R is 18 square units

[2 marks]

(d)

$$\begin{aligned} \mathbf{A}\,R &= S \\ \mathbf{A}^{-1}\mathbf{A}\,R &= \mathbf{A}^{-1}S \qquad \text{pre-multiply both sides by } \mathbf{A}^{-1} \\ R &= \frac{1}{4}\begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 0 & 8 & 12 \\ 4 & 16 & 4 \end{pmatrix} \\ &= \frac{1}{4}\begin{pmatrix} 8 & 56 & 44 \\ 8 & 40 & 20 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 14 & 11 \\ 2 & 10 & 5 \end{pmatrix} \end{aligned}$$

That is, the vertices of R are $(2,\ 2)$, $(14,\ 10)$ and $(11,\ 5)$

[4 marks]

Answer 3

(a)

$$det\,\mathbf{A} \;=\; det\begin{pmatrix} a & -5 \\ 2 & a+4 \end{pmatrix}$$

$$= \; a \times (\, a + 4 \,) - (-5) \times 2$$

$$= \; a^2 + 4a + 10$$

[2 marks]

(b)

$$det\,\mathbf{A} \;=\; \left[\, a^2 + 4a \,\right] + 10$$

$$= \; \left[\, (\, a + 2\,)^2 - 4 \,\right] + 10$$

$$= \; (\, a + 2\,)^2 + 6$$

As the square is $\geqslant 0$, and then 6 is added on, $det\,\mathbf{A} \; > \; 0$ for all real $\mathbf{A}$

$\therefore$ $\mathbf{A}$ is non-singular for all values of a ☐

[3 marks]

(c)

$$\mathbf{A}^{-1} \;=\; \begin{pmatrix} a & -5 \\ 2 & a+4 \end{pmatrix}$$

$$= \; \frac{1}{a^2 + 4a + 10}\begin{pmatrix} a+4 & 5 \\ -2 & a \end{pmatrix}$$

Told that $a \;=\; 0$,

$$\therefore \; \mathbf{A}^{-1} \;=\; \frac{1}{10}\begin{pmatrix} 4 & 5 \\ -2 & 0 \end{pmatrix}$$

[3 marks]

Answer 4

(i) As **A** is a singular matrix,

$$det\,\mathbf{A} \;=\; 0$$

$$det\begin{pmatrix} 5k & 3k-1 \\ -3 & k+1 \end{pmatrix} \;=\; 0$$

$$(5k) \times (\, k + 1 \,) - (3k - 1) \times (-3) \;=\; 0$$

$$5\,k^2 + 5k + 9k - 3 \;=\; 0$$

$$5\,k^2 + 14k - 3 \;=\; 0$$

$$(\, 5k - 1 \,)\;(\, k + 3 \,) \;=\; 0$$

Either $5k - 1 \;=\; 0$ or $k + 3 \;=\; 0$

$$k \;=\; \frac{1}{5} \quad \text{or} \quad k \;=\; -3$$

[4 marks]

(ii) **(a)**

$$\mathbf{B}^{-1} = \begin{pmatrix} 10 & 5 \\ -3 & 3 \end{pmatrix}^{-1}$$

$$= \frac{1}{10 \times 3 - 5 \times (-3)} \begin{pmatrix} 3 & -5 \\ 3 & 10 \end{pmatrix}$$

$$= \frac{1}{45} \begin{pmatrix} 3 & -5 \\ 3 & 10 \end{pmatrix}$$

[2 marks]

(b)

$$\mathbf{B}\,T = T'$$

$$\mathbf{B}^{-1}\,\mathbf{B}\,T = \mathbf{B}^{-1}\,T' \qquad \text{pre-multiply both sides by } \mathbf{A}^{-1}$$

$$T = \mathbf{B}^{-1}\,T'$$

$$T = \frac{1}{45} \begin{pmatrix} 3 & -5 \\ 3 & 10 \end{pmatrix} \begin{pmatrix} 0 & -20 & 10c \\ 0 & 6 & 6c \end{pmatrix}$$

$$= \frac{1}{45} \begin{pmatrix} 0 & -90 & 0 \\ 0 & 0 & 90c \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 2c \end{pmatrix}$$

That is, the vertices of T are $(0,\ 0)$, $(-2,\ 0)$ and $(0,\ 2c)$

[3 marks]

(c)

