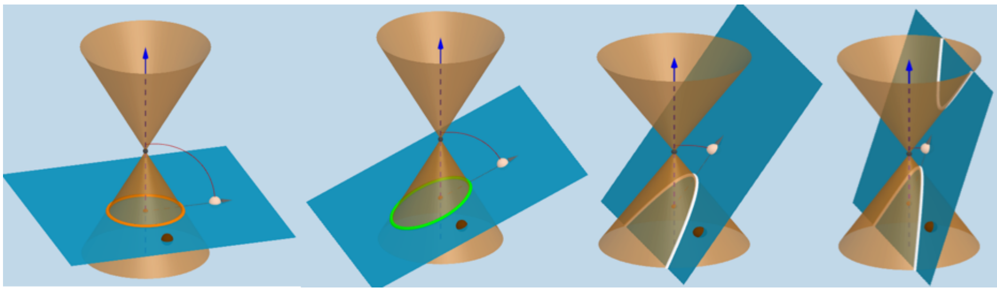
3.1 Circle and Hyperbola

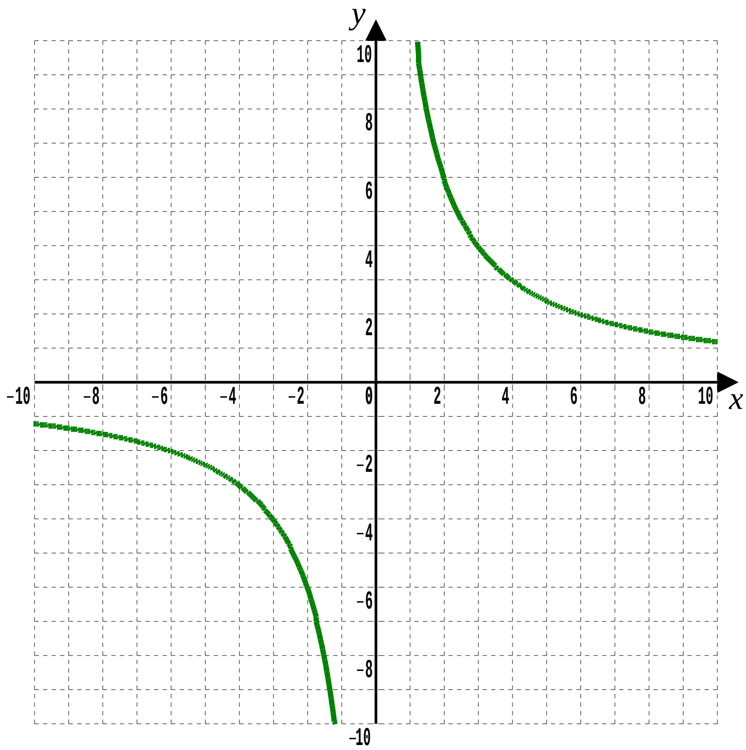
The circle, ellipse, parabola and the hyperbola form a category of curves known collectively as "Conic Sections". This name comes from the fact that they can all occur when a double cone (each of infinite extent) is sliced in various ways.



circle ellipse parabola hyperbola

The graph of a hyperbola is shown below. In this form it's known as the graph of inverse proportion or, more informally, as the "more is less" graph.

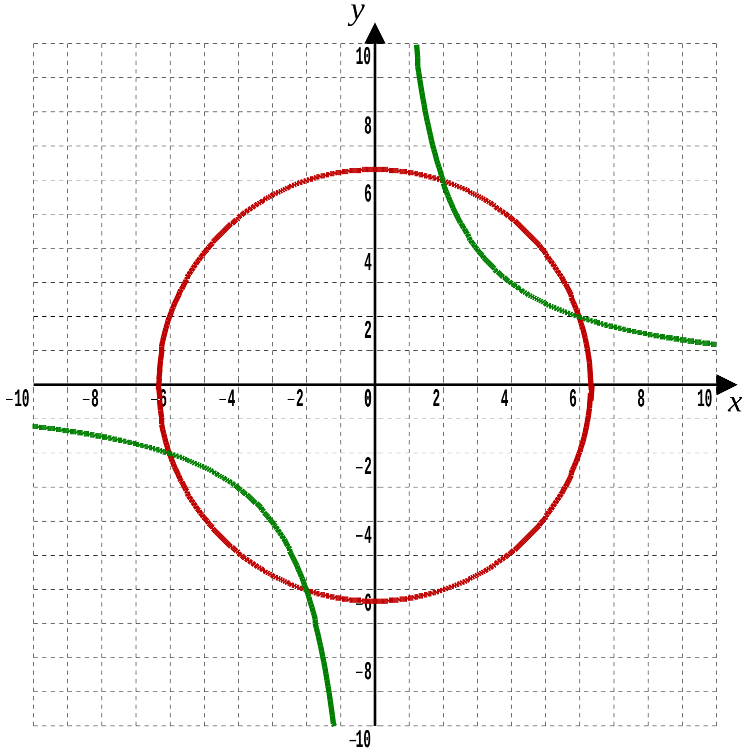
This particular hyperbola has the equation $y = \dfrac{12}{x}$



For GCSE and A-Level the hyperbolas considered are all of the form $y = \dfrac{k}{x}$

where k is a constant. In the graph above that constant is 12.

3.2 Quadratic In Disguise



Use algebra to find the points where $x^2 + y^2 = 40$ and $y = \dfrac{12}{x}$ intersect.

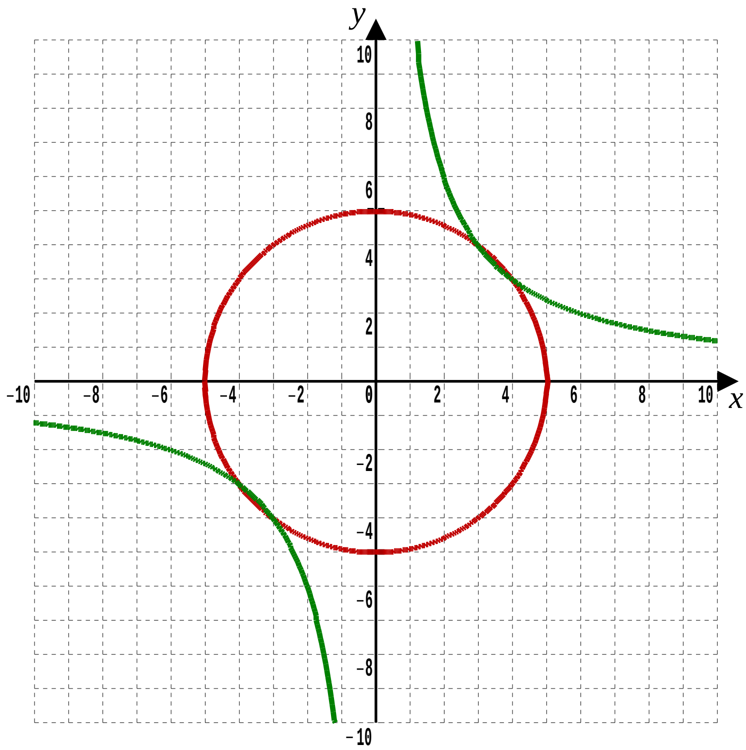
Teaching Video : http://www.NumberWonder.co.uk/v9091/3a.mp4 (Part 1)
http://www.NumberWonder.co.uk/v9091/3b.mp4 (Part 2)

⇐ Part 1 Part 2 ⇒

3.3 Exercise

Question 1

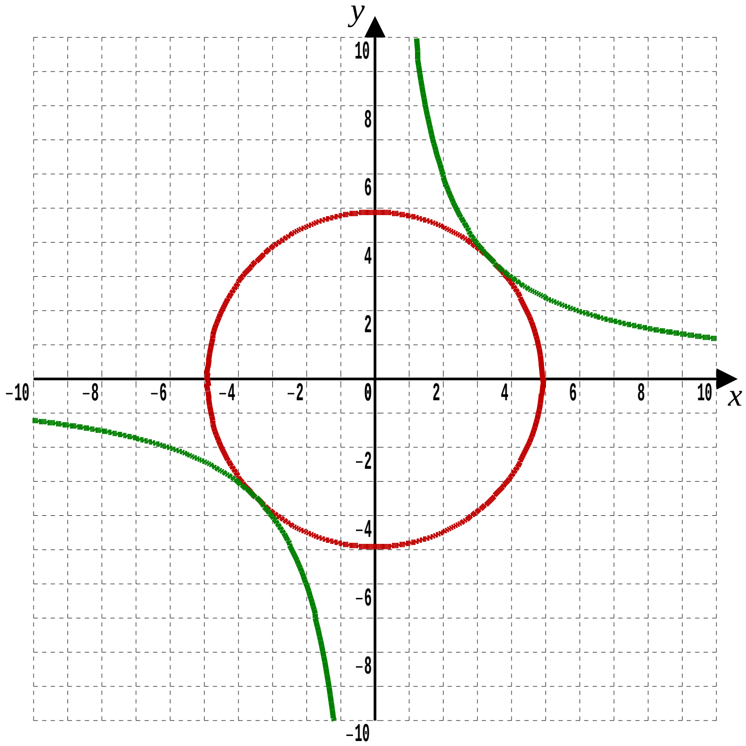
Fran has decided to shrink the circle to have a radius of 5.



Use algebra to find the points where $x^2 + y^2 = 25$ and $y = \dfrac{12}{x}$ intersect.

Question 2

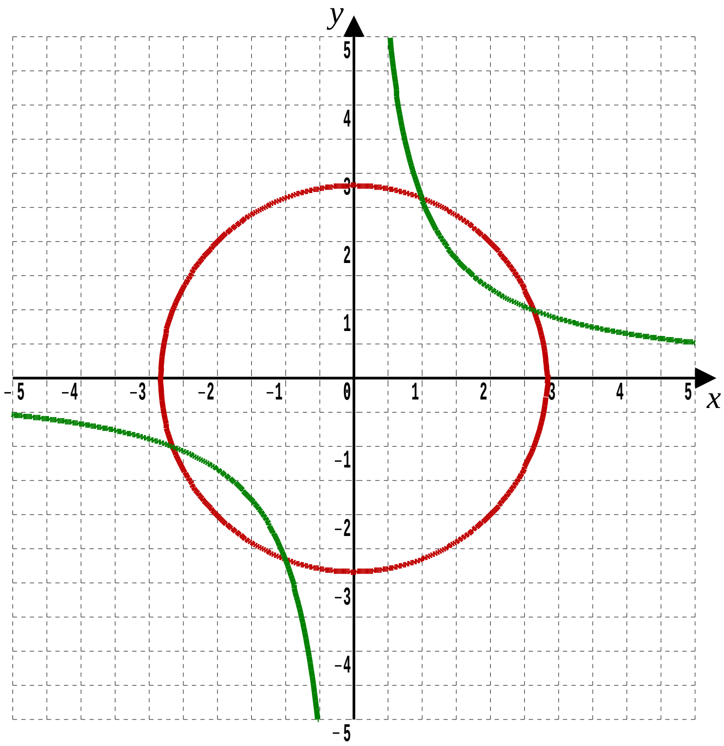
Oscar wonders if he can get a hyperbola to touch a circle in only two points. After playing around with a graph plotter, he arrives at the following graph.



