5.5 Answers

Answer 1

(i) and **(ii)**

The n^{th} red+white square will have a side of length $2n + 3$

From this is subtracted 4 white squares of side length n

$$(2n + 3)^2 - 4n^2 = 4n^2 + 12n + 9 - 4n^2$$
$$= 12n + 9$$

Term	1	2	3	4	...	n
Red Frame	21	33	45	57	...	$12n + 9$

[6 marks]

(iii) and **(iv)**

Term	1	2	3	4	...	n
Cumulative	21	54	99	156	...	
Rectangle	3×7	6×9	9×11	12×13	...	$(3n)(2n + 5)$

[6 marks]

(v) $\displaystyle\sum_{1}^{n} (12r + 9) = 3n(2n + 5)$

[3 marks]

(vi)

$$\text{LHS} = \sum_{1}^{n} (12n + 9)$$
$$= 12\sum_{1}^{n} r + 9\sum_{1}^{n} 1$$
$$= 12 \times \frac{1}{2} \times n(n + 1) + 9n$$
$$= 6n^2 + 6n + 9n$$
$$= 6n^2 + 15n$$
$$= (3n)(2n + 5)$$
$$= \text{RHS} \qquad \square$$

[4 marks]

Answer 2

(i)

$$\begin{aligned}
\text{LHS} &= \sum_{1}^{n} (4r - 3)^2 \\
&= \sum_{1}^{n} \left(16r^2 - 24r + 9\right) \\
&= 16\sum_{1}^{n} r^2 - 24\sum_{1}^{n} r + 9\sum_{1}^{n} 1 \\
&= 16 \times \frac{1}{6} n(n+1)(2n+1) - 24 \times \frac{1}{2} n(n+1) + 9n \\
&= \left(\frac{n}{3}\right)\left(8(n+1)(2n+1) - 36(n+1) + 27\right) \\
&= \left(\frac{n}{3}\right)\left(16n^2 + 24n + 8 - 36n - 36 + 27\right) \\
&= \frac{1}{3} n \left(16n^2 - 12n - 1\right) \\
&= \text{RHS} \qquad \square
\end{aligned}$$

[4 marks]

(ii)

$$\begin{aligned}
\text{LHS} &= \sum_{2}^{n} (4r - 5)^2 \\
&= \sum_{1}^{n} (4r - 5)^2 - 1 \\
&= \sum_{1}^{n} \left(16r^2 - 40r + 25\right) - 1 \\
&= 16\sum_{1}^{n} r^2 - 40\sum_{1}^{n} r + 25\sum_{1}^{n} 1 - 1 \\
&= 16 \times \frac{1}{6} n(n+1)(2n+1) - 40 \times \frac{1}{2} n(n+1) + 25n - 1 \\
&= \left(\frac{n}{3}\right)\left(8(n+1)(2n+1) - 60(n+1) + 75\right) - 1 \\
&= \left(\frac{n}{3}\right)\left(16n^2 + 24n + 8 - 60n - 60 + 75\right) - 1 \\
&= \frac{1}{3} n \left(16n^2 - 36n + 23\right) - 1 \\
&= \text{RHS} \qquad \square
\end{aligned}$$

[4 marks]

(iii)

$$\sum_{1}^{3}(4r-3)^2-\sum_{2}^{3}(4r-5)^2 = 1^2+5^2+9^2-3^2-7^2$$

$$= 1+25+81-9-49$$

$$= 49$$

[2 marks]

(iv) The outer frame, a square to side length 9, gives rise to a large square of area 9^2. From this could be subtracted a square of side length 7, and so the outer frame contains 9^2-7^2 individual red squares.
Similarly, the central frame contains 5^2-3^2 red squares.
Plus the single red square at the centre, 1^2

Total area of red frame $= \left(9^2-7^2\right)+\left(5^2-3^2\right)+1^2$

$$= 1^2+5^2+9^2-3^2-7^2$$

$$= \sum_{1}^{3}(4r-3)^2-\sum_{2}^{3}(4r-5)^2 \qquad \square$$

[4 marks]

(v)

$$\text{LHS} = \sum_{1}^{n}(4r-3)^2-\sum_{2}^{n}(4r-5)^2$$

$$= \frac{1}{3}n\left(16n^2-12n-1\right)-\left(\frac{1}{3}n\left(16n^2-36n+23\right)-1\right)$$

$$= \frac{1}{3}n\left(16n^2-12n-1-16n^2+36n-23\right)+1$$

$$= \frac{1}{3}n\left(24n-24\right)+1$$

$$= 8n(n-1)+1$$

$$= 8n^2-8n+1$$

$$= \text{RHS} \qquad \square$$

[2 marks]

(vi)

$$\sum_{1}^{n}\left(8r^2 - 8r + 1 \right)$$

$$= 8\sum_{1}^{n} r^2 - 8\sum_{1}^{n} r + \sum_{1}^{n} 1$$

$$= 8 \times \frac{1}{6} \times n\,(n+1)\,(2n+1) - 8 \times \frac{1}{2} \times n\,(n+1) + n$$

$$= \frac{4}{3} n\,(n+1)\,(2n+1) - 4n\,(n+1) + n$$

$$= \left(\frac{n}{3}\right)\left(4(n+1)(2n+1) - 12(n+1) + 3\right)$$

$$= \left(\frac{n}{3}\right)\left(8n^2 + 12n + 4 - 12n - 12 + 3\right)$$

$$= \left(\frac{n}{3}\right)\left(8n^2 - 5\right)$$

When $n = 5$,

$$\left(\frac{n}{3}\right)\left(8n^2 - 5\right) = \left(\frac{5}{3}\right)\left(8 \times 5^2 - 5\right)$$

$$= 325$$

$$= 5 \times 5 \times 13 \qquad \square$$

[5 marks]