Further Pure A-Level Mathematics
Compulsory Course Component
Core 1

SERIES

~ AND ~

VISUAL PROOF

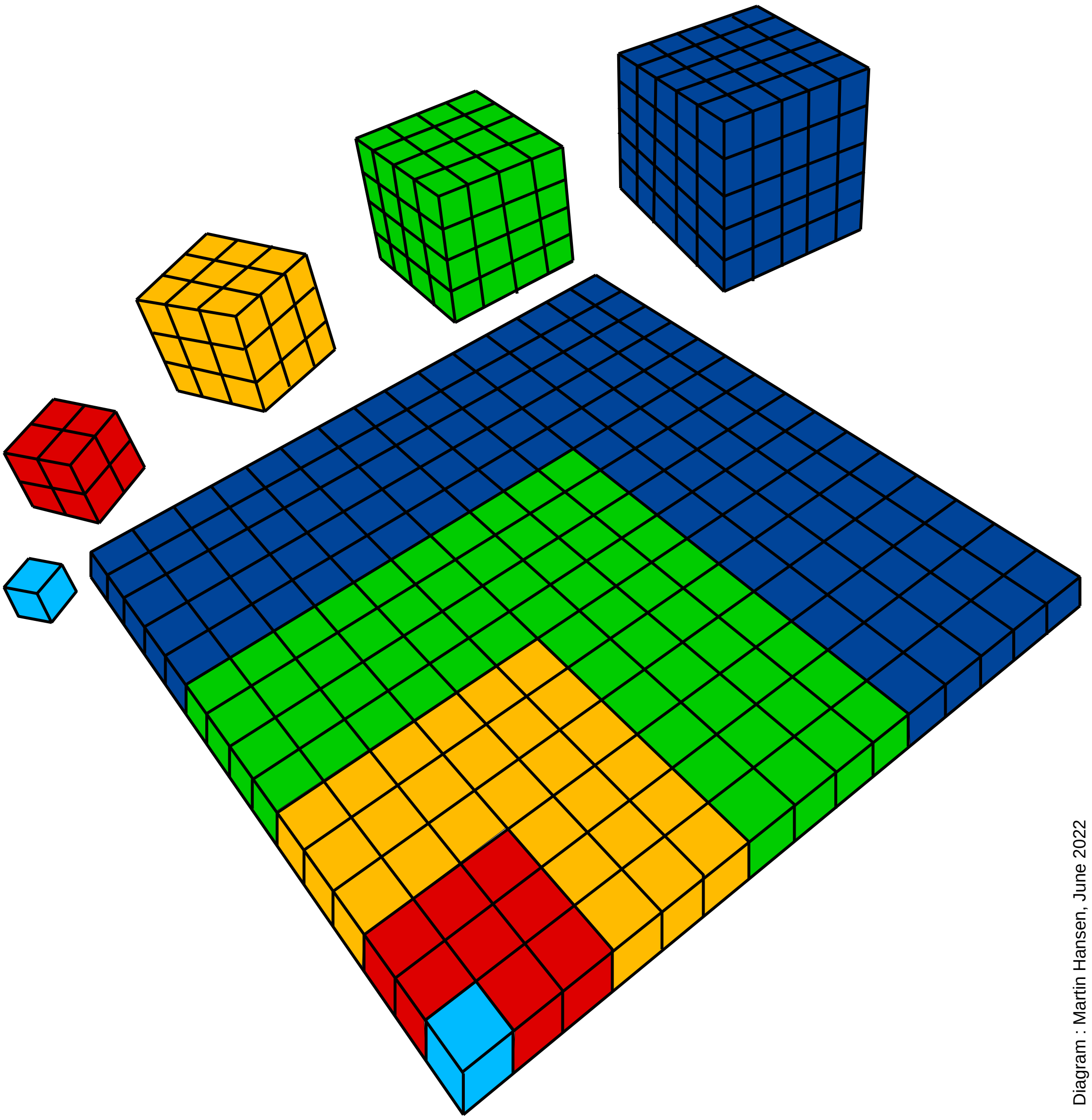


Diagram : Martin Hansen, June 2022

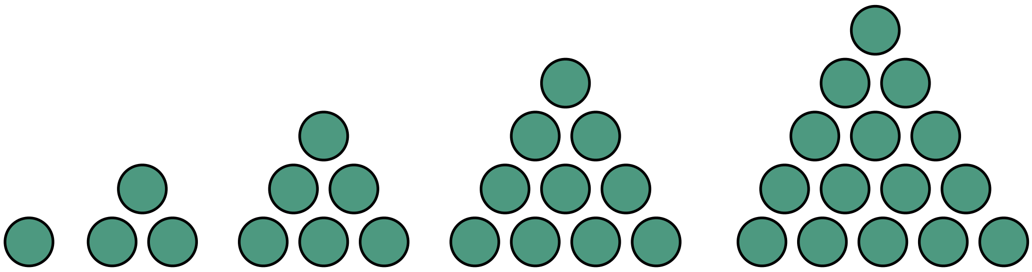
SERIES
AND
VISUAL PROOF

Lesson 1

Further A-Level Pure Mathematics : Core 1
Series and Visual Proof

1.1 Triangular Numbers

The triangular numbers are so called because the start of the sequence, which begins 1, 3, 6, 10, 15, ... can be visualised as dots arranged in equilateral triangles like so;



The sixth triangular number can be described in sigma notation as $\sum_{1}^{6} r$ and this can be found easily enough without drawing the next diagram as,

$$\sum_{1}^{6} r = 1 + 2 + 3 + 4 + 5 + 6$$

$$= 21$$

Notice that an item in a sequence has been found from a series.

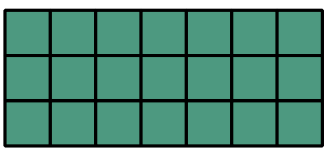
Sequence and Series

Sequence : A list of numbers, repetitions allowed, where order matters.

Series : The summation of a sequence.