T has an area of $135 \div 45 = 3$

$$\text{Area } \Delta = \frac{1}{2} \times b \times h$$

$$3 = \frac{1}{2} \times 2 \times 2c$$

$$c = 1.5$$

[3 marks]

Lesson 10

10.1 Pre or Post Multiply ?

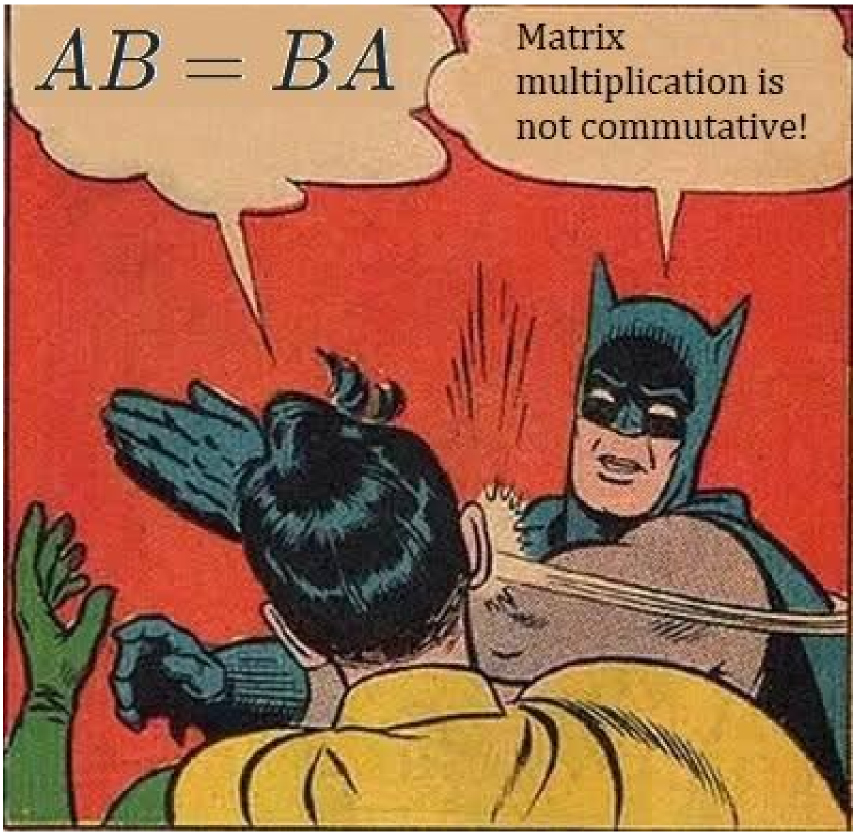
The non-commutative nature of matrix algebra demands care when multiplying both sides of a matrix equation. Suppose that **A**, **B**, and **C** are known matrices, with **X** being the unknown matrix to be found in terms of **A**, **B** and **C** and that,

$$\mathbf{AXB} = \mathbf{C}$$

There is no concept of "Matrix Division", so any idea of dividing both sides by **AB** is not going to end well. However, use can be made of the inverse matrices of the known matrices and the "Do Nothing" Identity matrix **I.**

$$\begin{aligned} \mathbf{AXB} &= \mathbf{C} & & \\ \mathbf{A}^{-1}\mathbf{AXB} &= \mathbf{A}^{-1}\mathbf{C} & & \text{Pre-multiply both sides by } \mathbf{A}^{-1} \\ \mathbf{XB} &= \mathbf{A}^{-1}\mathbf{C} & & \text{As } \mathbf{A}^{-1}\mathbf{A} = \mathbf{I} \\ \mathbf{XB}\,\mathbf{B}^{-1} &= \mathbf{A}^{-1}\mathbf{C}\,\mathbf{B}^{-1} & & \text{Post-multiply both sides by } \mathbf{B}^{-1} \\ \mathbf{X} &= \mathbf{A}^{-1}\mathbf{C}\,\mathbf{B}^{-1} & & \text{As } \mathbf{B}\,\mathbf{B}^{-1} = \mathbf{I} \end{aligned}$$

Sometimes "Pre-multiply" is called "Left-multiply" or "Front-multiply"
Similarly "Post-multiply" is also called "Right-multiply" or "Back-multiply"