Use algebra to find the points where $x^2 + y^2 = 18$ and $y = \dfrac{9}{x}$ intersect and so find out if Oscar has succeeded in his mission.

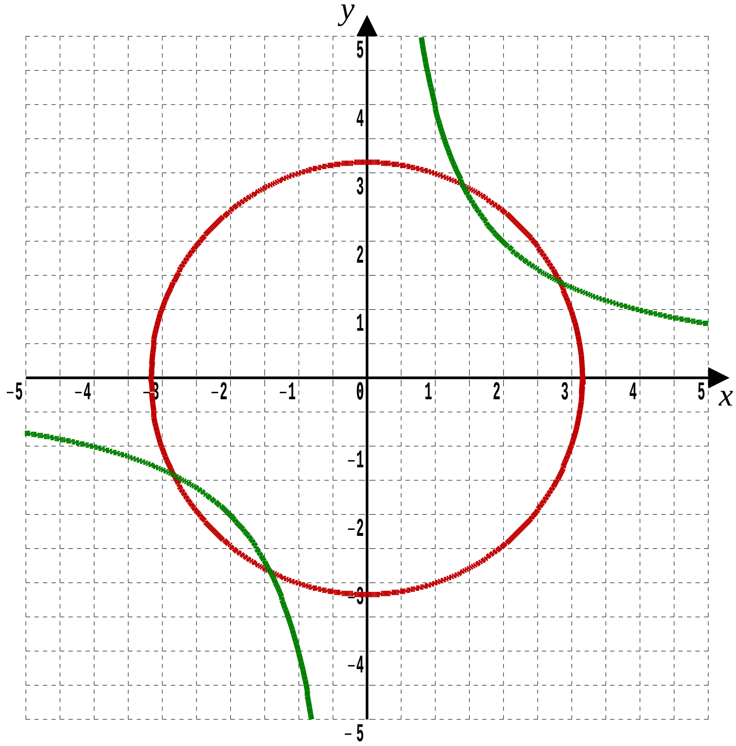
Question 3

In this question the intersections are not points with all integer coordinates.
One part of each point involves the square roots of an integer.
The graph is still useful to check that answers obtained are not obviously wrong !



Use algebra to find the points where $x^2 + y^2 = 8$ and $y = \frac{\sqrt{7}}{x}$ intersect.

Question 4



Use algebra to find the points where $x^2 + y^2 = 10$ and $y = \dfrac{4}{x}$ intersect

3.4 Answers

3.4.1 Solution to 3.2 Teaching Video Example

Square both sides of the equation of the hyperbola,

$$y = \frac{12}{x} \Rightarrow y^2 = \frac{144}{x^2}$$

Then substitute this in place of the y^2 in the equation for the circle,

$$x^2 + \left(\frac{144}{x^2} \right) = 40$$

Multiply through by x^2

$$x^2 \times x^2 + \frac{x^2}{1} \times \left(\frac{144}{x^2} \right) = x^2 \times 40$$

$$x^4 + 144 = 40\,x^2$$

$$x^4 - 40\,x^2 + 144 = 0$$

$$\text{Let } z = x^2 \Rightarrow z^2 = x^4$$

$$z^2 - 40\,z + 144 = 0$$

$$(z - 4)(z - 36) = 0$$

$$\text{Either} \quad z - 4 = 0 \quad \text{or} \quad z - 36 = 0$$

$$z = 4 \quad \text{or} \quad z = 36$$

$$\therefore \; x^2 = 4 \text{ or } x^2 = 36$$

$$x = \pm 2 \quad \text{or} \quad x = \pm 6$$

Use the equation for the hyperbola to get the y part of the points,

$$y = \frac{12}{x}$$

The four points of intersection are,

$$(-2, -6), \; (2, 6), (-6, -2) \text{ and } (6, 2)$$

3.4.2 Solution to 3.3 Exercise

Answer 1

Square both sides of the equation of the hyperbola,

$$y = \frac{12}{x} \Rightarrow y^2 = \frac{144}{x^2}$$

Then substitute this in place of the y^2 in the equation for the circle,

$$x^2 + \left(\frac{144}{x^2} \right) = 25$$

Multiply through by x^2

$$x^2 \times x^2 + \frac{x^2}{1} \times \left(\frac{144}{x^2} \right) = x^2 \times 25$$

$$x^4 + 144 = 25x^2$$

$$x^4 - 25x^2 + 144 = 0$$

$$\text{Let } z = x^2 \Rightarrow z^2 = x^4$$

$$z^2 - 25z + 144 = 0$$

$$(z - 9)(z - 16) = 0$$

$$\text{Either} \quad z - 9 = 0 \quad \text{or} \quad z - 16 = 0$$

$$z = 9 \quad \text{or} \quad z = 16$$

$$\therefore \; x^2 = 9 \text{ or } x^2 = 16$$

$$x = \pm 3 \quad \text{or} \quad x = \pm 4$$

Use the equation for the hyperbola to get the y part of the points,

$$y = \frac{12}{x}$$

The four points of intersection are,

$$(-3, -4), \quad (3, 4), (-4, -3) \text{ and } (4, 3)$$

Answer 2

Square both sides of the equation of the hyperbola,

$$y = \frac{9}{x} \Rightarrow y^2 = \frac{81}{x^2}$$

Then substitute this in place of the y^2 in the equation for the circle,

$$x^2 + \left(\frac{81}{x^2}\right) = 18$$

Multiply through by x^2

$$x^2 \times x^2 + \frac{x^2}{1} \times \left(\frac{81}{x^2}\right) = x^2 \times 18$$

$$x^4 + 81 = 18x^2$$

$$x^4 - 18x^2 + 81 = 0$$

$$\text{Let } z = x^2 \Rightarrow z^2 = x^4$$

$$z^2 - 18z + 81 = 0$$

$$(z - 9)(z - 9) = 0$$

$$\therefore \; z - 9 = 0$$

$$z = 9$$

$$\therefore \; x^2 = 9$$

$$x = \pm 3$$

Use the equation for the hyperbola to get the y part of the points,

$$y = \frac{9}{x}$$

The two points of intersection are,

$$(-3, -3), \quad (3, 3)$$

Oscar has succeeded in his mission !

Answer 3

Square both sides of the equation of the hyperbola,

$$y = \frac{\sqrt{7}}{x} \Rightarrow y^2 = \frac{7}{x^2}$$

Then substitute this in place of the y^2 in the equation for the circle,

$$x^2 + \left(\frac{7}{x^2} \right) = 8$$

Multiply through by x^2

$$x^2 \times x^2 + \frac{x^2}{1} \times \left(\frac{7}{x^2} \right) = x^2 \times 8$$

$$x^4 + 7 = 8x^2$$

$$x^4 - 8x^2 + 7 = 0$$

$$\text{Let } z = x^2 \Rightarrow z^2 = x^4$$

$$z^2 - 8z + 7 = 0$$

$$(z - 1)(z - 7) = 0$$

$$\text{Either} \quad z - 1 = 0 \quad \text{or} \quad z - 7 = 0$$

$$z = 1 \quad \text{or} \quad z = 7$$

$$\therefore \; x^2 = 1 \text{ or } x^2 = 7$$

$$x = \pm 1 \text{ or } x = \pm\sqrt{7}$$

Use the equation for the hyperbola to get the *y* part of the points,

$$y = \frac{\sqrt{7}}{x}$$

The four points of intersection are,

$$(-1, -\sqrt{7}), (1, \sqrt{7}), (-\sqrt{7}, -1) \text{ and } (\sqrt{7}, 1)$$

Answer 4

Square both sides of the equation of the hyperbola,

$$y = \frac{4}{x} \Rightarrow y^2 = \frac{16}{x^2}$$

Then substitute this in place of the y^2 in the equation for the circle,

$$x^2 + \left(\frac{16}{x^2} \right) = 10$$

Multiply through by x^2

$$x^2 \times x^2 + \frac{x^2}{1} \times \left(\frac{16}{x^2} \right) = x^2 \times 10$$

$$x^4 + 16 = 10\,x^2$$

$$x^4 - 10\,x^2 + 16 = 0$$

$$\text{Let } z = x^2 \Rightarrow z^2 = x^4$$

$$z^2 - 10z + 16 = 0$$

$$(z - 2)(z - 8) = 0$$

$$\text{Either} \quad z - 2 = 0 \quad \text{or} \quad z - 8 = 0$$

$$z = 2 \quad \text{or} \quad z = 8$$

$$\therefore \; x^2 = 2 \text{ or } x^2 = 8$$

$$x = \pm\sqrt{2} \text{ or } x = \pm\sqrt{8} \qquad \text{Note } \sqrt{8} = 2\sqrt{2}$$

Use the equation for the hyperbola to get the y part of the points,

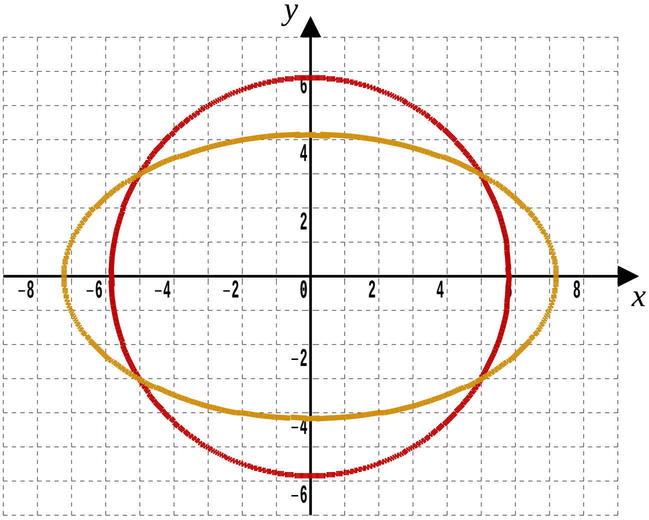
$$y = \frac{4}{x}$$

The four points of intersection are,

$$\left(-\sqrt{2}, -2\sqrt{2}\right), \left(\sqrt{2}, 2\sqrt{2}\right), \left(-2\sqrt{2}, -\sqrt{2}\right) \text{ and } \left(2\sqrt{2}, \sqrt{2}\right)$$

4.1 Circle and Ellipse

In appearance, a circle (in red) and an ellipse (in gold) have much in common. They are both smooth and continuous curves that form a graceful closed loop.



No surprise then that their algebraic equations also look similar.

$$\text{Red circle :} \quad x^2 + y^2 = 34$$

$$\text{Gold ellipse :} \quad x^2 + 3y^2 = 52$$

More generally,

The Equation of a Circle

$$x^2 + y^2 = r^2$$

This is a circle with centre (0, 0) and radius r

The Equation of an Ellipse

$$x^2 + a\,y^2 = w^2$$

This is an ellipse with centre (0, 0) and *half width*, w

In fact, when $a = 1$ in the equation of an ellipse, it becomes the equation of a circle.

This is because, $$a = \left(\frac{\textit{half width}}{\textit{half height}}\right)^2$$

and for a circle the *half width* and the *half height* are the same; the *radius*, r.