To remember which is which : "S, e, q, u, e, n, c, e and S + e + r + i + e + s"

When working with the Natural Numbers† it is useful to associate abutted unit squares with the numbers, such that, for example, 3×7, is associated with a rectangle of height 3 and width 7. The area of the rectangle, 21, is then both the answer to the multiplication, and the number of squares in the rectangle.



† The natural numbers, $\mathbb{N}$, are the set { 1, 2, 3, 4, 5, 6, 7, ... }
Notice that in the A-Level course $\mathbb{N}$ does not include zero, by definition.

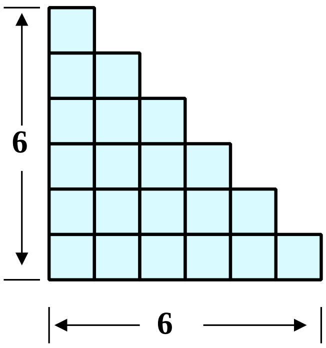
Formula for the n^{th} Triangular Number

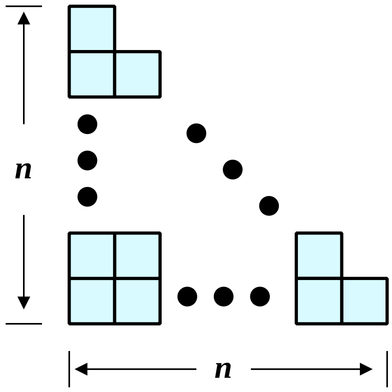
$$\sum_{1}^{n} r = \frac{1}{2} n (n + 1)$$

Proof

Watch the following teaching video then, in the space below, make use of the diagrams provided to write out the proof described.

Teaching Video : http://www.NumberWonder.co.uk/v9092/1.mp4





[6 marks]

1.2 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable.

