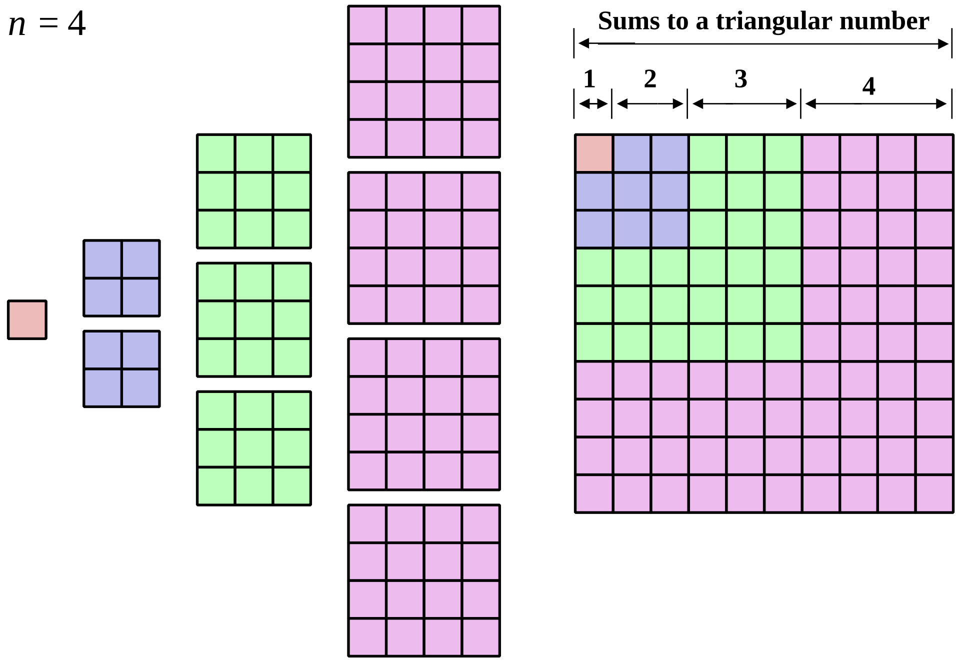
4.5 Answers

4.5.1 Solution to the Teaching Video Proof



$$\sum_{1}^{n} r^3 = \left(\sum_{1}^{n} r \right)^2$$

$$= \frac{1}{4} n^2 (n+1)^2$$

[6 marks]

4.5.2 Solutions to 4.4 Exercise

Answer 1

(i)

$$\sum_{1}^{n} r^3 = \frac{1}{4} n^2 \, (n + 1)^2$$

$$\sum_{1}^{12} r^3 = \frac{1}{4} \times 12^2 \times 13^2$$

$$= 6084$$

[2 marks]

(ii)

$$\sum_{r=50}^{75} r^3 = \sum_{1}^{75} r^3 - \sum_{1}^{49} r^3$$

$$= \frac{1}{4} \times 75^2 \times 76^2 - \frac{1}{4} \times 49^2 \times 50^2$$

$$= 8122500 - 1500625$$

$$= 6621875$$

[2 marks]

Answer 2

(i)

$$\text{LHS} = \sum_{r=n+1}^{3n} r^3$$

$$= \sum_{1}^{3n} r^3 - \sum_{1}^{n} r^3$$

$$= \frac{1}{4}(3n)^2(3n+1)^2 - \frac{1}{4}n^2(n+1)^2$$

$$= \left(\frac{n^2}{4}\right)\left(9(3n+1)^2 - (n+1)^2\right)$$

$$= \left(\frac{n^2}{4}\right)\left(9\left(9n^2 + 6n + 1\right) - \left(n^2 + 2n + 1\right)\right)$$

$$= \left(\frac{n^2}{4}\right)\left(81n^2 + 54n + 9 - n^2 - 2n - 1\right)$$

$$= \left(\frac{n^2}{4}\right)\left(80n^2 + 52n + 8\right)$$

$$= n^2\left(20n^2 + 13n + 2\right)$$

$$= n^2(4n+1)(5n+2)$$

$$= \text{RHS} \qquad \square$$

[4 marks]

(ii) This is part (a) with $n = 10$

$$\therefore \sum_{r=11}^{30} r^3 = 10^2 \times 41 \times 52$$

$$= 213200$$

[3 marks]