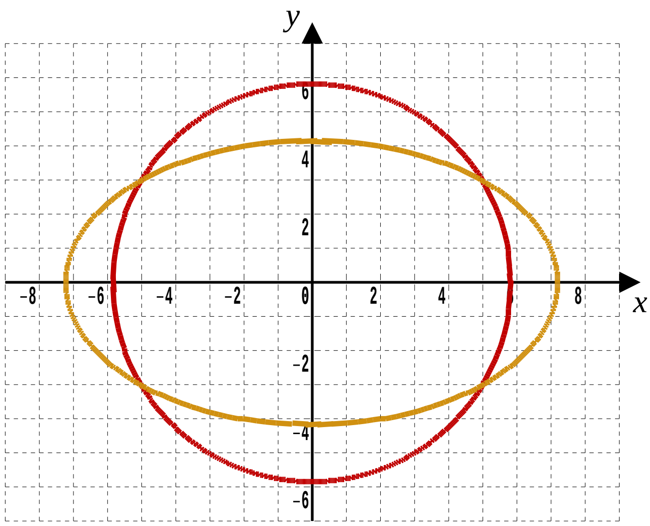
- The *half width* is the number where the ellipse crosses the positive x-axis.
- The *half height* is the number where the ellipse crosses the positive y-axis.

4.2 Where Gold meets Red

Use algebra to solve the simultaneous equations,

$$\text{Red circle}: \quad x^2 + y^2 = 34$$

$$\text{Gold ellipse}: \quad x^2 + 3y^2 = 52$$



Teaching Video : http://www.NumberWonder.co.uk/v9091/4.mp4

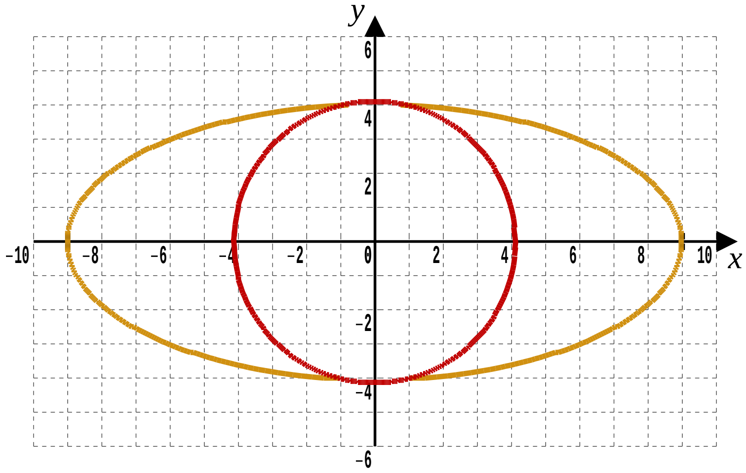
4.3 Exercise

Question 1

Use algebra to solve the simultaneous equations,

$$\text{Red circle: } x^2 + y^2 = 17$$

$$\text{Gold ellipse: } x^2 + 5y^2 = 81$$

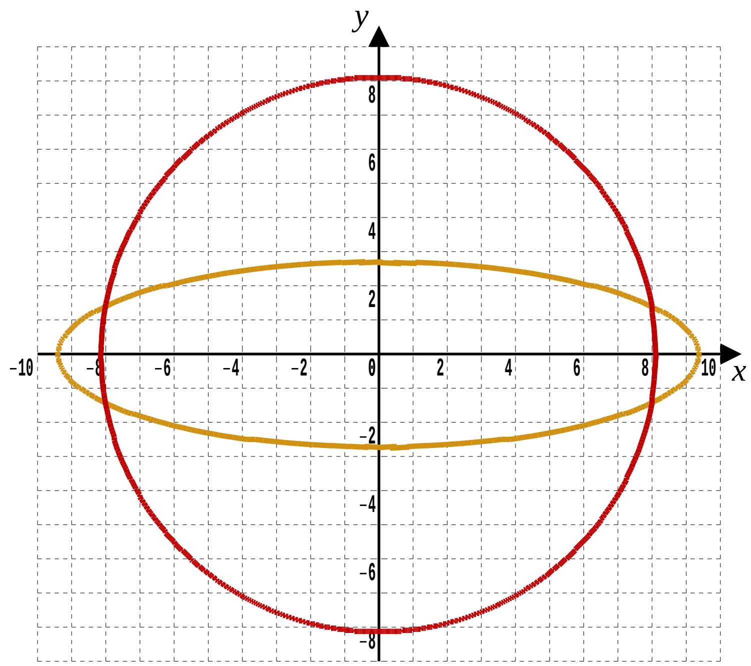


Question 2

Use algebra to solve the simultaneous equations,

$$\text{Red circle}: \quad x^2 + y^2 = 66$$

$$\text{Gold ellipse}: \quad x^2 + 12y^2 = 88$$



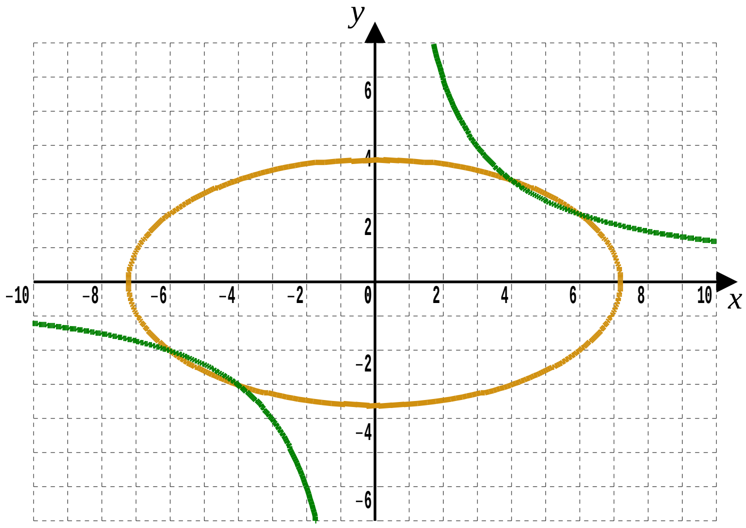
Give your points as exact coordinates, leaving square roots in your answers.

Question 3

Use algebra to solve the simultaneous equations,

$$\text{Gold ellipse}: \quad x^2 + 4y^2 = 52$$

$$\text{Green hyperbola}: \quad y = \frac{12}{x}$$



This list of pairs of positive integers that have a product of 576 may be of use !

1×576

2×288

3×192

4×144

6×96

8×72

9×64

12×48

16×36

18×32

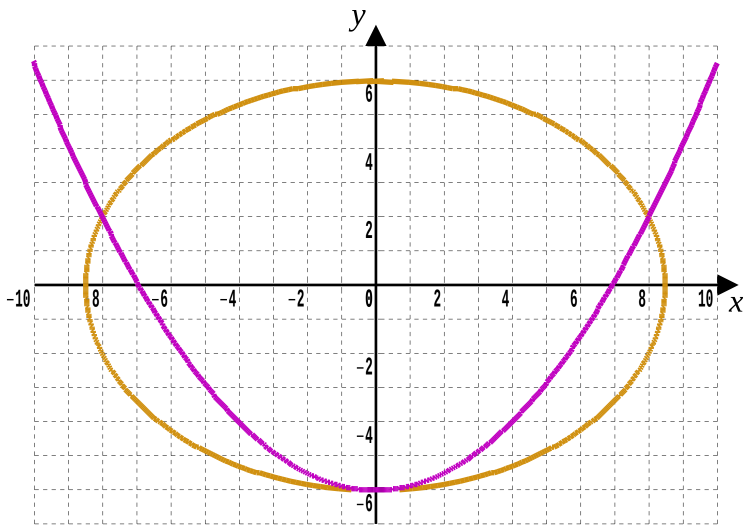
24×24

Question 4

Use algebra to solve the simultaneous equations,

$$\text{Gold ellipse : } \quad x^2 + 2y^2 = 72$$

$$\text{Purple parabola : } \quad y = \frac{x^2}{8} - 6$$



4.4 Answers

4.4.1 Solution to 4.2 Teaching Video Example

$$
\begin{aligned}
x^2 + 3y^2 &= 52 \\
x^2 + y^2 &= 34 \qquad \text{SUBTRACT} \\
\hline
2y^2 &= 18 \\
y^2 &= 9 \\
y &= \pm 3
\end{aligned}
$$

From the graph, four points are expected, so at this stage they are shaping up to be,

$(\quad, -3)$, $(\quad, -3)$, $(\quad, 3)$ and $(\quad, 3)$

The x part of these coordinates are easiest to find from the equation of the circle,

$$
\begin{aligned}
x^2 + y^2 &= 34 \\
x^2 &= 34 - y^2 \\
&= 34 - (\pm 3)^2 \\
&= 34 - 9 \\
&= 25 \\
\therefore\ x &= \pm 5
\end{aligned}
$$

The points are,

$(-5, -3)$, $(5, -3)$, $(-5, 3)$ and $(5, 3)$

4.4.2 Solution to 4.3 Exercise

Answer 1

$$\begin{aligned} x^2 + 5y^2 &= 81 \\ x^2 + y^2 &= 17 \qquad \text{SUBTRACT} \\ \hline 4y^2 &= 64 \\ y^2 &= 16 \\ y &= \pm 4 \end{aligned}$$

From the graph, four points are expected, so at this stage they are shaping up to be,

$$(\ \ ,-4),\ (\ \ ,-4),\ (\ \ ,\ 4) \text{ and } (\ \ ,\ 4)$$

The *x* part of these coordinates are easiest to find from the equation of the circle,

$$\begin{aligned} x^2 + y^2 &= 17 \\ x^2 &= 17 - y^2 \\ &= 17 - (\pm 4)^2 \\ &= 17 - 16 \\ &= 1 \\ \therefore\ x &= \pm 1 \end{aligned}$$

The points are,

$$(-1,-4),\ (1,-4),\ (-1,\ 4) \text{ and } (1,\ 4)$$

Answer 2

$$\begin{aligned} x^2 + 12y^2 &= 88 \\ x^2 + y^2 &= 66 \qquad \text{SUBTRACT} \\ \hline 11y^2 &= 22 \\ y^2 &= 2 \\ y &= \pm\sqrt{2} \end{aligned}$$

From the graph, four points are expected, so at this stage they are shaping up to be,

$(\quad , -\sqrt{2})$, $(\quad , -\sqrt{2})$, $(\quad , \sqrt{2})$ and $(\quad , \sqrt{2})$

The x part of these coordinates are easiest to find from the equation of the circle,

$$\begin{aligned} x^2 + y^2 &= 66 \\ x^2 &= 66 - y^2 \\ &= 66 - \left(\pm\sqrt{2}\right)^2 \\ &= 66 - 2 \\ &= 64 \\ \therefore\ x &= \pm 8 \end{aligned}$$

