GCSE Mathematics
Preparatory A-Level Mathematics

CONIC SECTIONS

~ Simultaneous Equations IV ~

Conic Sections

~ Simultaneous Equations IV ~

Lesson 1

1.1 Perspectives

A typical question from a GCSE examination on simultaneous equations might ask candidates to solve, for example, this pair of equations,

$$y = 2x - 3$$

$$x^2 + y^2 = 2$$

In the rush to complete such a question under time pressure, most candidates will opt for a purely algebraic method. However, the majority of the mistakes made with such questions comes from not thinking from other perspectives.

Does "Time Pressure = No time to consider other points of view" ?

For the examination, being able to do the algebra is essential but outside of the examination hall a computer (called a symbolic manipulator) could do that for you; it's not the most important skill to learn from doing such questions. What a computer can't do very well is see the problem from multiple perspectives.

Thinking geometrically, the upper of the two equations is a straight line; "obvious" because it is of the form $y = mx + c$.

The lower equation, as any A-Level mathematician will instantly tell you, is a circle. They would tell you it is centred at the origin and has a radius of $\sqrt{2}$, which is about 1.414.

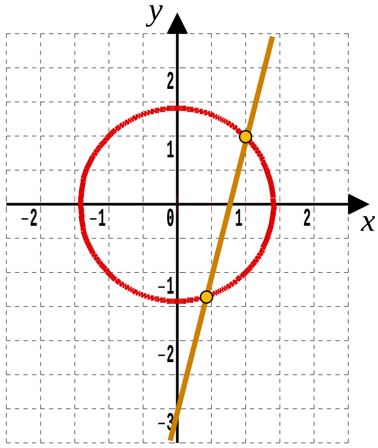
The Equation of a Circle

An algebraic equation of the form,

$$x^2 + y^2 = r^2$$

This is a circle with centre (0, 0) and radius r

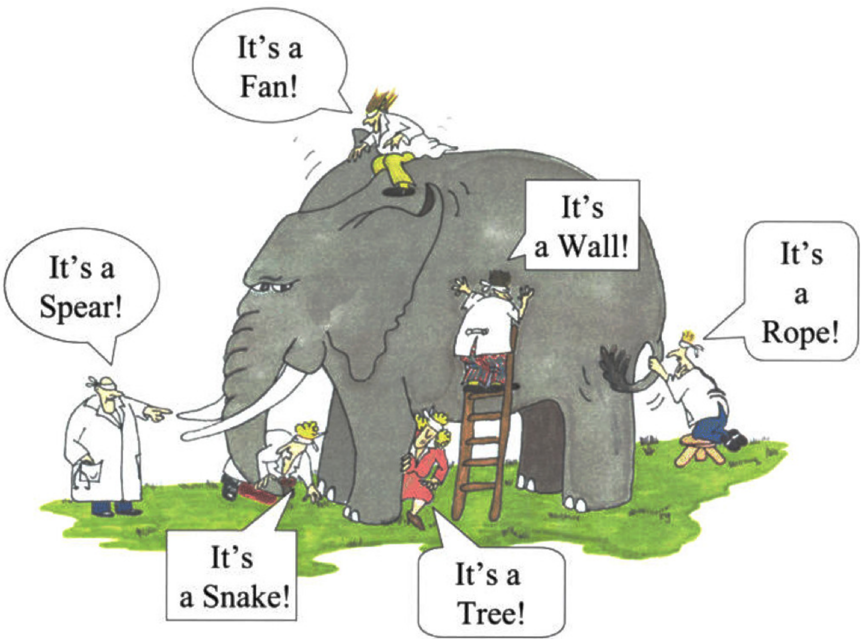
A knowledge of lines and circles allows us to sketch or plot the equations that are to be solved simultaneously; a new perspective on a algebraic problem.



The most obvious feature of the graph is that the line intersects the circle in two places. When solving these simultaneous equations the locations of those intersections are found. So if a candidate gives only one answer and not two it reveals that, not only have they made an algebraic error, they do not have a geometric perspective on the question !

Such intersections are points and a point has two parts, an x and a y. A common mistake is to find one part and forget to find the other. Again, this reveals a lack of a geometric understanding of what the algebra is doing; finding points.

From even a rough sketch, the approximate answers are apparent. Many "obviously wrong" answers go unchecked in examinations because the candidate has not thought about viewing an algebra question from the different perspective of geometry. In our question no x or y in the answer should be greater than $\sqrt{2}$ for example.