Marks available : 54

Question 1

Use the appropriate formula to determine the 100^{th} triangular number, $\sum_{1}^{100} r$

[2 marks]

Question 2

Evaluate, $\sum_{1}^{200} r$

[2 marks]

Question 3

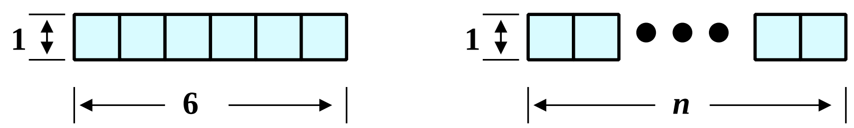
Find the sum of the first 500 natural numbers.

[2 marks]

Question 4

Use the diagram below to visually prove that,

(i) $\sum_{1}^{6} 1 = 6$ **(ii)** $\sum_{1}^{n} 1 = n$



[4 marks]

Question 5

Evaluate **(i)** $\sum_{1}^{50} 1$ **(ii)** $5 \sum_{1}^{23} 1$

[2 marks]

Question 6

(i) By first writing out all six terms in the series, determine the value of,

$$\sum_{1}^{6} (3r - 2)$$

[2 marks]

(ii) By using the appropriate formula determine,

$$3 \sum_{1}^{6} r - 2 \sum_{1}^{6} 1$$

[2 marks]

Question 7

By expanding $\sum_{1}^{14} (4r + 5)$ as $4 \sum_{1}^{14} r + 5 \sum_{1}^{14} 1$ and using appropriate formula, evaluate,

$$\sum_{1}^{14} (4r + 5)$$

[4 marks]

Question 8

Determine the value of $\sum_{1}^{30} (2r + 7)$

[4 marks]

Question 9

By expanding $\sum_{4}^{7} r$ as $\sum_{1}^{7} r - \sum_{1}^{3} r$ and using appropriate formula, evaluate,

$$\sum_{4}^{7} r$$

[4 marks]

Question 10

Determine the value of $\frac{1}{100} \sum_{80}^{120} r$

[4 marks]

Question 11

(i) Show that,

$$\sum_{r=1}^{n} (7r - 4) = \frac{1}{2} n (7n - 1)$$

[4 marks]

(ii) Hence evaluate,

$$\sum_{r=20}^{50} (7r - 4)$$

[4 marks]

Question 12

Given that $\sum_{r=1}^{n} r = 528$, find the value of n

[4 marks]

Question 13

Let the n^{th} triangular number be $T_n = \sum_{1}^{n} r$

(i) Using algebra, prove that $T_n - T_{n-1} = n$

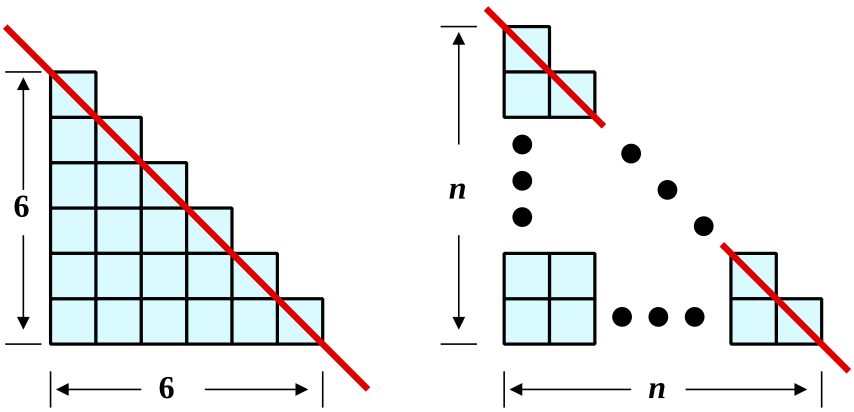
[4 marks]

(ii) By drawing triangles, find a visual proof that $T_n - T_{n-1} = n$

[6 marks]

1.3 Solutions

1.3.1 Answer to Teaching Video Example



The diagrams do most of the work !