The points are,

$(-8, -\sqrt{2})$, $(8, -\sqrt{2})$, $(-8, \sqrt{2})$ and $(8, \sqrt{2})$

Answer 3

Square both sides of the equation of the hyperbola,

$$y = \frac{12}{x} \Rightarrow y^2 = \frac{144}{x}$$

Then substitute this in place of the y^2 in the equation for the ellipse,

$$x^2 + 4\left(\frac{144}{x^2}\right) = 52$$

Multiply through by x^2

$$x^2 \times x^2 + \frac{x^2}{1} \times \left(\frac{576}{x^2}\right) = x^2 \times 52$$

$$x^4 + 576 = 52\,x^2$$

$$x^4 - 52\,x^2 + 576 = 0$$

$$\text{Let } z = x^2 \Rightarrow z^2 = x^4$$

$$z^2 - 52z + 576 = 0$$

$$(z - 16)(z - 36) = 0 \qquad \text{with help from list provided}$$

$$\text{Either} \quad z - 16 = 0 \quad \text{or} \quad z - 36 = 0$$

$$z = 16 \quad \text{or} \quad z = 36$$

$$\therefore \; x^2 = 16 \text{ or } x^2 = 36$$

$$x = \pm 4 \quad \text{or} \quad x = \pm 6$$

Use the equation for the hyperbola to get the y part of the points,

$$y = \frac{12}{x}$$

The four points of intersection are,

$$(-4, -3),\ (4,\ 3),\ (-6, -2) \text{ and } (6,\ 2)$$

Answer 4

Rearrange the equation of the quadratic,

$$y = \frac{x^2}{8} - 6$$

$$y + 6 = \frac{x^2}{8}$$

$$x^2 = 8y + 48$$

Then substitute this in place of the x^2 in the equation for the ellipse,

$$\left(x^2 \right) + 2y^2 = 72$$

$$\left(8y + 48 \right) + 2y^2 = 72$$

$$2y^2 + 8y - 24 = 0$$

$$y^2 + 4y - 12 = 0$$

$$(y + 6)(y - 2) = 0$$

Either $y + 6 = 0$ or $y - 2 = 0$

$$y = -6 \quad \text{or} \quad y = 2$$

Back with the equation for the quadratic,

$$x^2 = 8y + 48$$

square root both sides

$$x = \pm\sqrt{8y + 48}$$

So when $y = -6$, $x = \pm\sqrt{8 \times (-6) + 48} \Rightarrow x = 0$

And when $y = 2$, $x = \pm\sqrt{8 \times 2 + 48} \Rightarrow x = \pm 8$

The three points of intersection are,

$$(0, -6), (-8, 2) \text{ and } (8,2)$$

5.1 The Q Formula

In examinations candidates are allowed to use the Q Formula;
(The Q stands for Quadratic)
At GCSE level, it is given on a page along with several other useful formulae.
For A-Level candidates are expected to have memorised it.

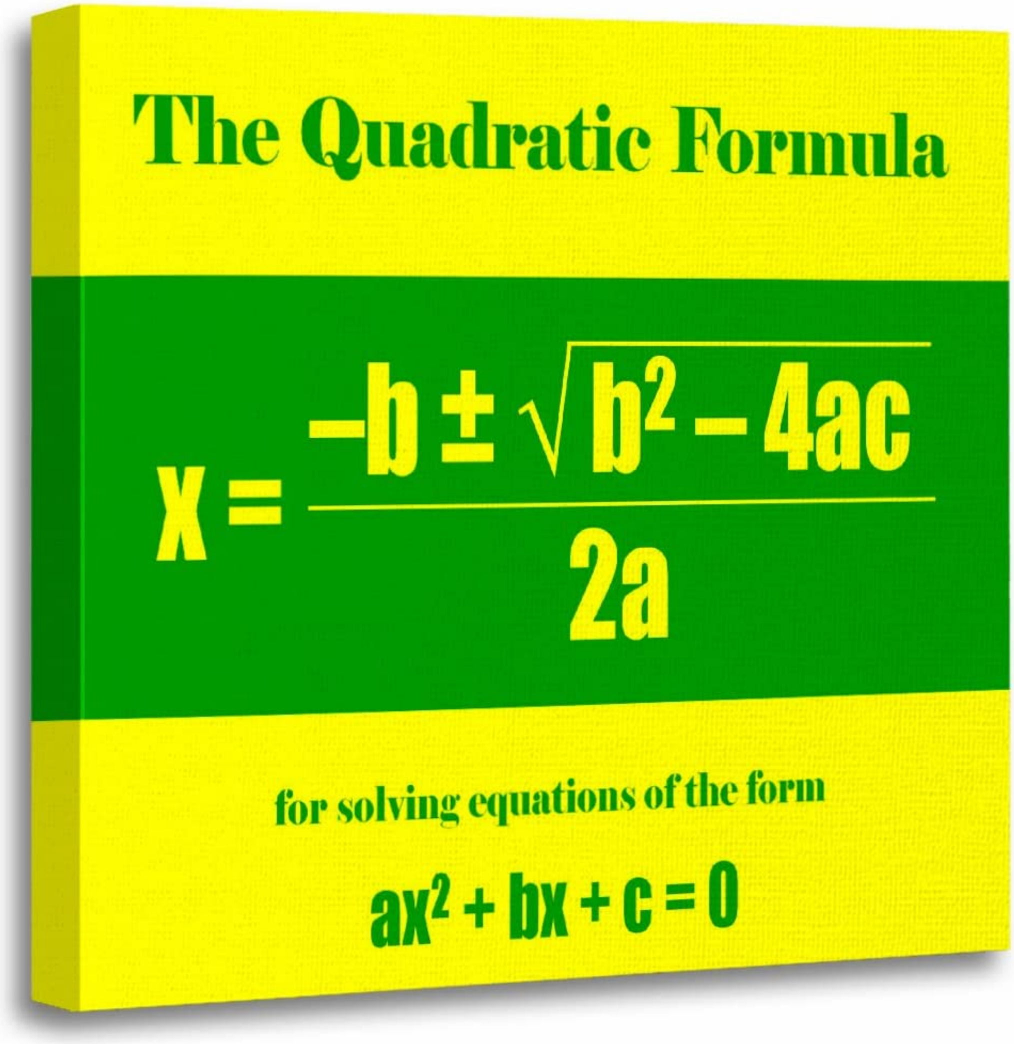
The Q Formula

A quadratic equation that is written in the form

$$a\,x^2 + bx + c = 0 \qquad \text{where } a, b \text{ and } c \text{ are constants}$$

has real solutions, if any exist, given by the formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

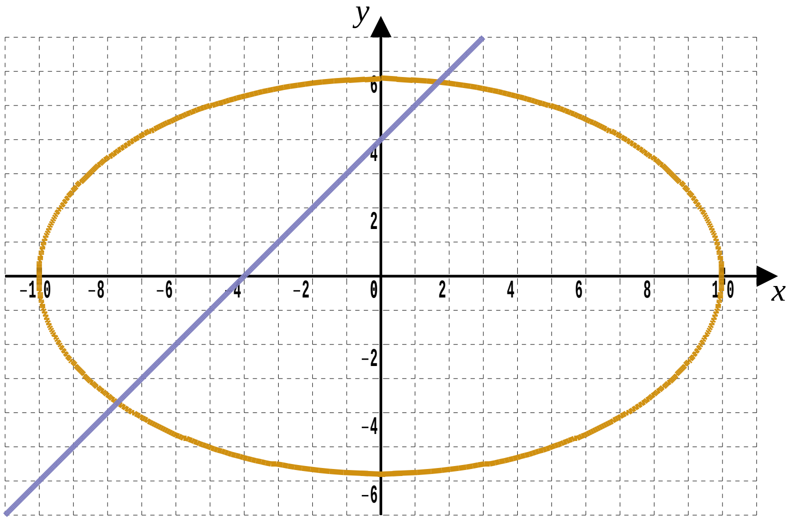


5.2 Q Formula In Use

Use algebra to solve the simultaneous equations,

Purple line : $y = x + 4$

Gold ellipse : $x^2 + 3y^2 = 100$



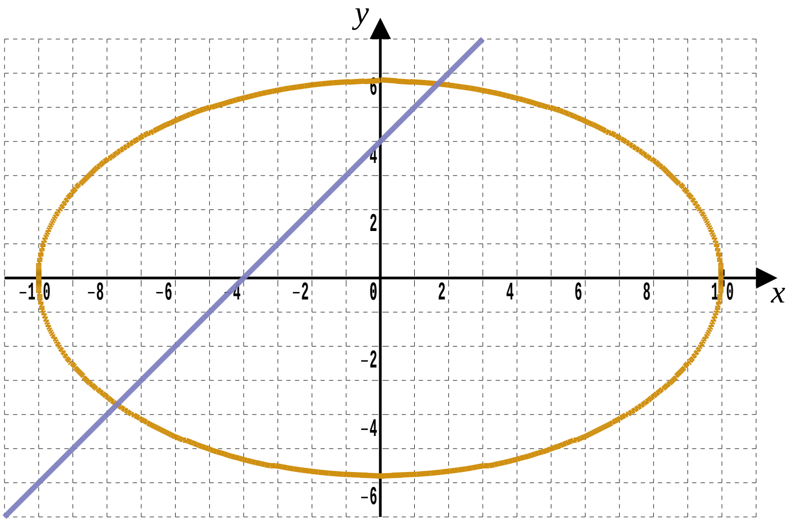
Teaching Video : http://www.NumberWonder.co.uk/v9091/5a.mp4 (Part 1)
http://www.NumberWonder.co.uk/v9091/5b.mp4 (Part 2)

<= Part 1 Part 2 =>

5.3 Exercise

Question 1

Here is the graph of the Teaching Video example, once more.



In the video the exact coordinates of the points of intersection were found to be,

$$\left(-3 - \sqrt{22},\ 1 - \sqrt{22}\right) \quad \left(-3 + \sqrt{22},\ 1 + \sqrt{22}\right)$$

Using a calculator, write these exact answers as decimal approximations accurate to one decimal place.

Do your decimal coordinates approximately match up with where the graph shows that the straight line and the ellipse cross ?

Question 2

Showing working, simplify the following numbers.

Your answers should all be of the form $p \pm \sqrt{q}$ where p and q are integers.

(i) $\dfrac{9 \pm \sqrt{18}}{3}$

(ii) $\dfrac{35 \pm \sqrt{50}}{5}$

(iii) $\dfrac{56 \pm \sqrt{48}}{4}$

(iv) $\dfrac{-24 \pm \sqrt{640}}{8}$

Question 3

Showing working, simplify the following numbers.

Your answers should all be of the form $r \pm s\sqrt{t}$ where r, s and t are integers.

(i) $\dfrac{10 \pm \sqrt{300}}{5}$

(ii) $\dfrac{-45 \pm \sqrt{360}}{3}$

Question 4

Show how to use the Q Formula to solve the following quadratic equations,
Your answers should all be of the form $p \pm \sqrt{q}$ where p and q are integers.

(i) $x^2 + 8x + 5 = 0$

(ii) $x^2 + 2x - 9 = 0$

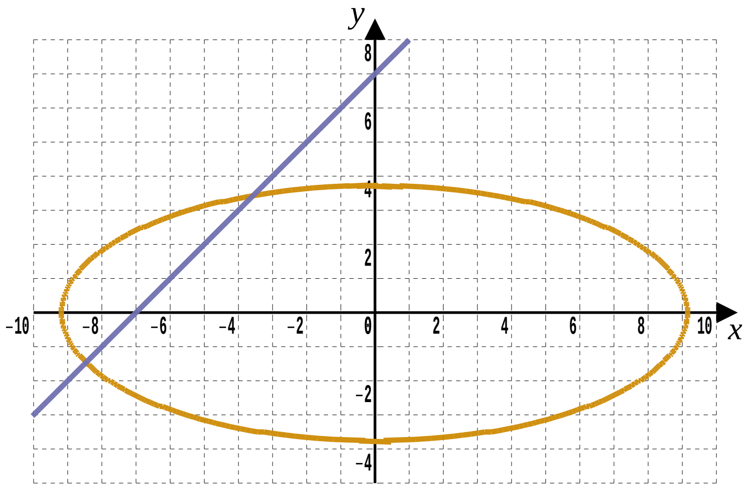
Question 5

Use algebra to solve the simultaneous equations,

$$\text{Purple line :} \qquad y = x + 7$$

$$\text{Gold ellipse :} \qquad x^2 + 6y^2 = 84$$

Write your answers as simplified exact points.



