6.2 Answers

Answer 1

The result to be proved is that,

$$\sum_{1}^{n}(2r - 1) = n^2$$

$$\begin{aligned}
\text{LHS} &= \sum_{1}^{n}(2r - 1) \\
&= 2\sum_{1}^{n} r - \sum_{1}^{n} 1 \\
&= 2 \times \frac{1}{2} n(n + 1) - n \\
&= n(n + 1) - n \\
&= n^2 + n - n \\
&= n^2 \\
&= \text{RHS} \qquad \square
\end{aligned}$$

[4 marks]

Answer 2

$$\begin{aligned}
\text{LHS} &= \sum_{r=1}^{n} r(r + 3) \\
&= \sum_{r=1}^{n}(r^2 + 3r) \\
&= \sum_{1}^{n} r^2 + 3\sum_{1}^{n} r \\
&= \frac{1}{6} \times n(n + 1)(2n + 1) + 3 \times \frac{1}{2} \times n(n + 1) \\
&= \left(\frac{n(n + 1)}{6}\right)(2n + 1 + 9) \\
&= \left(\frac{n(n + 1)}{6}\right)(2n + 10) \\
&= \frac{n}{3}(n + 1)(n + 5)
\end{aligned}$$

$$\therefore \ a = 3, \quad b = 5$$

[4 marks]

Answer 3

$$\sum_{1}^{n+1}(2r-1)+\sum_{1}^{n}(2r-1)=n^2+(n+1)^2$$

$$\begin{aligned}
\text{LHS} &= \sum_{1}^{n+1}(2r-1)+\sum_{1}^{n}(2r-1)\\
&= 2\sum_{1}^{n+1}r-\sum_{1}^{n+1}1+2\sum_{1}^{n}r-\sum_{1}^{n}1\\
&= 2\times\frac{1}{2}\times(n+1)(n+2)-(n+1)+2\times\frac{1}{2}\times n(n+1)-n\\
&= (n+1)(n+2)-(n+1)+n(n+1)-n\\
&= n^2+3n+2-n-1+n^2+n-n\\
&= 2n^2+2n+1\\
&= n^2+\left(n^2+2n+1\right)\\
&= n^2+(n+1)^2\\
&= \text{RHS} \qquad \square
\end{aligned}$$

[5 marks]

Answer 4

$$\begin{aligned}
\text{RHS} &= (n+1)^2\,T_{n-1}-(n-1)\,T_{n-1}+n\\
&= \left[(n+1)^2-(n-1)\right]T_{n-1}+n\\
&= \left[n^2+2n+1-n+1\right]\times\frac{1}{2}(n-1)n+n\\
&= \left(\frac{1}{2}n\right)\left(\left[n^2+n+2\right]\times(n-1)+2\right)\\
&= \left(\frac{1}{2}n\right)\left(n^3+n^2+2n-n^2-n-2+2\right)\\
&= \left(\frac{1}{2}n\right)\left(n^3+n\right)\\
&= \frac{1}{2}n^2\left(n^2+1\right)\\
&= T_{n^2}\\
&= \text{RHS} \qquad \square
\end{aligned}$$

[5 marks]

Answer 5

(a)

$$\text{LHS} = \sum_{r=1}^{n} \left(3r^2 + 8r + 3 \right)$$

$$= 3 \sum_{1}^{n} r^2 + 8 \sum_{1}^{n} r + 3 \sum_{1}^{n} 1$$

$$= 3 \times \frac{1}{6} \times n\,(n+1)\,(2n+1) + 8 \times \frac{1}{2} \times n\,(n+1) + 3n$$

$$= \left(\frac{n}{2} \right) \left((n+1)(2n+1) + 8(n+1) + 6 \right)$$

$$= \left(\frac{n}{2} \right) \left(2n^2 + 3n + 1 + 8n + 8 + 6 \right)$$

$$= \frac{1}{2} n \left(2n^2 + 11n + 15 \right)$$

$$= \frac{1}{2} n\,(2n+5)\,(n+3)$$

$$= \text{RHS} \qquad \square$$

[5 marks]

(b)

$$\sum_{r=1}^{12} \left(3r^2 + 8r + 3 + k\left(2^{r-1} \right) \right) = 3520$$

$$\sum_{1}^{12} \left(3r^2 + 8r + 3 \right) + k \sum_{1}^{12} \left(2^{r-1} \right) = 3520$$

$$\frac{1}{2} \times 12 \times 29 \times 15 + k\left(2^0 + 2^1 + \ldots + 2^{11} \right) = 3520$$

$$2610 + k \times \frac{2^0\,(2^{12} - 1)}{2 - 1} = 3520$$

$$k \times \frac{1 \times 4095}{1} = 910$$

$$k = \frac{2}{9}$$

[4 marks]

Answer 6

(i) The easiest way to obtain an expression for the number of red squares in any term in the sequence is to note that half of the "square less one" is red. This leads to $\frac{1}{2}\left(n^2 - 1\right)$ but there remains the issue that the first square are is of side length 3, and the side length then increases by 2 with each subsequent term. This leads to the conclusion that the n^{th} square has side length of ($2n + 1$). Putting this all together gives,

$$\begin{aligned}\frac{1}{2}\left((2n+1)^2 - 1\right) &= \frac{1}{2}\left(4n^2 + 4n + 1 - 1\right)\\ &= 2n^2 + 2n\\ &= (2n)(n+1)\end{aligned}$$

This is four times the corresponding triangular number, an observation that will be useful in the last part of the question, although you may like to already see if you can spot the four lots of a triangular number in each term in the photograph.