Being good at maths is being aware of different perspectives on the same problem

1.2 The Algebraic Solution

To solve the simultaneous equations

$$y = 2x - 3$$
$$x^2 + y^2 = 2$$

- Begin by using *the method of substitution.*

$$x^2 + y^2 = 2$$
$$x^2 + (2x - 3)^2 = 2$$

- Expand the brackets, FOIL

$$x^2 + (2x - 3)(2x - 3) = 2$$
$$x^2 + 4x^2 - 6x - 6x + 9 = 2$$

- Gather together like terms

$$5x^2 - 12x + 9 = 2$$

- Rearrange the equation into the form $f(x) = 0$

$$5x^2 - 12x + 7 = 0$$

- Factorise the quadratic

$$(5x - 7)(x - 1) = 0$$

- Apply The Product of Zero Theorem

$$\text{Either } 5x - 7 = 0 \quad \text{or} \quad x - 1 = 0$$
$$x = 1.4 \qquad x = 1$$

The answer is points where the line intersects the circle so we finally use the equation of the line $y = 2x - 3$ to work out y when x is 1.4, and 1.

The Final Answer

$$(1.4, -0.2) \text{ or } (1, -1)$$

- Don't forget to look back at the graph to check these are sensible answers.

Phew !

1.3 Exercise

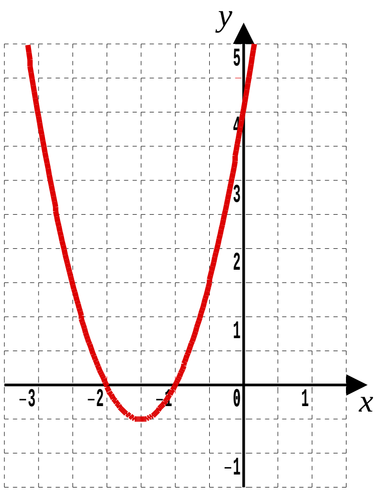
Question 1

(i) Solve the equation

$$2x^2 + 6x + 4 = 0$$

by first dividing throughout by 2, then factorising, and then applying The Product of Zero Theorem. You should end up with two values of x that make the original equation true.

(ii) Here is the graph of the equation, $y = 2x^2 + 6x + 4$



Explain how you can check your part (i) answers are likely to be correct from this graph.

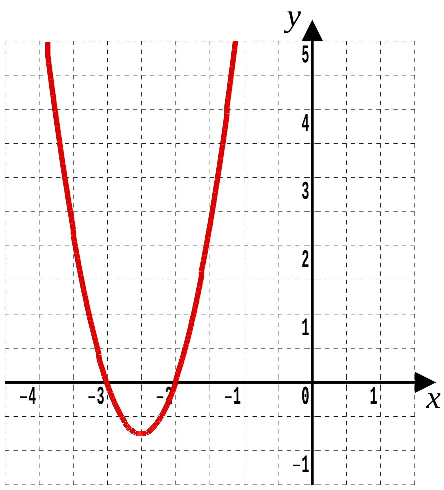
Question 2

(i) Solve the equation

$$3x^2 + 15x + 18 = 0$$

by first dividing throughout by 3, then factorising, and then applying The Product of Zero Theorem. You should end up with two values of x that make the original equation true.

(ii) Here is the graph of the equation, $y = 3x^2 + 15x + 18$



Explain how you can check your part (i) answers are likely to be correct from this graph.

Question 3

(i) Solve the simultaneous equations

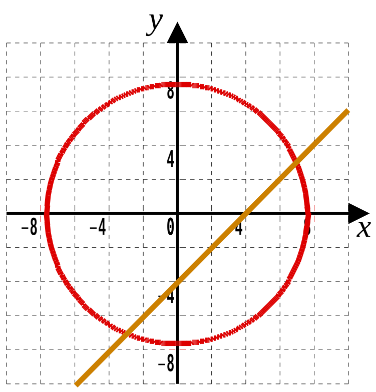
$$y = x - 4$$

$$x^2 + y^2 = 58$$

(ii) Here is the graph of the equations,

$$y = x - 4 \qquad \text{(the line)}$$

$$x^2 + y^2 = 58 \qquad \text{(the circle)}$$



Explain how you can check your part (i) answers are likely to be correct from this graph.