5.4 Answers

5.4.1 Solutions to Teaching Video "Q Formula In Use"

$$
\begin{aligned}
x^2 + 3y^2 &= 100 \\
x^2 + 3\left[(x+4)^2 \right] &= 100 \\
x^2 + 3\left[(x+4)(x+4) \right] &= 100 \\
x^2 + 3\left[x^2 + 8x + 16 \right] &= 100 \\
x^2 + 3x^2 + 24x + 48 &= 100 \\
4x^2 + 24x - 52 &= 0 \\
x^2 + 6x - 13 &= 0 \\
x &= \frac{-6 \pm \sqrt{6^2 - 4 \times 1 \times (-13)}}{2 \times 1} \\
&= \frac{-6 \pm \sqrt{36 + 52}}{2} \\
&= \frac{-6 \pm \sqrt{88}}{2} \\
&= \frac{-6 \pm \sqrt{4 \times 22}}{2} \\
&= \frac{-6 \pm 2\sqrt{22}}{2} \\
&= \frac{-6}{2} \pm \frac{2\sqrt{22}}{2} \\
&= -3 \pm \sqrt{22}
\end{aligned}
$$

Using $y = x + 4$ to get the y part of the coordinates

$$\left(-3 - \sqrt{22},\ 1 - \sqrt{22}\right) \quad \left(-3 + \sqrt{22},\ 1 + \sqrt{22}\right)$$

5.4.2 Solutions to 5.3 Exercise

Answer 1

$$\left(-3 - \sqrt{22},\ 1 - \sqrt{22}\right) \quad \left(-3 + \sqrt{22},\ 1 + \sqrt{22}\right)$$

These become, to one decimal place,

$$(-7.7,\ -3.7) \qquad (1.7,\ 5.7)$$

which do indeed match up with the graph.

Answer 2

(i)

$$\frac{9 \pm \sqrt{18}}{3} = \frac{9 \pm \sqrt{9 \times 2}}{3}$$

$$= 3 \pm \sqrt{2}$$

(ii)

$$\frac{35 \pm \sqrt{50}}{5} = \frac{35 \pm \sqrt{25 \times 2}}{5}$$

$$= 7 \pm \sqrt{2}$$

(iii)

$$\frac{56 \pm \sqrt{48}}{4} = \frac{56 \pm \sqrt{16 \times 3}}{4}$$

$$= 14 \pm \sqrt{3}$$

(iv)

$$\frac{-24 \pm \sqrt{640}}{8} = \frac{-24 \pm \sqrt{64 \times 10}}{8}$$

$$= -3 \pm \sqrt{10}$$

Answer 3

(i)

$$\frac{10 \pm \sqrt{300}}{5} = \frac{10 \pm \sqrt{100 \times 3}}{5}$$

$$= 2 \pm 2\sqrt{3}$$

(ii)

$$\frac{-45 \pm \sqrt{360}}{3} = \frac{-45 \pm \sqrt{36 \times 10}}{3}$$

$$= -15 \pm 2\sqrt{10}$$

Answer 4

(i)

$$x^2 + 8x + 5 = 0$$

$$x = \frac{-8 \pm \sqrt{8^2 - 4 \times 1 \times 5}}{2 \times 1}$$

$$= \frac{-8 \pm \sqrt{64 - 20}}{2}$$

$$= \frac{-8 \pm \sqrt{44}}{2}$$

$$= \frac{-8 \pm \sqrt{4 \times 11}}{2}$$

$$= -4 \pm \sqrt{11}$$

(ii)

$$x^2 + 2x - 9 = 0$$

$$x = \frac{-2 \pm \sqrt{2^2 - 4 \times 1 \times (-9)}}{2 \times 1}$$

$$= \frac{-2 \pm \sqrt{4 + 36}}{2}$$

$$= \frac{-2 \pm \sqrt{40}}{2}$$

$$= \frac{-2 \pm \sqrt{4 \times 10}}{2}$$

$$= -1 \pm \sqrt{10}$$

Answer 5

$$x^2 + 6y^2 = 84$$
$$x^2 + 6\left[\,(x+7)^2\,\right] = 84$$
$$x^2 + 6\left[\,(x+7)(x+7)\,\right] = 84$$
$$x^2 + 6\left[x^2 + 14x + 49\right] = 84$$
$$x^2 + 6x^2 + 84x + 294 = 84$$
$$7x^2 + 84x + 210 = 0$$
$$x^2 + 12x + 30 = 0$$
$$x = \frac{-12 \pm \sqrt{12^2 - 4 \times 1 \times 30}}{2 \times 1}$$
$$= \frac{-12 \pm \sqrt{144 - 120}}{2}$$
$$= \frac{-12 \pm \sqrt{24}}{2}$$
$$= \frac{-12 \pm \sqrt{4 \times 6}}{2}$$
$$= \frac{-12 \pm 2\sqrt{6}}{2}$$
$$= \frac{-12}{2} \pm \frac{2\sqrt{6}}{2}$$
$$= -6 \pm \sqrt{6}$$

Using $y = x + 7$ to get the y part of the coordinates

$$\left(-6 - \sqrt{6},\ 1 - \sqrt{6}\right) \quad \left(-6 + \sqrt{6},\ 1 + \sqrt{6}\right)$$

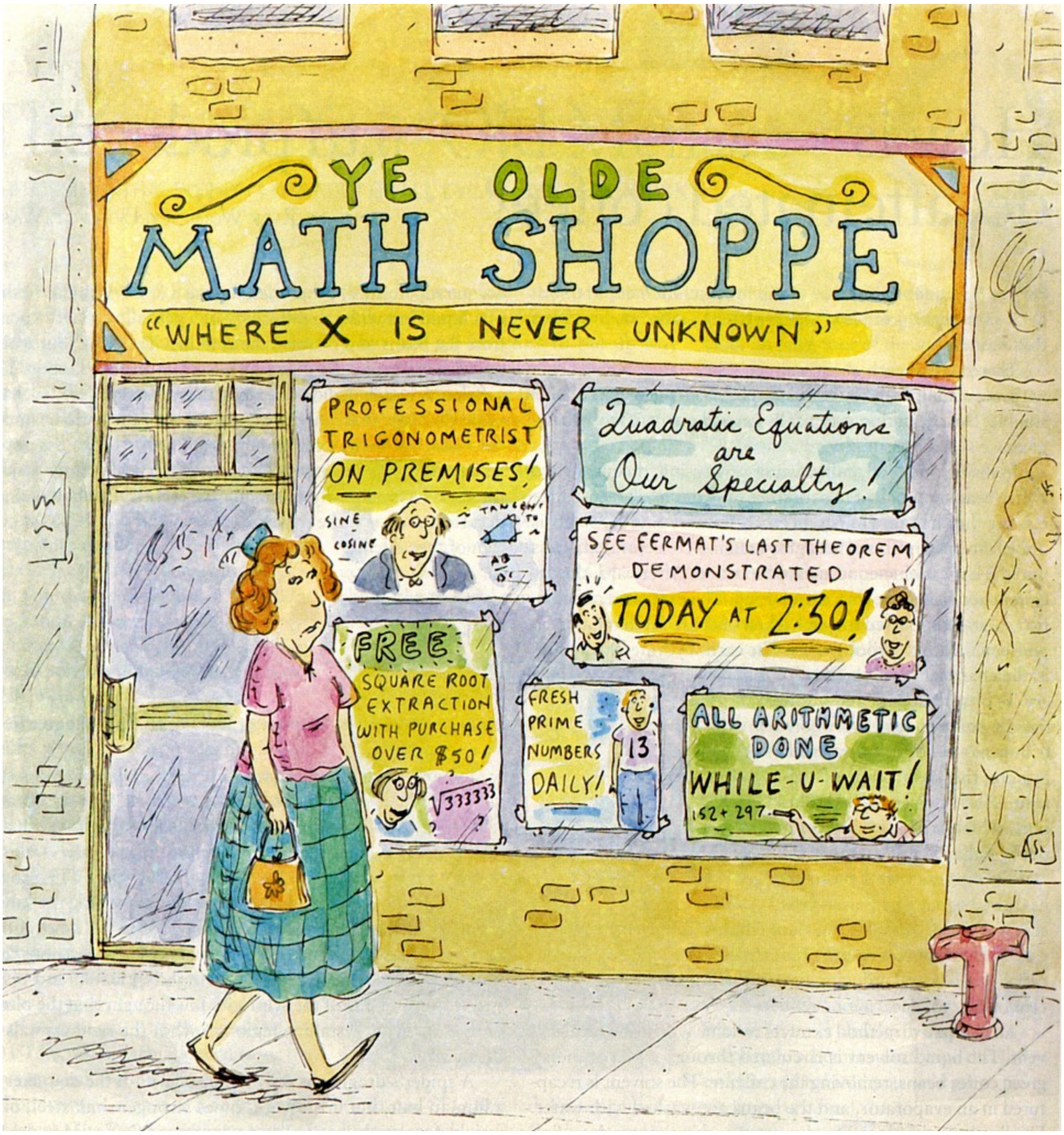
Lesson 6

GCSE and Preparatory A-Level Mathematics
Conic Sections (Simultaneous Equations IV)

6.1 Sweating The Q Formula

Last lesson you used the Q formula.
From memory, write it down.

☞



6.2 An Example Where $a \neq 1$

Question

Show how to use the Q formula to solve the equation,

$$3x^2 + 6x + 2 = 0$$

Answer

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

with $a = 3$, $b = 6$ and $c = 2$ gives

$$x = \frac{-6 \pm \sqrt{6^2 - 4 \times 3 \times 2}}{2 \times 3}$$

$$= \frac{-6 \pm \sqrt{36 - 24}}{6}$$

$$= \frac{-6 \pm \sqrt{12}}{6}$$

$$= \frac{-6}{6} \pm \frac{\sqrt{4 \times 3}}{6}$$

$$= -1 \pm \frac{2\sqrt{3}}{6}$$

$$= -1 \pm \frac{\sqrt{3}}{3}$$

Notes

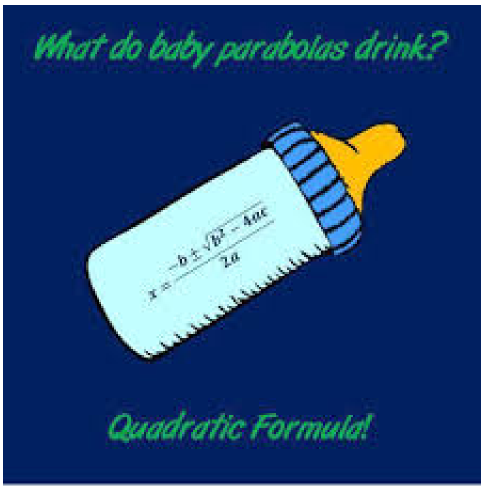
(i) You should begin by writing down the Q formula

(ii) Write down the value of *a*, of *b* and of *c*

(iii) You must show the numbers substituted into the formula.
Mark schemes give zero marks for the whole question if you don't do this.