10.2 A Pre-Multiply, Post-Multiply Example

(i) Given that **ABC** = **I**, prove that $\mathbf{B}^{-1} = \mathbf{CA}$

(ii) Given that $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & -6 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 2 & 1 \\ -3 & -1 \end{pmatrix}$, find **B**

Teaching Video : http://www.NumberWonder.co.uk/v9090/10a.mp4 (Part 1)
http://www.NumberWonder.co.uk/v9090/10b.mp4 (Part 2)

<= Part 1 Part 2 =>

[4, 4 marks]

10.3 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable

Marks Available : 40

Question 1

Further A-Level Examination Question from May 2017, IAL, FP1, Q2 (Edexcel)

$$\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 4 & 3 \end{pmatrix}, \quad \mathbf{P} = \begin{pmatrix} 3 & 6 \\ 11 & -8 \end{pmatrix}$$

(a) Find $\mathbf{A}^{-1}$

[2 marks]

The transformation represented by the matrix **B** followed by the transformation represented by the matrix **A** is equivalent to the transformation represented by the matrix **P**.

(b) Find **B**, giving your answer in its simplest form.

[4 marks]

Question 2

Further A-Level Examination Question from January 2012, IAL, FP1, Q8 (Edexcel)

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}$$

(a) Show that **A** is non-singular

[2 marks]

(b) Find **B** such that $\mathbf{B}\,\mathbf{A}^2 = \mathbf{A}$

[4 marks]

Question 3

(i) Given that **AXBA** = **I**, prove that $\mathbf{X}^{-1} = \mathbf{BAA}$

[4 marks]

(ii) Given that $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2 & 1 \\ -3 & -1 \end{pmatrix}$, find **X**

[4 marks]

Question 4

$$\mathbf{S} = \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix} \text{ and } \mathbf{B}^{-1} = \begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix}$$

(a) Work out,

(i) $\mathbf{S}^{-1}$ **(ii)** $\mathbf{B}$

(iii) $\mathbf{BS}$ **(iv)** $(\mathbf{BS})^{-1}$

[8 marks]

(b) Verify the quotable result that if **B** and **S** are non-singular matrices then the LHS of $\mathbf{S}^{-1}\mathbf{B}^{-1} = (\mathbf{BS})^{-1}$ is equal to its RHS.

[3 marks]

(c) If **S** represents "putting on socks" and **B** represents "putting on boots" interpret what the result $(\mathbf{BS})^{-1} = \mathbf{S}^{-1}\mathbf{B}^{-1}$ represents.

[2 marks]

Question 5

Further A-Level Examination Question from May 2018, IAL, F1, Q4 (Edexcel)

$$\mathbf{A} = \begin{pmatrix} 2p & 3q \\ 3p & 5q \end{pmatrix} \qquad \text{where } p \text{ and } q \text{ are non-zero real constants}$$

(a) Find $\mathbf{A}^{-1}$ in terms of p and q

[3 marks]

Given $\mathbf{XA} = \mathbf{B}$, where $\mathbf{B} = \begin{pmatrix} p & q \\ 6p & 11q \\ 5p & 8q \end{pmatrix}$

(b) find the matrix **X**, giving your answer in its simplest form.

[4 marks]

10.4 Answers

10.4.1 Solution to 10.2 Teaching Video (A Pre-Multiply, Post-Multiply Example)

(i)

$$\begin{aligned}
\mathbf{ABC} &= \mathbf{I} \\
\mathbf{A}^{-1}\mathbf{ABC} &= \mathbf{A}^{-1}\mathbf{I} && \text{Pre-multiply both sides by } \mathbf{A}^{-1} \\
\mathbf{BC} &= \mathbf{A}^{-1} && \text{As } \mathbf{A}^{-1}\mathbf{A} = \mathbf{I} \\
\mathbf{B}^{-1}\mathbf{BC} &= \mathbf{B}^{-1}\mathbf{A}^{-1} && \text{Pre-multiply both sides by } \mathbf{B}^{-1} \\
\mathbf{C} &= \mathbf{B}^{-1}\mathbf{A}^{-1} && \text{As } \mathbf{B}^{-1}\mathbf{B} = \mathbf{I} \\
\mathbf{CA} &= \mathbf{B}^{-1}\mathbf{A}^{-1}\mathbf{A} && \text{Post-multipiy both sides by } \mathbf{A} \\
\mathbf{CA} &= \mathbf{B}^{-1} && \text{As } \mathbf{A}^{-1}\mathbf{A} = \mathbf{I} \\
\therefore\ \mathbf{B}^{-1} &= \mathbf{CA} \quad \square
\end{aligned}$$