Question 4

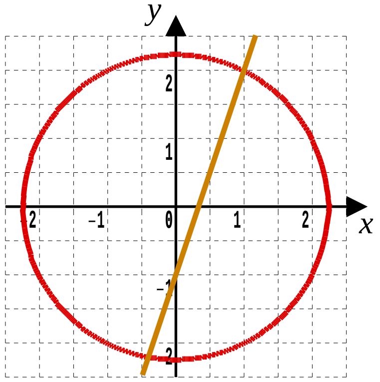
GCSE, June 2007, paper 3H, Q19

(i) Solve the simultaneous equations

$$y = 3x - 1$$
$$x^2 + y^2 = 5$$

(ii) Here is the graph of the equations,

$$y = 3x - 1 \qquad \text{(the line)}$$
$$x^2 + y^2 = 5 \qquad \text{(the circle)}$$



Explain how you can check your part (i) answers are likely to be correct from this graph.

Question 5

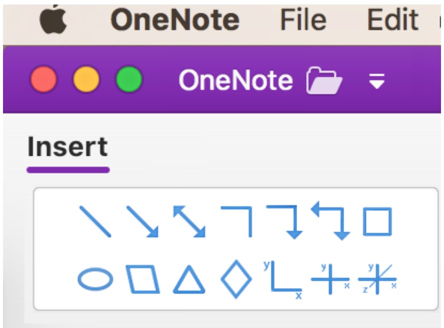
This is about using geometry to solve the simultaneous equations,

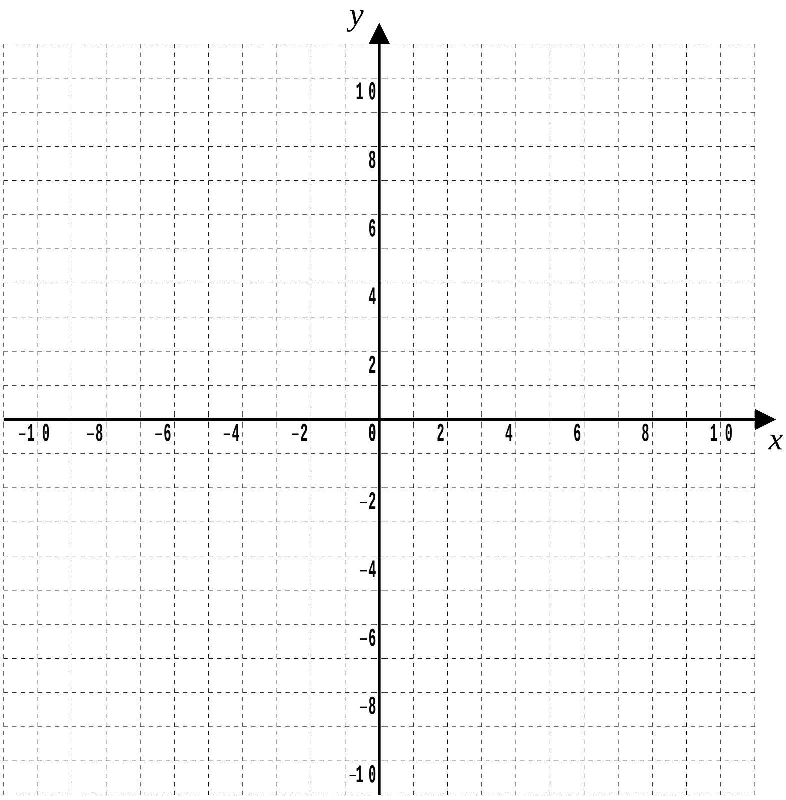
$$y = x - 7$$

$$x^2 + y^2 = 109$$

This is a valid method in this case because the points of intersection are integer and so a reasonably accurate plot will allow them to be read off the graph.

(i) Accurately plot the line and the circle on the graph.
On paper, use a compass to draw the circle.
On a computer, on OneNote for example, use the shapes tool.





(ii) From your graph, write down the solutions to the simultaneous equations.

1.4 Answers

1.4.1 Solutions (1.3 Exercise)

Answer 1

(i)

$$2x^2 + 6x + 4 = 0$$

$$x^2 + 3x + 2 = 0$$

$$(x + 2)(x + 1) = 0$$

Either $x + 2 = 0$ or $x + 1 = 0$

Solutions : $x = -2$ or $x = -1$

(ii) The graph has zero height, that is $y = 0$, when $x = -2$ or when $x = -1$

Answer 2

(i)

$$3x^2 + 15x + 18 = 0$$

$$x^2 + 5x + 6 = 0$$

$$(x + 3)(x + 2) = 0$$

Either $x + 3 = 0$ or $x + 2 = 0$

Solutions : $x = -3$ or $x = -2$

(ii) The graph has zero height, that is $y = 0$, when $x = -3$ or when $x = -2$

Answer 3

(i)

$$x^2 + (x - 4)^2 = 58$$

$$x^2 + x^2 - 8x + 16 = 58$$

$$2x^2 - 8x - 42 = 0$$

after dividing through by 2 $\quad x^2 - 4x - 21 = 0$

$$(x + 3)(x - 7) = 0$$

Either $x + 3 = 0$ or $x - 7 = 0$

Solutions : $x = -3$ or $x = 7$

$\therefore$ Points of intersection are $(-3, -7)$ or $(7, 3)$

(ii) The graph shows an intersection of the line and the curve at points that could be $(-3, -7)$ or $(7, 3)$ so there is nothing “obviously wrong” !