(iv) Questions often require the exact answer (square roots left as square roots).

(v) Or they may ask for the answer to a specified number of significant figures.



6.3 Exercise

Question 1

Show how to use the Q formula to solve the quadratic equation,

$$2x^2 + 8x + 5 = 0$$

and hence that $x = -2 \pm \frac{\sqrt{6}}{2}$ are the two solutions.

Question 2

Show how to use the Q formula to solve the quadratic equation,

$$3x^2 + 2x - 4 = 0$$

and hence that $x = -\frac{1}{3} \pm \frac{\sqrt{13}}{3}$ are the two solutions.

Question 3

GCSE Examination Question from May 2018, Paper 1H, Q11(b)

Solve $3x^2 + 6x - 5 = 0$

Show your working clearly.

Give your solutions correct to 3 significant figures.

[3 marks]

Question 4

GCSE Examination Question from June 2011, Paper 4H, Q21(b)

Solve $x^2 + 90x - 1200 = 0$

Give the value of x correct to 3 significant figures.

[3 marks]

Question 5

National 5 Examination Question from June 2024. Paper 2, Q8, SQA

Solve the equation $3x^2 + 8x + 1 = 0$

Give your answers to 2 decimal places.

[3 marks]

Question 6

IGCSE Examination Question from June 2023, Paper 2H, Q21, Edexcel

Solve the simultaneous equations,

$$2x^2 + 3y^2 = 11$$

$$x = 3y - 1$$

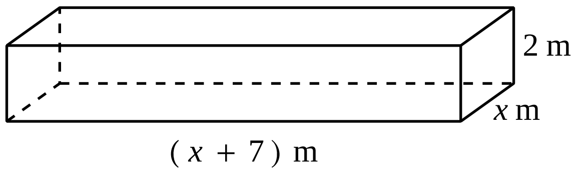
Show clear algebraic working

[5 marks]

Question 7

National 5 Examination Question from May 2023, Paper 2, Q14, SQA

A storage unit, built in the shape of a cuboid, is shown.



It has length $(x + 7)$ metres, breadth x metres and height 2 metres.

The volume of this unit is 45 cubic metres.

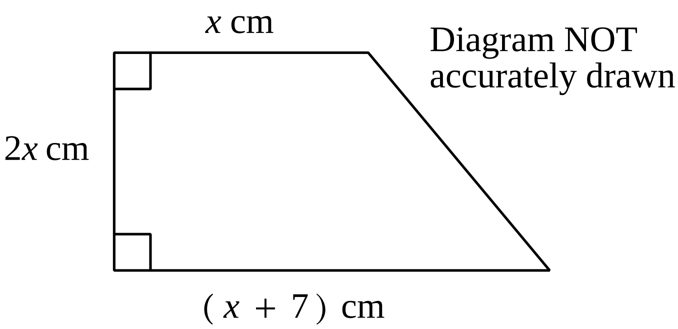
(a) Show that $2x^2 + 14x - 45 = 0$

[2 marks]

(b) Calculate x. the breadth of the storage unit.
Give your answer correct to 1 decimal place.

[4 marks]

Question 8
GCSE Examination Question from January 2015, 4H, Q17



The diagram shows a trapezium.
The trapezium has an area of 17 cm^2

(a) Show that $2x^2 + 7x - 17 = 0$

[3 marks]

(b) Work out the value of x
Give your answer correct to 3 significant figures,
Show your working clearly.

[3 marks]

6.4 Answers

Answer 1

$$2x^2 + 8x + 5 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

with $a = 2$, $b = 8$ and $c = 5$ gives

$$x = \frac{-8 \pm \sqrt{8^2 - 4 \times 2 \times 5}}{2 \times 2}$$

$$= \frac{-8 \pm \sqrt{64 - 40}}{4}$$

$$= \frac{-8 \pm \sqrt{24}}{4}$$

$$= \frac{-8}{4} \pm \frac{\sqrt{4 \times 6}}{4}$$

$$= -2 \pm \frac{2\sqrt{6}}{4}$$

$$= -2 \pm \frac{\sqrt{6}}{2}$$

Answer 2

$$3x^2 + 2x - 4 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

with $a = 3$, $b = 2$ and $c = -4$ gives

$$x = \frac{-2 \pm \sqrt{2^2 - 4 \times 3 \times (-4)}}{2 \times 3}$$

$$= \frac{-2 \pm \sqrt{4 + 48}}{6}$$

$$= \frac{-2 \pm \sqrt{52}}{6}$$

$$= \frac{-2}{6} \pm \frac{\sqrt{4 \times 13}}{6}$$

$$= -\frac{1}{3} \pm \frac{2\sqrt{13}}{6}$$

$$= -\frac{1}{3} \pm \frac{\sqrt{13}}{3}$$

Answer 3

$$3x^2 + 6x - 5 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

with $a = 3,\ b = 6$ and $c = -5$ gives

$$x = \frac{-6 \pm \sqrt{6^2 - 4 \times 3 \times (-5)}}{2 \times 3}$$

$$= \frac{-6 \pm \sqrt{36 + 60}}{6}$$

$$= \frac{-6 \pm \sqrt{96}}{6}$$

$$= \frac{-6}{6} \pm \frac{\sqrt{16 \times 6}}{6}$$

$$= -1 \pm \frac{4\sqrt{6}}{6}$$

$$= -1 \pm \frac{2\sqrt{6}}{3}$$

To 3 significant figures the solutions are -2.63 or 0.633

Answer 4

$$x^2 + 90x - 1200 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

with $a = 1,\ b = 90$ and $c = -1200$ gives

$$x = \frac{-90 \pm \sqrt{90^2 - 4 \times 1 \times (-1200)}}{2 \times 1}$$

$$= \frac{-90 \pm \sqrt{8100 + 4800}}{2}$$

$$= \frac{-90 \pm \sqrt{12900}}{2}$$

$$= \frac{-90}{2} \pm \frac{\sqrt{100 \times 129}}{2}$$

$$= -45 \pm \frac{10\sqrt{129}}{2}$$

$$= -45 \pm 5\sqrt{129}$$

To 3 significant figures the solutions are -102 or 11.8

Answer 5

$$3x^2 + 8x + 1 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

with $a = 3$, $b = 8$ and $c = 1$ gives

$$x = \frac{-8 \pm \sqrt{8^2 - 4 \times 3 \times 1}}{2 \times 3}$$

$$= \frac{-8 \pm \sqrt{64 - 12}}{6}$$

$$= \frac{-8 \pm \sqrt{52}}{6}$$

$$= \frac{-8}{6} \pm \frac{\sqrt{4 \times 13}}{2}$$

$$= -\frac{4}{3} \pm \frac{2\sqrt{13}}{6}$$

$$= -\frac{4}{3} \pm \frac{\sqrt{13}}{3}$$

To 2 decimal places the solutions are -0.13 or -2.54

Answer 6

$$2x^2 + 3y^2 = 11$$

$$2(3y - 1)^2 + 3y^2 = 11$$

$$2\left[(3y - 1)(3y - 1)\right] + 3y^2 = 11$$

$$2\left[9y^2 - 6y + 1\right] + 3y^2 = 11$$

$$21y^2 - 12y - 9 = 0$$

$$7y^2 - 4y - 3 = 0$$

(could use formula here)

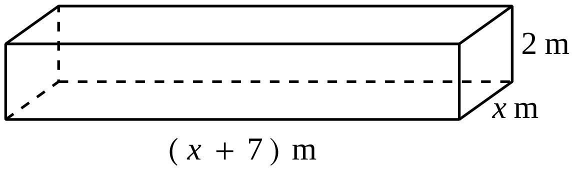
$$(7y + 3)(y - 1) = 0$$

Either $y = -\frac{3}{7}$ or $y = 1$

Using $x = 3y - 1$ to get the x part of the coordinates

$$\left(-\frac{16}{7}, -\frac{3}{7}\right) \quad (2,\ 1)$$

Answer 7



(a)

$$Volume = length \times breadth \times height$$

$$45 = 2x(x + 7)$$

$$45 = 2x^2 + 14x$$

$$2x^2 + 14x - 45 = 0 \qquad \square$$

(b)

$$2x^2 + 14x - 45 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

with $a = 2$, $b = 14$ and $c = -45$ gives

$$x = \frac{-14 \pm \sqrt{14^2 - 4 \times 2 \times (-45)}}{2 \times 2}$$

$$= \frac{-14 \pm \sqrt{196 + 360}}{4}$$

$$= \frac{-14 \pm \sqrt{556}}{4}$$

$$= \frac{-14}{4} \pm \frac{\sqrt{4 \times 139}}{4}$$

$$= -\frac{7}{2} \pm \frac{\sqrt{139}}{2}$$

To 1 decimal places the solutions are -9.4 or 2.4

Breadth is 2.4 metres (as it cannot be negative)

Answer 8

(a)

Area of a trapezium is half the sum of the parallel sides time the distance in between

$$\therefore \; 17 = \frac{1}{2}\left(x + (x + 7) \right) \times 2x$$

$$2 \times 17 = (2x + 7) \times 2x$$

$$34 = 4x^2 + 14x$$

$$4x^2 + 14x - 34 = 0$$

$$2x^2 + 7x - 17 = 0 \qquad \square$$

(b)

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

with $a = 2$, $b = 7$ and $c = -17$ gives

$$x = \frac{-7 \pm \sqrt{7^2 - 4 \times 2 \times (-17)}}{2 \times 2}$$

$$= \frac{-7 \pm \sqrt{49 + 136}}{4}$$

$$= \frac{-7 \pm \sqrt{185}}{4}$$

To 3 significant figures the solutions are -5.15 or 1.65

In the context of the problem only $x = 1.65$ is a valid solution.

Conic Sections (Simultaneous Equations IV)

7.1 Animated Conic Sections

Conic Sections are formed when a double cone is sliced in a variety of ways. The possible slices obtained can be circular, elliptical, parabolic or hyperbolic in shape. The following two minute video is an animation of the slicing process along with the corresponding graph. It shows that a few other surprising results can occur; watch out for the graph being a single point or a single straight line. The most interesting of these “surprise results” is the double straight line.

Teaching Video : http://www.NumberWonder.co.uk/v9091/7.mp4

7.2 The Double Straight Line (Hyperbola)

The recent GCSE examination papers have included some tough questions on conic sections, tackled from an algebraic point of view. As an example of the sort of question being asked, consider the following;

The Question

Use algebra to solve the following equations simultaneously,

$$x^2 - 4xy + 3y^2 = 0$$

$$y = 2x - 5$$

Give It A Go

Your job now is to tackle this on a scrap piece of paper using algebra.

The important thing to notice is that the y in

$$x^2 - 4xy + 3y^2 = 0$$

is in two places.

