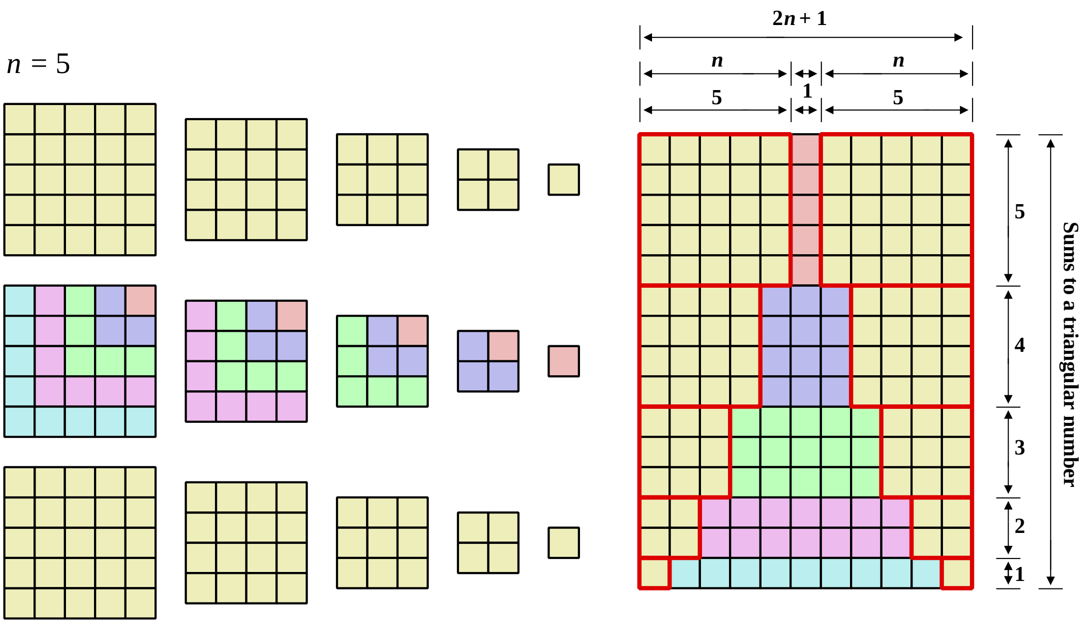
3.4 Answers

3.4.1 Solution to 3.2 Teaching Video Proof

$n = 5$



$$3 \sum_{r=1}^{n} r^2 = \frac{1}{2} n (n+1)(2n+1)$$

$$\therefore \sum_{r=1}^{n} r^2 = \frac{1}{6} n (n+1)(2n+1)$$

[6 marks]

3.4.2 Solution to 3.3 Exercise

Answer 1

(i)

$$\sum_{1}^{n} r^2 = \frac{1}{6} n (n + 1)(2n + 1)$$

$$\sum_{1}^{10} r^2 = \frac{1}{6} \times 10 \times 11 \times 21$$

$$= 385$$

[2 marks]

(ii)

$$\sum_{1}^{n} r^2 = \frac{1}{6} n (n + 1)(2n + 1)$$

$$\sum_{r=25}^{50} r^2 = \sum_{1}^{50} r^2 - \sum_{1}^{24} r^2$$

$$= \frac{1}{6} \times 50 \times 51 \times 101 - \frac{1}{6} \times 24 \times 25 \times 49$$

$$= 42925 - 4900$$

$$= 38025$$

[2 marks]

Answer 2

(i)

$$\text{LHS} = \sum_{r=n+1}^{2n} r^2$$

$$= \sum_{1}^{2n} r^2 - \sum_{1}^{n} r^2$$

$$= \frac{1}{6} \times 2n (2n + 1)(4n + 1) - \frac{1}{6} \times n (n + 1)(2n + 1)$$

$$= \left(\frac{n (2n + 1)}{6} \right) \left(2 (4n + 1) - (n + 1) \right)$$

$$= \frac{1}{6} n (2n + 1)(7n + 1)$$

$$= \text{RHS} \qquad \square$$

[4 marks]

(ii) Use the part (i) result with $n = 10$

$$\sum_{r=11}^{20} r^2 = \frac{1}{6} \times 10 \times 21 \times 71$$

$$= 2485$$

[3 marks]

Answer 3

(i)

$$\begin{aligned}
\text{LHS} &= \sum_{r=1}^{n}\left(r^2 + r - 2\right) \\
&= \sum_{1}^{n} r^2 + \sum_{1}^{n} r - 2\sum_{1}^{n} 1 \\
&= \frac{1}{6} \times n\,(n+1)\,(2n+1) + \frac{1}{2} \times n\,(n+1) - 2n \\
&= \left(\frac{n}{6}\right)\left(\,(n+1)\,(2n+1) + 3(n+1) - 12\right) \\
&= \left(\frac{n}{6}\right)\left(2n^2 + 3n + 1 + 3n + 3 - 12\right) \\
&= \left(\frac{n}{6}\right)\left(2n^2 + 6n - 8\right) \\
&= \left(\frac{n}{3}\right)\left(n^2 + 3n - 4\right) \\
&= \frac{1}{3}\,n\,(n+4)\,(n-1) \\
&= \text{RHS} \qquad \square
\end{aligned}$$