Answer 4

(i)

$$x^2 + (3x - 1)^2 = 5$$

$$x^2 + 9x^2 - 6x + 1 = 5$$

$$10x^2 - 6x - 4 = 0$$

after dividing through by 2 $\quad 5x^2 - 3x - 2 = 0$

$$(5x + 2)(x - 1) = 0$$

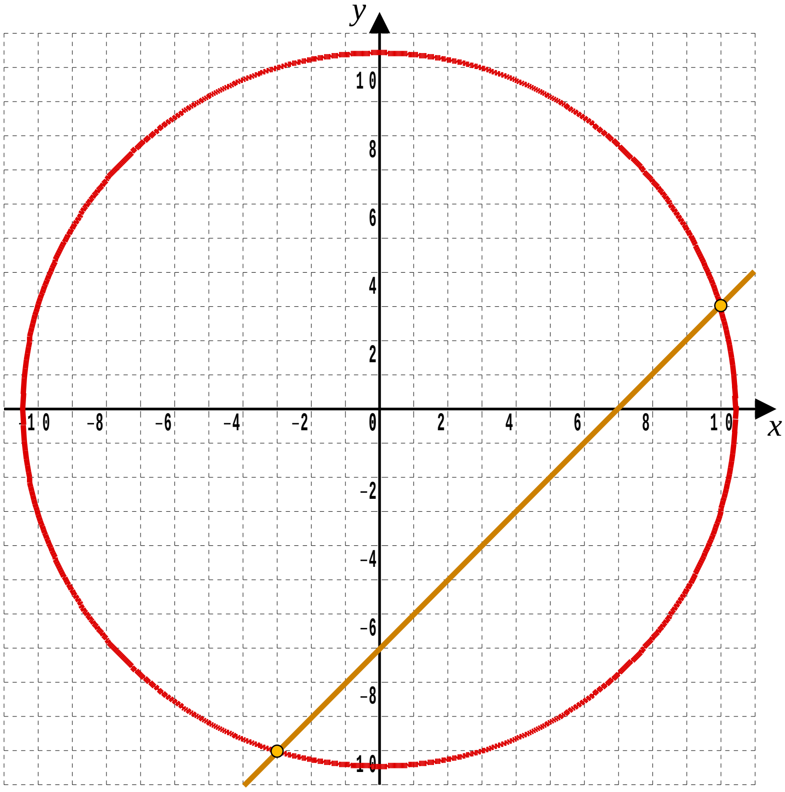
Either $\quad 5x + 2 = 0 \quad$ or $\quad x - 1 = 0$

Solutions : $x = -0.4$ or $x = 1$

$\therefore$ Points of intersection are $(-0.4, -2.2)$ or $(1, 2)$

(ii) The graph shows an intersection of the line and the curve at points that could be $(-0.4, -2.2)$ or $(1, 2)$ so there is nothing "obviously wrong" !

Answer 5



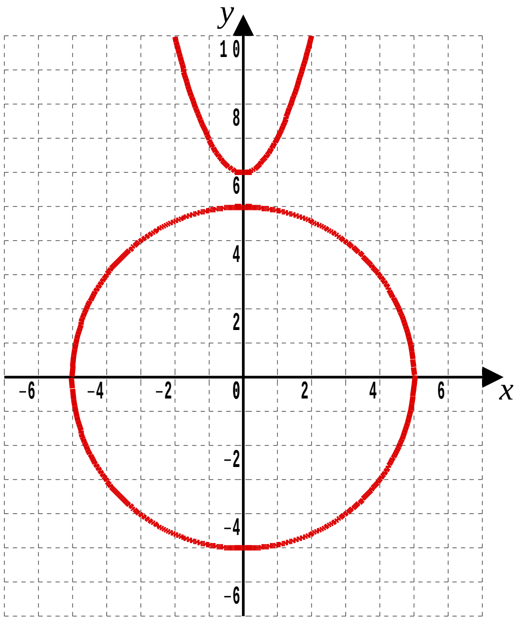
From the graph the solutions are $(-3. -10)$ or $(3, 10)$