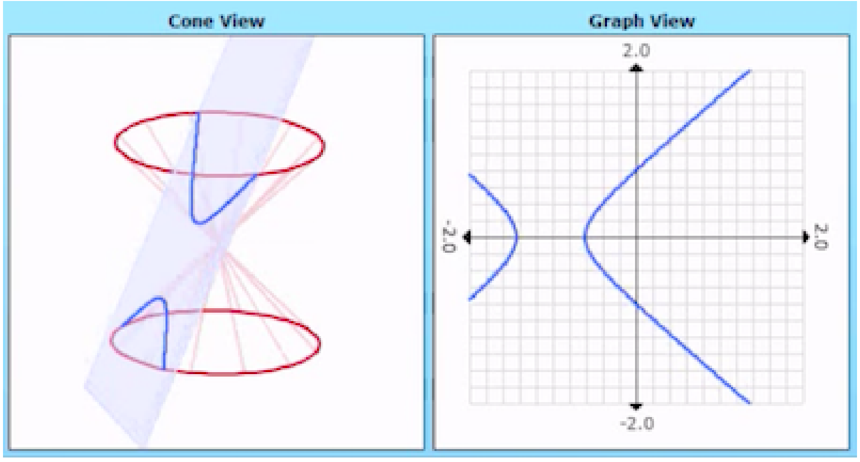
Each of those needs to be replaced with $2x - 5$ from the other equation.

Over the page, what the graph of $x^2 - 4xy + 3y^2 = 0$ looks like will be revealed and a full solution to the question will also be provided.

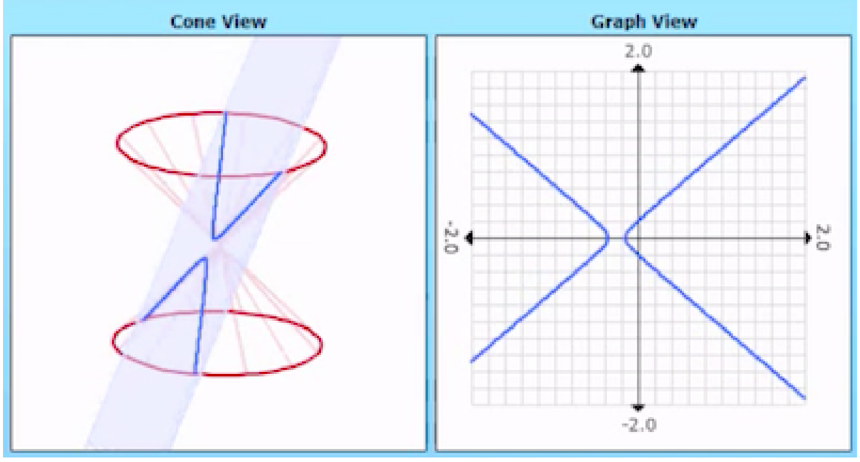
The Graph

The animation video showed how a hyperbola can degenerate into a graph with two straight lines.

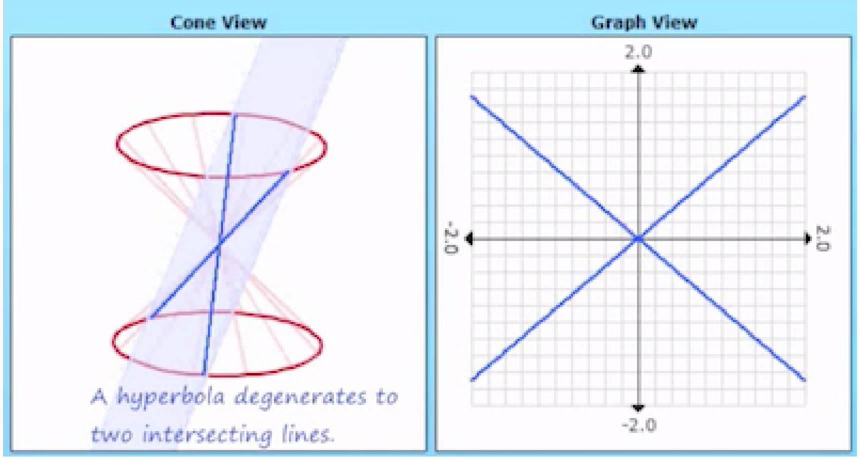
Frame 1



Frame 2



Frame 3



The equation of this example

$$x^2 - 4xy + 3y^2 = 0$$

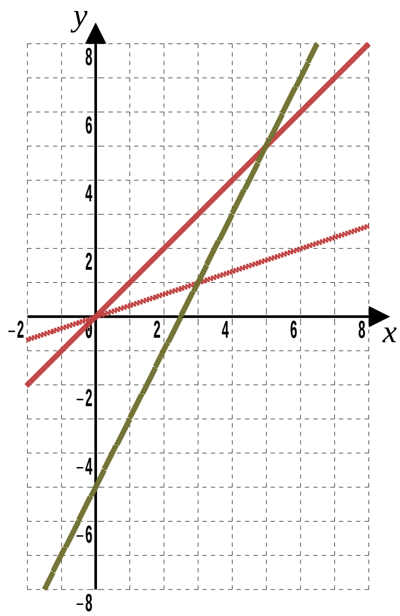
is such a double straight line graph.

The “more is less” hyperbola that you know and love,

$$y = \frac{k}{x} \quad \text{for } k \text{ a constant}$$

has been rotated and squashed. If anything, it's amazing that after such a mangling the resulting equation is not far more complicated.

Here is an accurate graph of the double straight line $x^2 - 4xy + 3y^2 = 0$ in in pink, with the straight line $y = 2x - 5$ in green.



From this graph you can see the two points of intersection.

They are (3, 1) and (5, 5) : Did you get them from your algebra ?

If you need help sorting out the algebra, take a look at section 14.4 which presents a detailed worked solution. (It's after the 14.3 Exercise)

7.3 Exercise

Question 1

GCSE Examination Question from January 2020, Paper 2H, Q22

The line with equation $y = x + 2$ intersects the curve with equation $x^2 + y^2 - 2y = 24$ at the points *A* and *B*.

Find the coordinates of *A* and *B*.

Show clear algebraic working.

[5 marks]

Question 2

GCSE Examination Question from June 2018, Paper 2HR, Q18

Solve the simultaneous equations,

$$2x^2 + 3y^2 = 14$$

$$x = 2y - 3$$

Show clear algebraic working,

[5 marks]

Question 3

GCSE Question from May 2018, Paper 1H, Q11(b)

Solve $3x^2 + 6x - 5 = 0$

Show your working clearly

Give your solutions correct to 3 significant figures.

[3 marks]

Question 4

GCSE Examination Question from June 2019, Paper 2H, Q20

The equation of the line **L** is $y = 9 - x$

The equation of the curve C is $x^2 - 3xy + 2y^2 = 0$

L and **C** intersect at two points.

Find the coordinates of those two points.

Show clear algebraic working.

[5 marks]

Question 5

IGCSE Examination Question from January 2023, Paper 2HR, Q22, Edexcel

Solve the simultaneous equations,

$$2y^2 + x^2 = -6x + 42$$

$$2x + y = -3$$

Show clear algebraic working.

[5 marks]

7.4 The 7.2 Example's Worked Solution

The Question

Use algebra to solve the following equations simultaneously,

$$x^2 - 4xy + 3y^2 = 0$$

$$y = 2x - 5$$

The Answer

$$x^2 - 4x(2x - 5) + 3\left[(2x - 5)^2\right] = 0$$

$$x^2 - 8x^2 + 20x + 3\left[4x^2 - 20x + 25\right] = 0$$

$$x^2 - 8x^2 + 20x + 12x^2 - 60x + 75 = 0$$

$$5x^2 - 40x + 75 = 0$$

Divide through by 5

$$x^2 - 8x + 15 = 0$$

$$(x - 3)(x - 5) = 0$$

Either $x - 3 = 0$ or $x - 5 = 0$

$x = 3$ or $x = 5$

Use $y = 2x - 5$ to get the y-part of the coordinates

Answer : Points of intersection are $(3, 1)$ or $(5, 5)$

Answer 1

$$x^2 + y^2 - 2y = 24$$

$$y = x + 2$$

METHOD 1

$$x^2 + \left[(x + 2)^2 \right] - 2(x + 2) = 24$$

$$x^2 + \left[x^2 + 4x + 4 \right] - 2x - 4 = 24$$

$$x^2 + x^2 + 4x + 4 - 2x - 4 = 24$$

$$2x^2 + 2x - 24 = 0$$

Divide through by 2

$$x^2 + x - 12 = 0$$

$$(x + 4)(x - 3) = 0$$

Either $x + 4 = 0$ or $x - 3 = 0$

$x = -4$ or $x = 3$

Use $y = x + 2$ to get the *y*-part of the coordinates

Answer : Points of intersection are $(-4, -2)$ or $(3, 5)$

METHOD 2

$$\left[(y - 2)^2 \right] + y^2 - 2y = 24$$

$$y^2 - 4y + 4 + y^2 - 2y = 24$$

$$2y^2 - 6y - 20 = 0$$

Divide through by 2

$$y^2 - 3y - 10 = 0$$

$$(y - 5)(y + 2) = 0$$

Either $y - 5 = 0$ or $y + 2 = 0$

$y = 5$ or $y = -2$

Use $x = y - 2$ to get the *y*-part of the coordinates

Answer : Points of intersection are $(-4, -2)$ or $(3, 5)$

Answer 2

$$2x^2 + 3y^2 = 14$$
$$x = 2y - 3$$

$$2\left[\,(2y - 3)^2\,\right] + 3y^2 = 14$$
$$2\left[4y^2 - 12y + 9\right] + 3y^2 = 14$$
$$8y^2 - 24y + 18 + 3y^2 - 14 = 0$$
$$11y^2 - 24y + 4 = 0$$
$$(11y - 2)(y - 2) = 0$$
$$\text{Either } 11y - 2 = 0 \text{ or } y - 2 = 0$$
$$y = \frac{2}{11} \text{ or } y = 2$$

Use $x = 2y - 3$ to get the y-part of the coordinates

Answer : Points of intersection are $\left(-\frac{29}{11}, \frac{2}{11}\right)$ or $(1, 2)$

Answer 3

$$3x^2 + 6x - 5 = 0$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

with $a = 3$, $b = 6$ and $c = -5$ gives

$$x = \frac{-6 \pm \sqrt{6^2 - 4 \times 3 \times (-5)}}{2 \times 3}$$
$$= \frac{-6 \pm \sqrt{36 + 60}}{6}$$
$$= \frac{-6 \pm \sqrt{96}}{6}$$
$$= \frac{-6}{6} \pm \frac{\sqrt{16 \times 6}}{6}$$
$$= -1 \pm \frac{4\sqrt{6}}{6}$$
$$= -1 \pm \frac{2\sqrt{6}}{3}$$

To 3 significant figures the solutions are -2.63 or 0.633

Answer 4

$$x^2 - 3xy + 2y^2 = 0$$
$$y = 9 - x$$

$$x^2 - 3x(9 - x) + 2\left[(9 - x)^2\right] = 0$$
$$x^2 - 27x + 3x^2 + 2\left[81 - 18x + x^2\right] = 0$$
$$x^2 - 27x + 3x^2 + 162 - 36x + 2x^2 = 0$$
$$6x^2 - 63x + 162 = 0$$
$$2x^2 - 21x + 54 = 0$$
$$(x - 6)(2x - 9) = 0$$

Either $x - 6 = 0$ or $2x - 9 = 0$

$$x = 6 \text{ or } x = \frac{9}{2}$$

Use $y = 9 - x$ to get the y-part of the coordinates

Answer : Points of intersection are $(6, 3)$ or $(4.5, 4.5)$

Answer 5

$$2x + y = -3 \Leftrightarrow y = -2x - 3$$
$$y = (-1)(2x + 3)$$
$$y^2 = (2x + 3)^2$$

$$2y^2 + x^2 = -6x + 42$$
$$2(2x + 3)^2 + x^2 = -6x + 42$$
$$2\left(4x^2 + 12x + 9\right) + x^2 = -6x + 42$$
$$8x^2 + 24x + 18 + x^2 + 6x - 42 = 0$$
$$9x^2 + 30x - 24 = 0$$
$$3x^2 + 10x - 8 = 0$$

$(3x - 2)(x + 4) = 0$ (Or use the Q formula)

Either $x = \dfrac{2}{3}$ or $x = -4$

Use $y = (-1)(2x + 3$ to get the y-part of the coordinates

Points of intersection are $\left(\dfrac{2}{3}, -\dfrac{13}{3}\right)$ or $(-4, 5)$

8.1 Test

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 40

Question 1

The Q Formula

A quadratic equation that is written in the form

$$a x^2 + bx + c = 0 \qquad \text{where } a, b \text{ and } c \text{ are constants}$$

has real solutions, if any exist, given by the formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Show how to use the Q formula to solve the quadratic equation,

$$2x^2 + 10x + 5 = 0$$

and hence that $x = -\frac{5}{2} \pm \frac{\sqrt{15}}{2}$ are the two solutions.