For the specific case of the sixth triangular number, the idea is to obtain the answer in a way that can then be generalised,

$$\sum_{1}^{6} r = 1 + 2 + 3 + 4 + 5 + 6 \quad \text{from which the diagram is drawn}$$

$$= \frac{1}{2} \times 6 \times 6 + 6 \times \frac{1}{2} \quad \text{areas from the diagram}$$

$$= 18 + 3$$

$$= 21$$

From which the general case is deduced,

$$\sum_{1}^{n} r = 1 + 2 + \ldots + (n - 1) + n \quad \text{from which the diagram is drawn}$$

$$= \frac{1}{2} \times n \times n + n \times \frac{1}{2} \quad \text{areas from the diagram}$$

$$= \frac{1}{2} n (n + 1) \qquad \square$$

[6 marks]

1.3.2 Answers to 1.2 Exercise

Answer 1

$$\sum_{1}^{100} r = \frac{1}{2} \times 100 \times 101$$

$$= 5050$$

[2 marks]

Answer 2

$$\sum_{1}^{200} r = \frac{1}{2} \times 200 \times 201$$

$$= 20100$$

[2 marks]

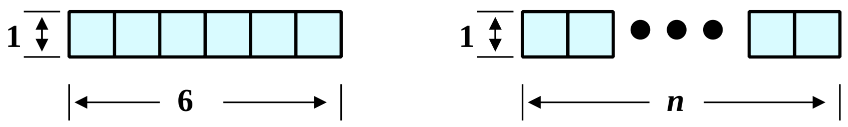
Answer 3

$$\sum_{1}^{500} r = \frac{1}{2} \times 500 \times 501$$

$$= 125250$$

[2 marks]

Answer 4



For the specific case,

$$\sum_{1}^{6} 1 = 1 + 1 + 1 + 1 + 1 + 1 \quad \text{from which the diagram is drawn}$$

$$= 1 \times 6 \quad \text{area from the diagram}$$

$$= 6$$

From which the general case is deduced,

$$\sum_{1}^{n} 1 = 1 + 1 + \ldots + 1 + 1 \quad \text{from which the diagram is drawn}$$

$$= 1 \times n \quad \text{area from the diagram}$$

$$= n \qquad \square$$

[4 marks]

Answer 5

(i) 50

(ii) 115

[2 marks]

Answer 6

(i)

$$\sum_{1}^{6} (3r - 2) = 1 + 4 + 7 + 10 + 13 + 16$$

$$= 51$$

(ii)

$$3 \sum_{1}^{6} r - 2 \sum_{1}^{6} 1 = 3 \times \frac{1}{2} \times 6 \times 7 - 2 \times 6$$

$$= 63 - 12$$

$$= 51$$

[4 marks]

Answer 7

$$\sum_{1}^{14} (4r + 5) = 4 \sum_{1}^{14} r + 5 \sum_{1}^{14} 1$$

$$= 4 \times \frac{1}{2} \times 14 \times 15 + 5 \times 14$$

$$= 420 + 70$$

$$= 490$$

[4 marks]

Answer 8

$$\sum_{1}^{30} (2r + 7) = 2 \sum_{1}^{30} r + 7 \sum_{1}^{30} 1$$

$$= 2 \times \frac{1}{2} \times 30 \times 31 + 7 \times 30$$

$$= 930 + 210$$

$$= 1140$$

[4 marks]

Answer 9

$$\sum_{4}^{7} r = \sum_{1}^{7} r - \sum_{1}^{3} r$$

$$= \frac{1}{2} \times 7 \times 8 - \frac{1}{2} \times 3 \times 4$$

$$= 28 - 6$$

$$= 22$$

[4 marks]