2.1 Circle and Parabola

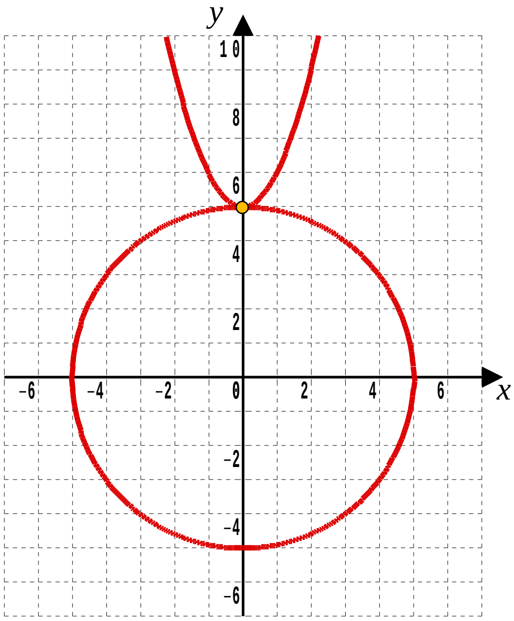
The circle with equation $x^2 + y^2 = 25$ has centre (0, 0) and radius 5 units.

Into this circle is going to be lowered a parabola, essentially $y = x^2$

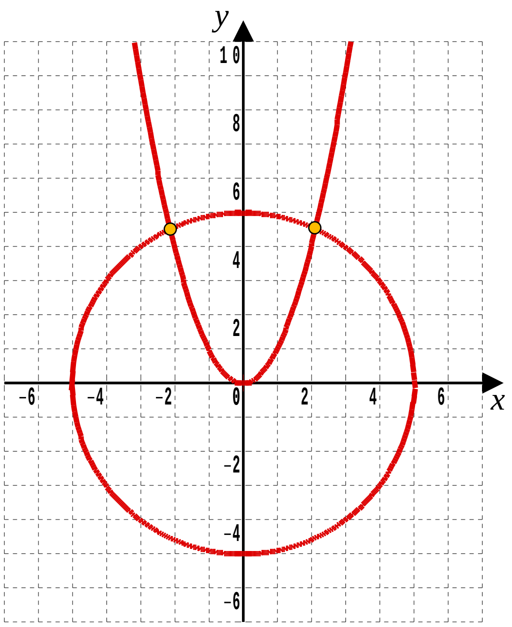
The lowering is achieved by varying the value of c in the equation $y = x^2 + c$



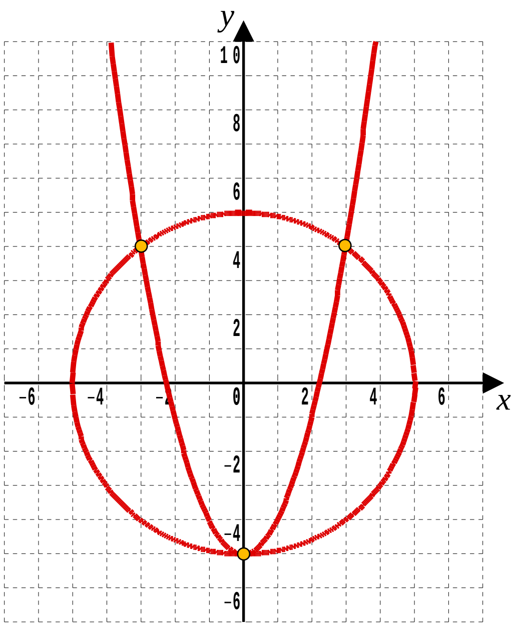
$x^2 + y^2 = 25$ with $y = x^2 + 6$
Zero solutions