[4 marks]

(ii) To find the number of terms,

$$n^2 + n - 2 = 418$$

$$n^2 + n - 420 = 0$$

$$\begin{bmatrix} FACT\{420\} = 2 \times 2 \times 3 \times 5 \times 7 \\ \sqrt{420} = 20.5 \end{bmatrix}$$

$$(n+21)\,(n-20) = 0$$

Either $n = -21$ (reject) or $n = 20$

Use the part (i) formula with $n = 20$

$$\begin{aligned}
0 + 4 + 10 + 18 + \dots + 418 &= \sum_{1}^{20}\left(r^2 + r - 2\right) \\
&= \frac{1}{3} \times 20 \times 24 \times 19 \\
&= 3040
\end{aligned}$$

[3 marks]

Answer 4

(a)

$$\begin{aligned}
\text{LHS} &= \sum_{r=1}^{n} \left(9r^2 - 4r \right) \\
&= 9\sum_{1}^{n} r^2 - 4\sum_{1}^{n} r \\
&= 9 \times \frac{1}{6} \times n(n+1)(2n+1) - 4 \times \frac{1}{2} n(n+1) \\
&= \left(\frac{n(n+1)}{6} \right)\left(9(2n+1) - 12 \right) \\
&= \left(\frac{n(n+1)}{6} \right)\left(18n - 3 \right) \\
&= \frac{1}{2} n(n+1)(6n-1) \\
&= \text{RHS} \qquad \square
\end{aligned}$$

[4 marks]

(b)

$$\begin{aligned}
\sum_{r=1}^{12} \left(9r^2 - 4r + k\left(2^r\right) \right) &= 6630 \\
\sum_{1}^{12} \left(9r^2 - 4r \right) + k\sum_{1}^{12} \left(2^r \right) &= 6630 \\
\frac{1}{2} \times 12 \times 13 \times 71 + k\left(2^1 + 2^2 + \ldots + 2^{12} \right) &= 6630 \\
5538 + k \times \frac{2\left(2^{12} - 1\right)}{2 - 1} &= 6630 \\
k \times \frac{2 \times 4095}{1} &= 1092 \\
k &= \frac{2}{15}
\end{aligned}$$

[4 marks]

Answer 5

(a)

$$\text{LHS} = \sum_{r=1}^{n} (2r + 5)^2$$

$$= \sum_{1}^{n} \left(4r^2 + 20r + 25\right)$$

$$= 4 \sum_{1}^{n} r^2 + 20 \sum_{1}^{n} r + 25 \sum_{1}^{n} 1$$

$$= 4 \times \frac{1}{6} \times n (n + 1)(2n + 1) + 20 \times \frac{1}{2} n (n + 1) + 25n$$

$$= \left(\frac{n}{6}\right)\left(4 (n + 1)(2n + 1) + 60 (n + 1) + 150\right)$$

$$= \left(\frac{n}{6}\right)\left(8n^2 + 12n + 4 + 60n + 60 + 150\right)$$

$$= \left(\frac{n}{6}\right)\left(8n^2 + 72n + 214\right)$$

$$= \frac{n}{3}\left(4n^2 + 36n + 107\right)$$

$$= \frac{n}{3}\left(4\left[n^2 + 9n\right] + 107\right)$$

$$= \frac{n}{3}\left(4\left[\left(n + \frac{9}{2}\right)^2 - \frac{81}{4}\right] + 107\right)$$

$$= \frac{n}{3}\left(4\left(n + \frac{9}{2}\right)^2 - 81 + 107\right)$$

$$= \frac{n}{3}\left((2n + 9)^2 + 26\right)$$

$$\therefore\ a = 2, \quad b = 9 \ \text{and} \ c = 26$$

[5 marks]

(b)

$$\sum_{r=0}^{100} (2r + 5)^2 = \sum_{1}^{100} (2r + 5)^2 + 5^2$$

$$= \frac{100}{3}\left((2 \times 100 + 9)^2 + 26\right) + 5^2$$

$$= 1456925$$

[2 marks]

Answer 6

(i) The series of triangular numbers to be summed is,

$$1 + 3 + 6 + 10 + 15 + 21 + ... + \frac{1}{2} n (n + 1)$$

$$= \sum_{1}^{n} \frac{1}{2} r (r + 1)$$

$$= \frac{1}{2} \sum_{1}^{n} r^2 + \frac{1}{2} \sum_{1}^{n} r$$

$$= \frac{1}{2} \times \frac{1}{6} n (n + 1) (2n + 1) + \frac{1}{2} \times \frac{1}{2} n (n + 1)$$

$$= \left(\frac{n (n + 1)}{12} \right) \left((2n + 1) + 3 \right)$$

$$= \frac{n (n + 1) (2n + 4)}{12}$$

$$= \frac{n (n + 1) (n + 2)}{6}$$

[5 marks]

(ii) The photograph corresponds to the case $n = 4$

$$\frac{4 \times 5 \times 6}{6} = 20 \text{ tins}$$

[2 marks]