[4 marks]

(ii)

$$\begin{aligned}
\mathbf{B}^{-1} &= \mathbf{CA} \\
&= \begin{pmatrix} 2 & 1 \\ -3 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & -6 \end{pmatrix} \\
&= \begin{pmatrix} -1 & -4 \\ 1 & 3 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\therefore\ \mathbf{B} &= \begin{pmatrix} -1 & -4 \\ 1 & 3 \end{pmatrix}^{-1} \\
&= \frac{1}{(-1)\times(3) - (-4)\times 1}\begin{pmatrix} 3 & 4 \\ -1 & -1 \end{pmatrix} \\
&= \frac{1}{(-3) + 4}\begin{pmatrix} 3 & 4 \\ -1 & -1 \end{pmatrix} \\
&= \begin{pmatrix} 3 & 4 \\ -1 & -1 \end{pmatrix}
\end{aligned}$$

[4 marks]

10.4.2 Solution to 10.3 Exercise

Answer 1

(a)

$$\mathbf{A}^{-1} = \begin{pmatrix} 2 & -1 \\ 4 & 3 \end{pmatrix}^{-1}$$

$$= \frac{1}{2 \times 3 - (-1) \times 4} \begin{pmatrix} 3 & 1 \\ -4 & 2 \end{pmatrix}$$

$$= \frac{1}{10} \begin{pmatrix} 3 & 1 \\ -4 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 0.3 & 0.1 \\ -0.4 & 0.2 \end{pmatrix}$$

[2 marks]

(b) Told that,

$$\mathbf{P} = \mathbf{AB}$$

$$\mathbf{A}^{-1}\mathbf{P} = \mathbf{A}^{-1}\mathbf{AB} \qquad \text{Pre-multiply both sides by } \mathbf{A}^{-1}$$

$$\mathbf{A}^{-1}\mathbf{P} = \mathbf{B} \qquad \text{As } \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

$$\therefore \mathbf{B} = \mathbf{A}^{-1}\mathbf{P}$$

$$= \frac{1}{10} \begin{pmatrix} 3 & 1 \\ -4 & 2 \end{pmatrix} \begin{pmatrix} 3 & 6 \\ 11 & -8 \end{pmatrix}$$

$$= \frac{1}{10} \begin{pmatrix} 20 & 10 \\ 10 & -40 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 \\ 1 & -4 \end{pmatrix}$$

[4 marks]

Answer 2

(a) To show that **A** is non-singular, show that $det\ \mathbf{A} \neq 0$

$$\begin{aligned} det\ \mathbf{A} &= det \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} \\ &= 0 \times 3 - 1 \times 2 \\ &= -2 \\ &\neq 0 \end{aligned}$$

$\therefore$ **A** is non-singular

[2 marks]

(b)

$$\begin{aligned} \mathbf{B}\,\mathbf{A}^2 &= \mathbf{A} \\ \mathbf{BAA} &= \mathbf{A} \\ \mathbf{BAA}\,\mathbf{A}^{-1} &= \mathbf{A}\,\mathbf{A}^{-1} && \text{Post-multiply both sides by } \mathbf{A}^{-1} \\ \mathbf{BA} &= \mathbf{I} && \text{As } \mathbf{A}\,\mathbf{A}^{-1} = \mathbf{I} \\ \mathbf{BA}\,\mathbf{A}^{-1} &= \mathbf{I}\,\mathbf{A}^{-1} && \text{Post-multiply both sides by } \mathbf{A}^{-1} \\ \mathbf{B} &= \mathbf{A}^{-1} && \text{As } \mathbf{A}\,\mathbf{A}^{-1} = \mathbf{I} \\ &= \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}^{-1} \\ &= \frac{1}{(-2)} \begin{pmatrix} 3 & -1 \\ -2 & 0 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} -3 & 1 \\ 2 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -1.5 & 0.5 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

[4 marks]