$x^2 + y^2 = 25$ with $y = x^2 + 5$
One solution



$x^2 + y^2 = 25$ with $y = x^2$
Two solutions



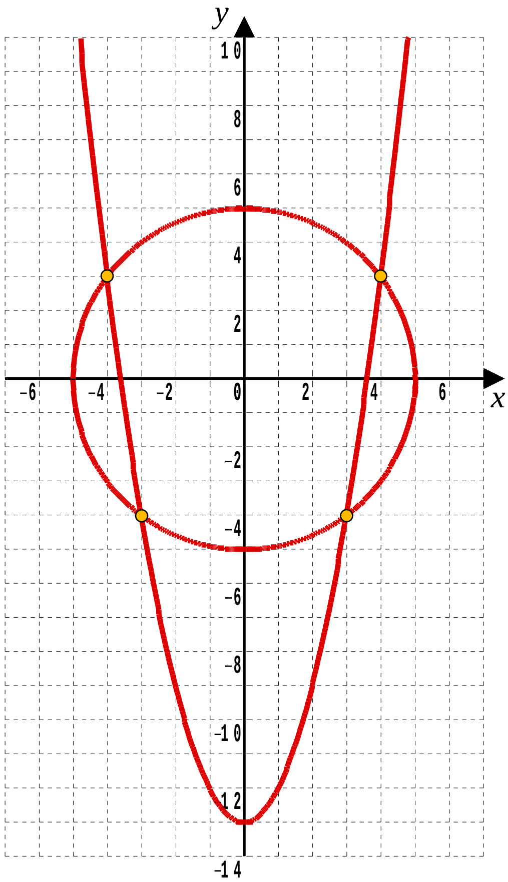
$x^2 + y^2 = 25$ with $y = x^2 - 5$
Three solutions

The fifth and final possibility is that there are four solutions.

$$x^2 + y^2 = 25 \quad \text{with} \quad y = x^2 - 13$$

Geometrically, we can see from a graph that there are four solutions. The challenge now is to use algebra to find all four solutions.

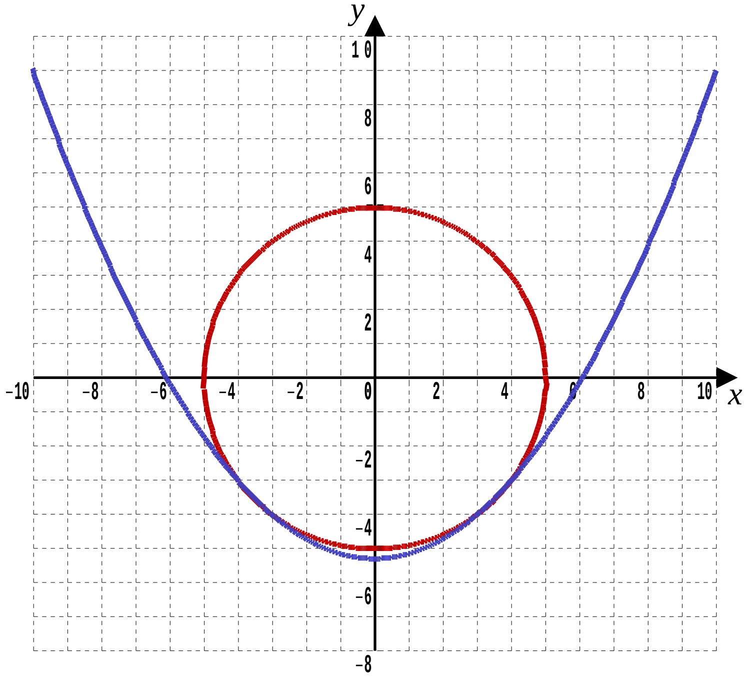
Teaching Videos : http://www.NumberWonder.co.uk/v9091/2.mp4



2.2 Exercise

Question 1

The number of intersection points is often not obvious from a graph !

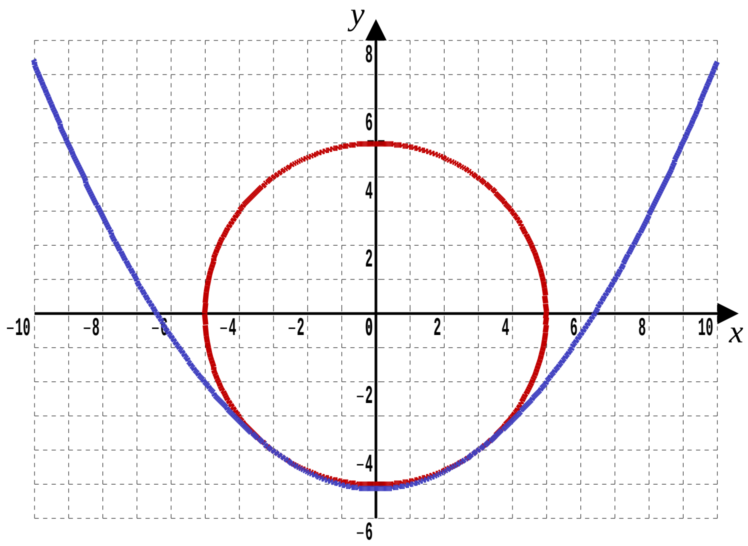


Use algebra to find all points of intersection of the circle $x^2 + y^2 = 25$

with the parabolic curve, $y = \dfrac{x^2 - 37}{7}$

Question 2

Sophia is trying to get more than four points of intersection by running the quadratic curve along an arc of the circle. Here is her first attempt.