[3 marks]

(ii)

$$\begin{aligned}\sum_1^n\left(2r^2 + 2r\right) &= 2\sum_1^n r^2 + 2\sum_1^n r\\ &= 2 \times \frac{1}{6}n(n+1)(2n+1) + 2 \times \frac{1}{2}n(n+1)\\ &= \left(\frac{n(n+1)}{3}\right)(2n+1+3)\\ &= \frac{2}{3}n(n+1)(n+2)\end{aligned}$$

[3 marks]

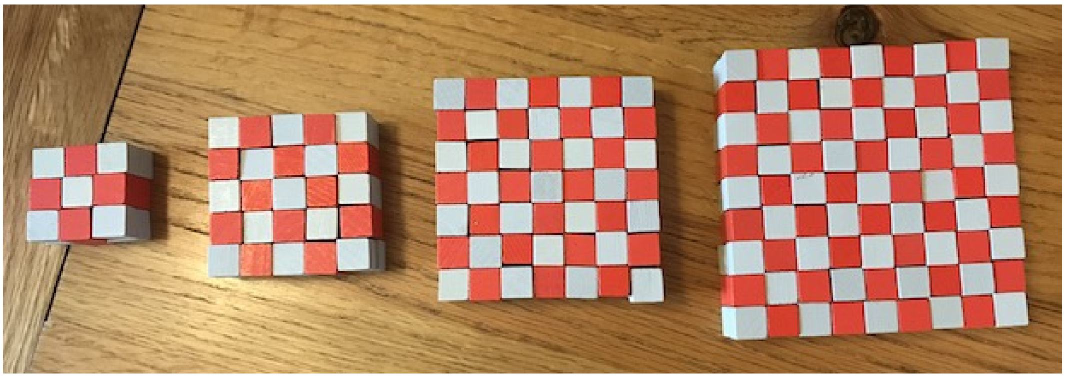
(iii) The sum of the first n triangular numbers is,

$$\sum_1^n \frac{1}{2}r(r+1) = \frac{1}{2}\sum_1^n r^2 + \frac{1}{2}\sum_1^n r$$

$$\begin{aligned}\therefore\ \frac{2}{3}n(n+1)(n+2) &= 4 \times \left(\frac{1}{2}\sum_1^n r^2 + \frac{1}{2}\sum_1^n r\right)\\ &= 2 \times 2 \times \text{Sum of first } n \text{ triangular numbers} \qquad \square\end{aligned}$$

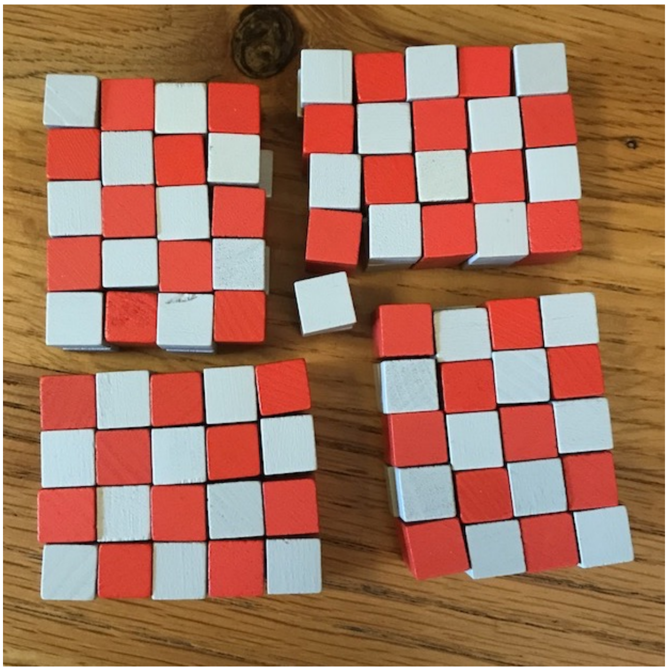
[4 marks]

(iv)



The very first term gives the 2×2 base.
Every subsequent term, n, adds on four times the n^{th} triangular number.
These can be seen in the terms in the photograph.

For example, the fourth term can be partitioned as shown, and by looking carefully at each of the four pieces (ignore the central 1×1 square which is always white) the fourth triangular number $1 + 2 + 3 + 4$ can be seen in the diagonals.



[3 marks]