Answer 3
(i)

$$
\begin{aligned}
\mathbf{AXBA} &= \mathbf{I} && \\
\mathbf{A}^{-1}\,\mathbf{AXBA} &= \mathbf{A}^{-1}\,\mathbf{I} && \text{Pre-multiply both sides by } \mathbf{A}^{-1} \\
\mathbf{XBA} &= \mathbf{A}^{-1} && \text{As } \mathbf{A}^{-1}\mathbf{A} = \mathbf{I} \\
\mathbf{X}^{-1}\,\mathbf{XBA} &= \mathbf{X}^{-1}\,\mathbf{A}^{-1} && \text{Pre-multiply both sides by } \mathbf{X}^{-1} \\
\mathbf{BA} &= \mathbf{X}^{-1}\,\mathbf{A}^{-1} && \text{As } \mathbf{X}^{-1}\mathbf{X} = \mathbf{I} \\
\mathbf{BAA} &= \mathbf{X}^{-1}\,\mathbf{A}^{-1}\,\mathbf{A} && \text{Post-multiply both sides by } \mathbf{A} \\
\mathbf{BAA} &= \mathbf{X}^{-1} && \text{As } \mathbf{A}^{-1}\mathbf{A} = \mathbf{I} \\
\therefore\ \mathbf{X}^{-1} &= \mathbf{BAA} \quad \square &&
\end{aligned}
$$

[4 marks]

(ii)

$$
\begin{aligned}
\mathbf{X}^{-1} &= \begin{pmatrix} 2 & 1 \\ -3 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 2 & 1 \\ -3 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 2 & 5 \\ -3 & -7 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{X} &= \begin{pmatrix} 2 & 5 \\ -3 & -7 \end{pmatrix}^{-1} \\
&= \frac{1}{2 \times (-7) - 5 \times (-3)} \begin{pmatrix} -7 & -5 \\ 3 & 2 \end{pmatrix} \\
&= \frac{1}{1} \begin{pmatrix} -7 & -5 \\ 3 & 2 \end{pmatrix} \\
&= \begin{pmatrix} -7 & -5 \\ 3 & 2 \end{pmatrix}
\end{aligned}
$$

[4 marks]

Answer 4

(a) **(i)**

$$\mathbf{S}^{-1} = \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix}^{-1}$$

$$= \frac{1}{2} \begin{pmatrix} 3 & -4 \\ -1 & 2 \end{pmatrix}$$

(ii)

$$\mathbf{B} = \begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix}^{-1}$$

$$= \frac{1}{2} \begin{pmatrix} 1 & 0 \\ -3 & 2 \end{pmatrix}$$

(iii)

$$\mathbf{BS} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 2 & 4 \\ -4 & -6 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 \\ -2 & -3 \end{pmatrix}$$

(iv)

$$(\mathbf{BS})^{-1} = \begin{pmatrix} 1 & 2 \\ -2 & -3 \end{pmatrix}^{-1}$$

$$= \frac{1}{1} \begin{pmatrix} -3 & -2 \\ 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -3 & -2 \\ 2 & 1 \end{pmatrix}$$

[2 marks each = 8 marks in Total]

(b)

$$\text{LHS} = \mathbf{S}^{-1}\mathbf{B}^{-1}$$

$$= \frac{1}{2} \begin{pmatrix} 3 & -4 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} -6 & -4 \\ 4 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} -3 & -2 \\ 2 & 1 \end{pmatrix}$$

$$= (\mathbf{BS})^{-1}$$

$$= \text{RHS} \qquad \square$$

[3 marks]

(c) The opposite of putting on socks then putting on boots is first taking off boots then taking off socks.

[2 marks]

Answer 5

(a)

$$\mathbf{A}^{-1} = \begin{pmatrix} 2p & 3q \\ 3p & 5q \end{pmatrix}^{-1}$$

$$= \frac{1}{2p \times 5q - 3p \times 3q} \begin{pmatrix} 5q & -3q \\ -3p & 2q \end{pmatrix}$$

$$= \frac{1}{10pq - 9pq} \begin{pmatrix} 5q & -3q \\ -3p & 2q \end{pmatrix}$$

$$= \frac{1}{pq} \begin{pmatrix} 5q & -3q \\ -3p & 2q \end{pmatrix}$$

[3 marks]