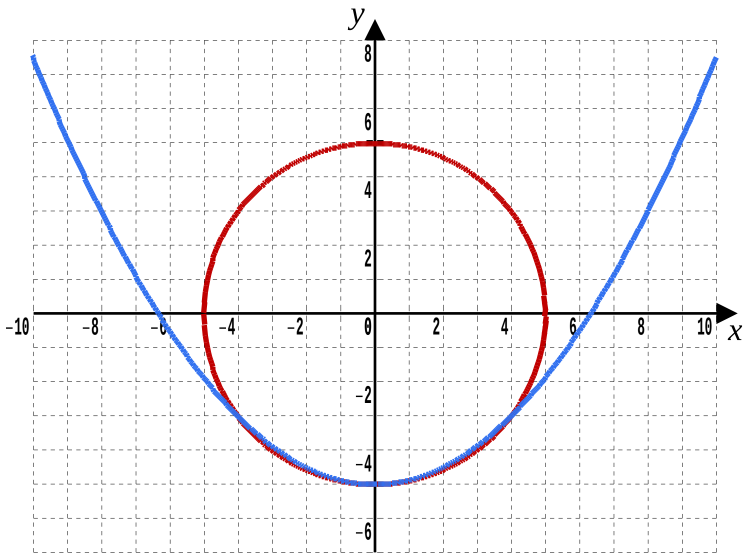


Use algebra to find all points of intersection of the circle $x^2 + y^2 = 25$

with the quadratic curve, $y = \dfrac{x^2 - 41}{8}$

Question 3

Sophia is does not give up easily. Here is her next attempt.

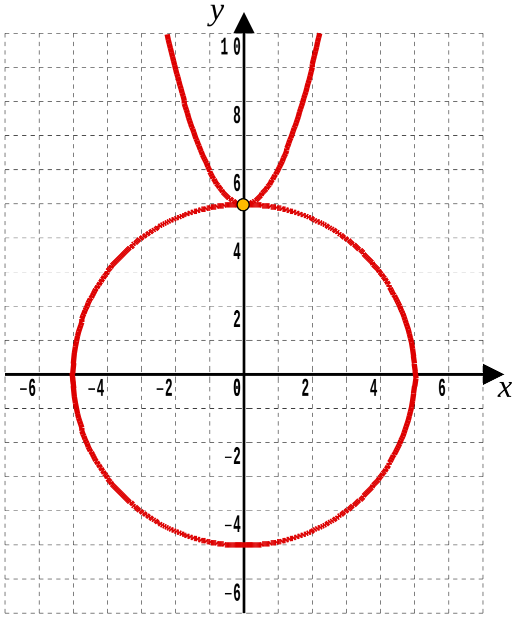


(i) Use algebra to find all points of intersection of the circle $x^2 + y^2 = 25$ with the quadratic curve, $y = \dfrac{x^2 - 40}{8}$

(ii) Do you think Sophia's idea can be made to work ?
Yes or No ?

Question 4

In the introduction it was claimed that the circle $x^2 + y^2 = 25$ would only have one point of intersection with the parabola $y = x^2 + 5$



Verify this claim by using algebra to solve the two equations simultaneously.

2.3 Answers

2.3.1 Solutions to 2.1 Teaching Video Example

Rearrange the equation of the quadratic,

$$y = x^2 - 13$$
$$x^2 = y + 13$$

Then substitute this in place of the x^2 in the equation of the circle,

$$(x^2) + y^2 = 25$$
$$(y + 13) + y^2 = 25$$
$$y^2 - y - 12 = 0$$
$$(y + 4)(y - 3) = 0$$
$$\text{Either} \quad y + 4 = 0 \quad \text{or} \quad y - 3 = 0$$
$$y = -4 \quad \text{or} \quad y = 3$$

Back with the equation for the quadratic,

$$x^2 = y + 13$$

square root both sides

$$x = \pm\sqrt{y + 13}$$
$$\text{So when } y = -4, \quad x = \pm\sqrt{-4 + 13} \Rightarrow x = \pm 3$$
$$\text{And when } y = 3, \quad x = \pm\sqrt{3 + 13} \Rightarrow x = \pm 4$$