Answer 10

$$\frac{1}{100}\sum_{80}^{120} r = \frac{1}{100}\left(\sum_{1}^{120} r - \sum_{1}^{79} r\right)$$

$$= \frac{1}{100}\left(\frac{1}{2} \times 120 \times 121 - \frac{1}{2} \times 79 \times 80\right)$$

$$= \frac{1}{100}\,(\,7260 - 3160\,)$$

$$= \frac{1}{100} \times 4100$$

$$= 41$$

[4 marks]

Answer 11

(i)

$$\text{LHS} = \sum_{r=1}^{n} (\,7r - 4\,)$$

$$= 7\sum_{1}^{n} r - 4\sum_{1}^{n} 1$$

$$= 7 \times \frac{1}{2} n\,(\,n + 1\,) - 4n$$

$$= \frac{7}{2} n^2 + \frac{7}{2} n - \frac{8}{2} n$$

$$= \frac{7}{2} n^2 - \frac{1}{2} n$$

$$= \frac{1}{2} n\,(\,7n - 1\,)$$

$$= \text{RHS} \qquad \square$$

[4 marks]

(ii)

$$\sum_{r=20}^{50} (\,7r - 4\,) = \sum_{1}^{50} (\,7r - 4\,) - \sum_{1}^{19} (\,7r - 4\,)$$

$$= \frac{1}{2} \times 50 \times 349 - \frac{1}{2} \times 19 \times 132$$

$$= 8725 - 1254$$

$$= 7471$$

[4 marks]

Answer 12

$$\sum_{r=1}^{n} r = 528$$

$$\frac{1}{2}n(n+1) = 528$$

$$n^2 + n = 1056$$

$$n^2 + n - 1056 = 0$$

$$\begin{bmatrix} FACT\{1056\} = 2^5 \times 3 \times 11 \\ \sqrt{1056} = 32.5 \end{bmatrix}$$

$$(n - 32)(n + 33) = 0$$

Either $n = 32$ or $n = -33$ (rejected)

$$\therefore n = 32$$

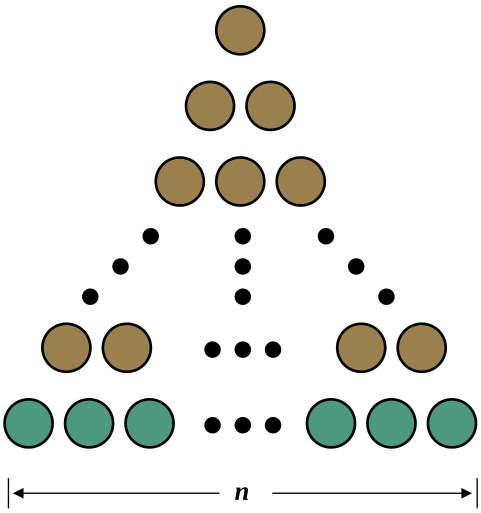
[4 marks]

Answer 13
(i)

$$
\begin{aligned}
\text{LHS} &= T_n - T_{n-1} \\
&= \sum_{1}^{n} r - \sum_{1}^{n-1} r \\
&= \frac{1}{2} n(n+1) - \frac{1}{2}(n-1)n \\
&= \frac{1}{2} n((n+1) - (n-1)) \\
&= \frac{1}{2} n(2) \\
&= n \\
&= \text{RHS} \qquad \square
\end{aligned}
$$

[4 marks]

(ii)



Any triangular number T_n (brown and green) take away the triangular number immediately before T_{n-1} (brown) leaves only the base row (green) of n spots.
Thus,

$$T_n - T_{n-1} = n \qquad \square$$

[6 marks]

Lesson 2

Further A-Level Pure Mathematics : Core 1
Series and Visual Proof

2.1 Formalising Series Manipulations

Here is a formal statement of the rules that were used without being explicitly stated in Lesson 1,