(b)

$$\mathbf{XA} = \mathbf{B}$$

$$\mathbf{XA}\,\mathbf{A}^{-1} = \mathbf{B}\,\mathbf{A}^{-1} \qquad \text{Post-multiple both sides by } \mathbf{A}^{-1}$$

$$\mathbf{X} = \mathbf{BA}^{-1} \qquad \text{As } \mathbf{AA}^{-1} = \mathbf{I}$$

$$= \begin{pmatrix} p & q \\ 6p & 11q \\ 5p & 8q \end{pmatrix} \frac{1}{pq} \begin{pmatrix} 5q & -3q \\ -3p & 2q \end{pmatrix}$$

$$= \frac{1}{pq} \begin{pmatrix} 2pq & -pq \\ -3pq & 4pq \\ pq & pq \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -1 \\ -3 & 4 \\ 1 & 1 \end{pmatrix}$$

[4 marks]

Lesson 11

Further A-Level Pure Mathematics : Core 1
Matrix Transformations

11.1 Test

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 40

Question 1

Further A-Level Examination Question from June 2013, IAL, FP1, Q1 (Edexcel)

$$\mathbf{M} = \begin{pmatrix} x & x - 2 \\ 3x - 6 & 4x - 11 \end{pmatrix}$$

Given that the matrix **M** is singular, find the possible values of x

[4 marks]

Question 2

Further A-Level Examination Question from June 2009, IAL, FP1, Q5 (Edexcel)

$$\mathbf{R} = \begin{pmatrix} a & 2 \\ a & b \end{pmatrix} \text{ where } a \text{ and } b \text{ are constants and } a > 0$$

(a) Find $\mathbf{R}^2$ in terms of a and b

[3 marks]

Given that $\mathbf{R}^2$ represents an enlargement with centre (0, 0) and scale factor 15

(b) find the value of a and the value of b

[5 marks]

Question 3

Further A-Level Examination Question from June 2014(R), IAL, FP1, Q6 (Edexcel)

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$$

Given that $\mathbf{M} = (\mathbf{A} + \mathbf{B})(2\mathbf{A} - \mathbf{B})$,

(a) calculate the matrix **M**

[6 marks]

(b) find the matrix **C** such that **MC** = **A**

[4 marks]

Question 4

Further A-Level Examination Question from May 2017, IAL, FP1, Q5 (Edexcel)

(i) $\mathbf{A} = \begin{pmatrix} p & 2 \\ 3 & p \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} -5 & 4 \\ 6 & -5 \end{pmatrix}$ where p is a constant

(a) Find, in terms of p, the matrix **AB**

[2 marks]

Given that $\mathbf{AB} + 2\mathbf{A} = k\mathbf{I}$ where k is a constant and **I** is the 2×2 identity matrix

(b) find the value of p and the value of k

[4 marks]

(ii) $\mathbf{M} = \begin{pmatrix} a & -9 \\ 1 & 2 \end{pmatrix}$ where a is a real constant

Triangle T has an area of 15 square units

Triangle T is transformed to the triangle T' by the transformation represented by the matrix **M**.

Given that the area of triangle T' is 270 square units, find the possible values of a

[5 marks]

Question 5

Further A-Level Examination Question from June 2014, IAL, FP1, Q4 (Edexcel)

(i) Given that $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & -1 \\ 4 & 5 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2 & -1 & 4 \\ 1 & 3 & 1 \end{pmatrix}$,

(a) find **AB**

[3 marks]

(b) Explain why $\mathbf{AB} \neq \mathbf{BA}$

[1 mark]

(ii) Given that $\mathbf{C} = \begin{pmatrix} 2k & -2 \\ 3 & k \end{pmatrix}$ find $\mathbf{C}^{-1}$, giving your answer in terms of k

[3 marks]

11.2 Answers

Answer 1

As matrix **M** is singular,

$$\det \mathbf{M} = 0$$

$$\det \begin{pmatrix} x & x-2 \\ 3x-6 & 4x-11 \end{pmatrix} = 0$$

$$x \times (4x - 11) - [(x-2)(3x-6)] = 0$$

$$4x^2 - 11x - [3x^2 - 12x + 12] = 0$$

$$4x^2 - 11x - 3x^2 + 12x - 12 = 0$$

$$x^2 + x - 12 = 0$$

$$(x+4)(x-3) = 0$$

Either $x + 4 = 0$ or $x - 3 = 0$

$x = -4$ or $x = 3$

[4 marks]

Answer 2

(a) $\mathbf{R}^2 = \begin{pmatrix} a & 2 \\ a & b \end{pmatrix} \begin{pmatrix} a & 2 \\ a & b \end{pmatrix} = \begin{pmatrix} a^2 + 2a & 2a + 2b \\ a^2 + ab & 2a + b^2 \end{pmatrix}$