The four points of intersection are,

$$(-3, -4), \ (3, -4), (-4, 3) \text{ and } (4, 3)$$

2.3.2 Solutions to 2.2 Exercise

Answer 1

Rearrange the equation of the quadratic,

$$y = \frac{x^2 - 37}{7}$$

$$7y = x^2 - 37$$

$$x^2 = 7y + 37$$

Then substitute this in place of the x^2 in the equation of the circle,

$$\left(x^2\right) + y^2 = 25$$

$$\left(7y + 37\right) + y^2 = 25$$

$$y^2 + 7y + 12 = 0$$

$$(y + 4)(y + 3) = 0$$

$$\text{Either} \quad y + 4 = 0 \quad \text{or} \quad y + 3 = 0$$

$$y = -4 \quad \text{or} \quad y = -3$$

Back with the equation for the quadratic,

$$x^2 = 7y + 37$$

square root both sides

$$x = \pm\sqrt{7y + 37}$$

$$\text{So when } y = -4, \quad x = \pm\sqrt{7 \times (-4) + 37} \Rightarrow x = \pm 3$$

$$\text{And when } y = -3, \quad x = \pm\sqrt{7 \times (-3) + 37} \Rightarrow x = \pm 4$$

The four points of intersection are,

$$(-3, -4), \quad (3, -4), (-4, -3) \text{ and } (4, -3)$$

Answer 2

Rearrange the equation of the quadratic,

$$y = \frac{x^2 - 41}{8}$$

$$8y = x^2 - 41$$

$$x^2 = 8y + 41$$

Then substitute this in place of the x^2 in the equation of the circle,

$$\left(x^2\right) + y^2 = 25$$

$$\left(8y + 41\right) + y^2 = 25$$

$$y^2 + 8y + 16 = 0$$

$$(y + 4)(y + 4) = 0$$

$$\text{Either} \quad y + 4 = 0 \quad \text{or} \quad y + 4 = 0$$

$$y = -4$$

Back with the equation for the quadratic,

$$x^2 = 8y + 41$$

square root both sides

$$x = \pm\sqrt{8y + 41}$$

$$\text{So when } y = -4, \quad x = \pm\sqrt{8 \times (-4) + 41} \Rightarrow x = \pm 3$$

The two points of intersection are,

$$(-3, -4), \quad (3, -4)$$

Answer 3

(i) Rearrange the equation of the quadratic,

$$y = \frac{x^2 - 40}{8}$$

$$8y = x^2 - 40$$

$$x^2 = 8y + 40$$

Then substitute this in place of the x^2 in the equation of the circle,

$$\left(x^2 \right) + y^2 = 25$$

$$\left(8y + 40 \right) + y^2 = 25$$

$$y^2 + 8y + 15 = 0$$

$$(y + 3)(y + 5) = 0$$

$$\text{Either} \quad y + 3 = 0 \quad \text{or} \quad y + 5 = 0$$

$$y = -3 \quad \text{or} \quad y = -5$$

Back with the equation for the quadratic,

$$x^2 = 8y + 40$$

square root both sides

$$x = \pm\sqrt{8y + 40}$$

So when $y = -3$, $x = \pm\sqrt{8 \times (-3) + 40} \Rightarrow x = \pm 4$

And when $y = -5$, $x = \pm\sqrt{8 \times (-5) + 40} \Rightarrow x = 0$

The three points of intersection are,

$$(-4, -3), \quad (4, -3) \quad \text{and} \quad (0, -5)$$

(ii) Sophia's idea cannot be made to work.
The maximum possible number of points of intersection is four.
(Could be proved using The Fundamental Theorem of Algebra)

Answer 4

Rearrange the equation of the quadratic,

$$y = x^2 + 5$$

$$x^2 = y - 5$$

Then substitute this in place of the x^2 in the equation of the circle,

$$\left(x^2 \right) + y^2 = 25$$

$$\left(y - 5 \right) + y^2 = 25$$

$$y^2 + y - 30 = 0$$

$$(y + 6)(y - 5) = 0$$

$$\text{Either} \quad y + 6 = 0 \quad \text{or} \quad y - 5 = 0$$

$$y = -6 \quad \text{or} \quad y = 5$$

Back with the equation for the quadratic,

$$x^2 = y - 5$$

square root both sides

$$x = \pm\sqrt{y - 5}$$

$$\text{So when } y = -6, \quad x = \pm\sqrt{(-6) - 5} \Rightarrow x = \text{No real solutions}$$

$$\text{And when } y = 5, \quad x = \pm\sqrt{5 - 5} \Rightarrow x = 0$$

The only points of intersection is,

$$(0, \ 5)$$