Answer 3

(i)

$$\text{LHS} = \sum_{r=1}^{n} r^2 (r - 1)$$

$$= \sum_{1}^{n} r^3 - \sum_{1}^{n} r^2$$

$$= \frac{1}{4} n^2 (n + 1)^2 - \frac{1}{6} n (n + 1) (2n + 1)$$

$$= \left(\frac{n (n + 1)}{12} \right) \left(3n (n + 1) - 2 (2n + 1) \right)$$

$$= \left(\frac{n (n + 1)}{12} \right) \left(3 n^2 + 3n - 4n - 2 \right)$$

$$= \left(\frac{n (n + 1)}{12} \right) \left(3 n^2 - n - 2 \right)$$

$$= \frac{1}{12} n (n + 1) \left(3 n^2 - n - 2 \right)$$

$$= \frac{1}{12} n (n + 1)(3n + 2)(n - 1)$$

$$= \text{RHS} \qquad \square$$

[4 marks]

(ii) To work out how many terms there are,

$$n^2 (n - 1) = 3840$$

$$FACT \{3840\} = 2^8 \times 3 \times 5$$

$$\therefore \quad 3840 = \left(2^4 \right)^2 \times 3 \times 5$$

$$= 16^2 \times 15$$

That is, $n = 16$

$$0 + 4 + 18 + 48 + 100 + ... + 3840 = \sum_{1}^{16} r^2 (r - 1)$$

$$= \frac{1}{12} \times 16 \times 17 \times 50 \times 15$$

$$= 17000$$

[3 marks]

Answer 4

(a)

$$\text{LHS} = \sum_{r=1}^{n} \left(r^3 + 6r - 3 \right)$$

$$= \sum_{1}^{n} r^3 + 6 \sum_{1}^{n} r - 3 \sum_{1}^{n} 1$$

$$= \frac{1}{4} n^2 (n + 1)^2 + 6 \times \frac{1}{2} n \, (n + 1) - 3n$$

$$= \left(\frac{n}{4} \right) \left(n (n + 1)^2 + 12 \, (n + 1) - 12 \right)$$

$$= \left(\frac{n}{4} \right) \left(n \left(n^2 + 2n + 1 \right) + 12n + 12 - 12 \right)$$

$$= \left(\frac{n^2}{4} \right) \left(n^2 + 2n + 1 + 12 \right)$$

$$= \left(\frac{n^2}{4} \right) \left(n^2 + 2n + 13 \right)$$

$$= \frac{1}{4} n^2 \left(n^2 + 2n + 13 \right)$$

$$= \text{RHS} \qquad \square$$

[5 marks]

(b)

$$\sum_{r=16}^{30} \left(r^3 + 6r - 3 \right) = \sum_{1}^{30} \left(r^3 + 6r - 3 \right) - \sum_{1}^{15} \left(r^3 + 6r - 3 \right)$$

$$= \frac{1}{4} \times 30^2 \times 973 - \frac{1}{4} \times 15^2 \times 268$$

$$= 218925 - 15075$$

$$= 203850$$

[2 marks]

Answer 5

(a)

$$\text{LHS} = \sum_{r=1}^{n} r\left(r^2 - 3\right)$$

$$= \sum_{1}^{n} r^3 - 3\sum_{1}^{n} r$$

$$= \frac{1}{4} n^2 (n+1)^2 - 3 \times \frac{1}{2} n (n+1)$$

$$= \left(\frac{n(n+1)}{4}\right)\left(n(n+1) - 6\right)$$

$$= \left(\frac{n(n+1)}{4}\right)\left(n^2 + n - 6\right)$$

$$= \frac{1}{4} n (n+1)(n+3)(n-2)$$

$$= \text{RHS} \qquad \square$$

[5 marks]

(b)

$$\sum_{10}^{50} r\left(r^2 - 3\right) = \sum_{1}^{50} r\left(r^2 - 3\right) - \sum_{1}^{9} r\left(r^2 - 3\right)$$

$$= \frac{1}{4} \times 50 \times 51 \times 53 \times 48 - \frac{1}{4} \times 9 \times 10 \times 12 \times 7$$

$$= 1621800 - 1890$$

$$= 1619910$$

[3 marks]

Answer 6

(i) **(c)** $3T_n + T_{n-1} = T_{2n}$

[2 marks]

(ii)

$$\begin{aligned} \text{LHS} &= 3T_n + T_{n-1} \\ &= 3\sum_1^n r + \sum_1^{n-1} r \\ &= 3 \times \frac{1}{2} \times n(n+1) + \frac{1}{2} \times (n-1)n \\ &= \left(\frac{n}{2}\right)\left(3(n+1) + (n-1)\right) \\ &= \left(\frac{n}{2}\right)\left(3n + 3 + n - 1\right) \\ &= \frac{1}{2}n(4n+2) \\ &= \frac{1}{2}(2n)(2n+1) \\ &= T_{2n} \\ &= \text{RHS} \qquad \square \end{aligned}$$

[5 marks]