[4 marks]

Question 2

GCSE Examination Question from May 2019, Paper 1HR, Q22

Solve the simultaneous equations

$$2x^2 + 3y^2 = 5$$

$$y = 2x + 1$$

Show clear algebraic working

[5 marks]

Question 3

GCSE Examination from June 2019, Paper 2HR, Q5

Factorise $x^2 - 5x - 36$

[2 marks]

Question 4

GCSE Examination Question from June 2019, Paper 2H, Q9

Solve the simultaneous equations,

$$x + 2y = -0.5$$

$$3x - y = 16$$

Show clear algebraic working.

[3 marks]

Question 5

GCSE Examination Question from May 2019, Paper 1H, Q12(a)

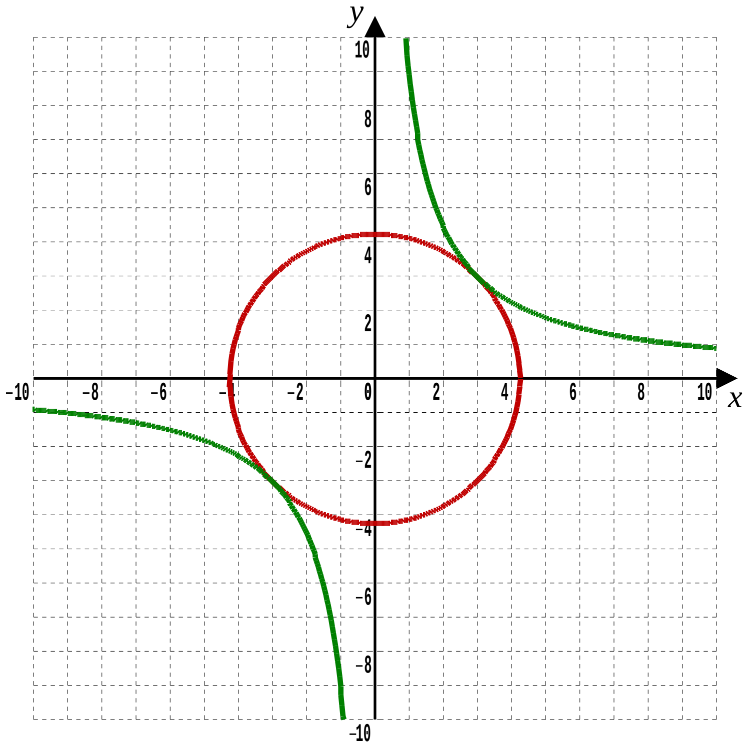
Factorise $2x^2 - 7x + 6$

[2 marks]

Question 6

Vanessa has plotted a graph of a circle and a hyperbola,

Red Circle : $x^2 + y^2 = 24$ Green Hyperbola : $y = \dfrac{12}{x}$



Vanessa is not sure if the circle and hyperbola intersect or, they do, if they intersect in one, two, three or four points. Use algebra to find all points (if any) of intersection.

[8 marks]

Question 7

National 5 Examination Question from May 2022, Paper 2, Q7, SQA

Solve the equation $4x^2 + 2x - 7 = 0$

Give your answer correct to 2 significant figures.

[4 marks]

Question 8

GCSE Examination Question from January 2019, Paper 2H, Q15

The diagram shows a trapezium.

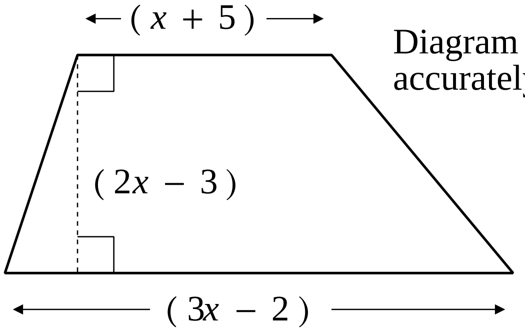


Diagram NOT accurately drawn

All measurements shown on the diagram are in centimetres.

The area of the trapezium is 133 cm^2

(a) Show that $8x^2 - 6x - 275 = 0$

[3 marks]

(b) Find the value of x

Show your working clearly

[3 marks]

Question 9

GCSE Examination Question from January 2019, Paper 2HR, Q21

The curve with equation $y = (10x - 3)(x + 1)$ and the line with equation $y - 6x = 0$ intersect at the points A and B

Find the coordinates of the midpoint of AB

Show your working clearly.

[6 marks]

8.2 Answers

Answer 1

$$2x^2 + 10x + 5 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

with $a = 2,\ b = 10$ and $c = 5$ gives

$$x = \frac{-10 \pm \sqrt{10^2 - 4 \times 2 \times 5}}{2 \times 2}$$

$$= \frac{-10 \pm \sqrt{100 - 40}}{4}$$

$$= \frac{-10 \pm \sqrt{60}}{4}$$

$$= \frac{-10}{4} \pm \frac{\sqrt{4 \times 15}}{4}$$

$$= -\frac{10}{4} \pm \frac{2\sqrt{15}}{4}$$

$$= -\frac{5}{2} \pm \frac{\sqrt{15}}{2} \qquad \square$$

Answer 2

$$2x^2 + 3y^2 = 5$$

$$y = 2x + 1$$

$$2x^2 + 3\left[(2x + 1)^2\right] = 5$$

$$2x^2 + 3\left[4x^2 + 4x + 1\right] = 5$$

$$2x^2 + 12x^2 + 12x + 3 - 5 = 0$$

$$14x^2 + 12x - 2 = 0$$

Divide through by 2

$$7x^2 + 6x - 1 = 0$$

$$(7x - 1)(x + 1) = 0$$

Either $7x - 1 = 0$ or $x + 1 = 0$

$$x = \frac{1}{7} \text{ or } x = -1$$

Use $y = 2x + 1$ to get the y-part of the coordinates

Answer : Points of intersection are $\left(\frac{1}{7}, \frac{9}{7}\right)$ and $(-1, -1)$

Answer 3

$x^2 - 5x - 36 = (x - 9)(x + 4)$

Answer 4

$x + 2y = -0.5$

$3x - y = 16$

$$\begin{aligned} x + 2y &= -0.5 \\ 6x - 2y &= 32 \qquad \text{ADD} \\ \hline 7x &= 31.5 \\ x &= 4.5 \qquad \therefore y = -2.5 \end{aligned}$$

Answer 5

$2x^2 - 7x + 6 = (2x - 3)(x - 2)$

Answer 6

Square both sides of the equation of the hyperbola,

$$y = \frac{12}{x} \Rightarrow y^2 = \frac{144}{x^2}$$

Then substitute this in place of the y^2 in the equation of the circle,

$$x^2 + \left(\frac{144}{x^2}\right) = 24$$

Multiply through by x^2

$$x^2 \times x^2 + \frac{x^2}{1} \times \left(\frac{144}{x^2}\right) = x^2 \times 24$$

$$x^4 + 144 = 24x^2$$

$$x^4 - 24x^2 + 144 = 0$$

$$\text{Let } z = x^2 \Rightarrow z^2 = x^4$$

$$z^2 - 24z + 144 = 0$$

$$(z - 12)(z - 12) = 0$$

$$\therefore \quad z - 12 = 0 \quad \Rightarrow z = 12$$

$$\therefore \quad x^2 = 12 \quad \Rightarrow x = \pm\sqrt{12}$$

Use the equation for the hyperbola to get the y part of the points,

$$y = \frac{12}{x}$$

The two points of intersection are,

$$\left(-\sqrt{12},\ -\sqrt{12}\right),\ \left(\sqrt{12},\ \sqrt{12}\right)$$

Answer 7

$$4x^2 + 2x - 7 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

with $a = 4$, $b = 2$ and $c = -7$ gives

$$x = \frac{-2 \pm \sqrt{2^2 - 4 \times 4 \times (-7)}}{2 \times 4}$$

$$= \frac{-2 \pm \sqrt{4 + 112}}{8}$$

$$= \frac{-2 \pm \sqrt{116}}{8}$$

$$= \frac{-2}{8} \pm \frac{\sqrt{4 \times 29}}{8}$$

$$= -\frac{1}{4} \pm \frac{\sqrt{29}}{4}$$

To 2 significant figures the solutions are 1.1 or -1.6

Answer 8

(a)

Area of a trapezium is half the sum of the parallel sides time the distance in between

$$\therefore \quad 133 = \frac{1}{2}\left((3x - 2) + (x + 5) \right) \times (2x - 3)$$

$$2 \times 133 = (4x + 3) \times (2x - 3)$$

$$266 = 8x^2 - 12x + 6x - 9$$

$$8x^2 - 6x - 9 - 266 = 0$$

$$8x^2 - 6x - 275 = 0 \qquad \square$$

(b)

Guessing the brackets is not impossible but also not easy !

Possibly it's quicker to use the Q formula, being careful with minus signs...

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

with $a = 8$, $b = -6$ and $c = -275$ gives

$$x = \frac{6 \pm \sqrt{6^2 - 4 \times 8 \times (-275)}}{2 \times 8}$$

$$= \frac{6 \pm \sqrt{36 + 8800}}{16}$$

$$= \frac{6 \pm \sqrt{8836}}{16}$$

$$= \frac{6 \pm 94}{16}$$

$$= -\frac{11}{2}, \ \frac{25}{4}$$

Only the positive value of x makes sense in the context of the question

Note : The factorisation into brackets would be $(2x + 11) (4x - 25)$

Answer 9

$$(10x - 3)(x + 1) = 6x$$

$$10x^2 + 10x - 3x - 3 - 6x = 0$$

$$10x^2 + x - 3 = 0$$

$$(5x + 3)(2x - 1) = 0$$

Either $5x + 3 = 0$ or $2x - 1 = 0$

$$x = -\frac{3}{5} \quad \text{or} \quad x = \frac{1}{2}$$

Use $y = 6x$ to get the y-part of the coordinates

Points of intersection are $\left(-\frac{3}{5}, -\frac{18}{5}\right)$ or $\left(\frac{1}{2}, 3\right)$

or, as decimals, $(-0.6, -3.6)$ or $(0.5, 3)$

Midpoint $\left(\frac{-0.6 + 0.5}{2}, \frac{-3.6 + 3}{2}\right) = (-0.05, -0.3)$