[3 marks]

(b) The question is telling us that,

$$\begin{pmatrix} a^2 + 2a & 2a + 2b \\ a^2 + ab & 2a + b^2 \end{pmatrix} = \begin{pmatrix} 15 & 0 \\ 0 & 15 \end{pmatrix}$$

Upper right : $2a + 2b = 0 \Rightarrow a = -b$

Upper left : $a^2 + 2a = 15$

$$a^2 + 2a - 15 = 0$$

$(a+5)(a-3) = 0 \Rightarrow a = 3$ as told $a > 0$

$$\therefore b = -3$$

[5 marks]

Answer 3

(a)

$$\mathbf{M} = (\mathbf{A} + \mathbf{B})(2\mathbf{A} - \mathbf{B})$$

$$= \left(\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \right) \left(2\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \right)$$

$$= \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 5 & 1 \\ -2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -1 \\ -7 & -2 \end{pmatrix}$$

[6 marks]

(b)

$$\mathbf{MC} = \mathbf{A}$$

$$\mathbf{M}^{-1}\mathbf{MC} = \mathbf{M}^{-1}\mathbf{A} \qquad \text{Pre-multiply both sides by } \mathbf{M}^{-1}$$

$$\mathbf{C} = \mathbf{M}^{-1}\mathbf{A} \qquad \text{As } \mathbf{M}^{-1}\mathbf{M} = \mathbf{I}$$

$$= -\frac{1}{9}\begin{pmatrix} -2 & 1 \\ 7 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$$

$$= -\frac{1}{9}\begin{pmatrix} -5 & -2 \\ 13 & 7 \end{pmatrix}$$

$$= \frac{1}{9}\begin{pmatrix} 5 & 2 \\ -13 & -7 \end{pmatrix}$$

[4 marks]

Answer 4

(i) **(a)**

$$\mathbf{AB} = \begin{pmatrix} p & 2 \\ 3 & p \end{pmatrix} \begin{pmatrix} -5 & 4 \\ 6 & -5 \end{pmatrix}$$

$$= \begin{pmatrix} -5p + 12 & 4p - 10 \\ -15 + 6p & 12 - 5p \end{pmatrix}$$

[2 marks]

(b)

$$\mathbf{AB} + 2\mathbf{A} = k\mathbf{I}$$

$$\begin{pmatrix} -5p + 12 & 4p - 10 \\ -15 + 6p & 12 - 5p \end{pmatrix} + 2\begin{pmatrix} p & 2 \\ 3 & p \end{pmatrix} = k\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} -3p + 12 & 4p - 6 \\ -9 + 6p & 12 - 3p \end{pmatrix} = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$$

$$\text{Upper Right : } 4p - 6 = 0 \Rightarrow p = \frac{3}{2}$$

$$\text{Lower Left : } 12 - 3p = k \Rightarrow k = \frac{15}{2}$$

[4 marks]

(ii) The question is telling us that,

$$det\,\mathbf{M} = \pm\frac{270}{15}$$

$$det\begin{pmatrix} a & -9 \\ 1 & 2 \end{pmatrix} = \pm 18$$

$$2a + 9 = \pm 18$$

$$2a = \pm 18 - 9$$

$$\text{Either } a = 4.5 \text{ or } a = -13.5$$

[5 marks]

Answer 5

(i) **(a)**

$$\mathbf{AB} = \begin{pmatrix} 1 & 2 \\ 3 & -1 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} 2 & -1 & 4 \\ 1 & 3 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 5 & 6 \\ 5 & -6 & 11 \\ 13 & 11 & 21 \end{pmatrix}$$

[3 marks]

(b)

$$\mathbf{BA} = \begin{pmatrix} 2 & -1 & 4 \\ 1 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & -1 \\ 4 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 15 & 25 \\ 14 & 4 \end{pmatrix}$$

$$\therefore \ \mathbf{AB} \neq \mathbf{BA} \qquad \square$$

[1 mark]

(ii)

$$\mathbf{C} = \begin{pmatrix} 2k & -2 \\ 3 & k \end{pmatrix}$$

$$\mathbf{C}^{-1} = \frac{1}{2k^2 + 6}\begin{pmatrix} k & 2 \\ -3 & 2k \end{pmatrix}$$

[3 marks]