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Non Start From 1

To find the sum of a series that does not start at $r = 1$,

$$\sum_{r=k}^{n} f(r) = \sum_{r=1}^{n} f(r) - \sum_{r=1}^{k-1} f(r)$$

Factor Extractor

$$\sum_{r=1}^{n} k f(r) = k \sum_{r=1}^{n} f(r) \qquad \text{where } k \text{ is a constant}$$

The Linearity Property

$$\sum_{r=1}^{n} \left(f(r) + g(r) \right) = \sum_{r=1}^{n} f(r) + \sum_{r=1}^{n} g(r)$$

2.2 Example

The following question makes use of all of the above rules,

Given that $\displaystyle\sum_{r=n}^{2n} (15 - 2r) = 0$, find n

Teaching Video : http://www.NumberWonder.co.uk/v9092/2a.mp4
http://www.NumberWonder.co.uk/v9092/2b.mp4

<= Part 1 Part 2 =>

Watch the Part 1 video, writing out your own version of the initial steps below.
The part 1 video invites you to complete the solution before watching Part 2.

Given that $\sum_{r=n}^{2n} (15 - 2r) = 0$, find n

☞

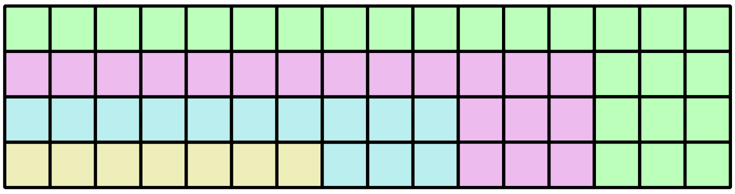
[6 marks]

2.3 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable.

Marks available : 40

Question 1



The diagram suggests the pattern,

$$7 = 1 \times 7 \qquad : \qquad n = 1$$

$$7 + 13 = 2 \times 10 \qquad : \qquad n = 2$$

$$7 + 13 + 19 = 3 \times 13 \qquad : \qquad n = 3$$

$$7 + 13 + 19 + 25 = 4 \times 16 \qquad : \qquad n = 4$$

(i) Write down the next line of the pattern corresponding to $n = 5$

[1 mark]

The pattern suggests that,

$$\sum_{r=1}^{n} (6r + 1) = n(3n + 4)$$

(ii) Prove the suggested result by using the standard results for $\sum_{1}^{n} 1$ and $\sum_{1}^{n} r$

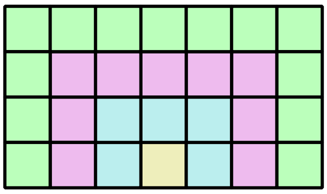
The start of the proof is given below,

$$\text{LHS} = \sum_{r=1}^{n} (6r + 1)$$

$$=$$

[4 marks]

Question 2



The diagram suggests the pattern,

$$1 = 1 \times 1 \qquad : \qquad n = 1$$

$$1 + 5 = 2 \times 3 \qquad : \qquad n = 2$$

$$1 + 5 + 9 = 3 \times 5 \qquad : \qquad n = 3$$

$$1 + 5 + 9 + 13 = 4 \times 7 \qquad : \qquad n = 4$$

(i) Write down the next line of the pattern corresponding to $n = 5$

[1 mark]

(ii) The pattern suggests a relationship of the form,

$$\sum_{r=1}^{n} (ar + b) = n (cn + d)$$

Write down the suggested relationship with the integer constants a, b, c and d replaced with their numerical values.

[2 marks]

(iii) Prove the suggested result by using the standard results for $\sum_{1}^{n} 1$ and $\sum_{1}^{n} r$

[4 marks]

Question 3

Prove $\sum_{r=1}^{2n} (5r - 4) = n(10n - 3)$ using the standard results for $\sum_{1}^{n} 1$ and $\sum_{1}^{n} r$

[4 marks]

Question 4

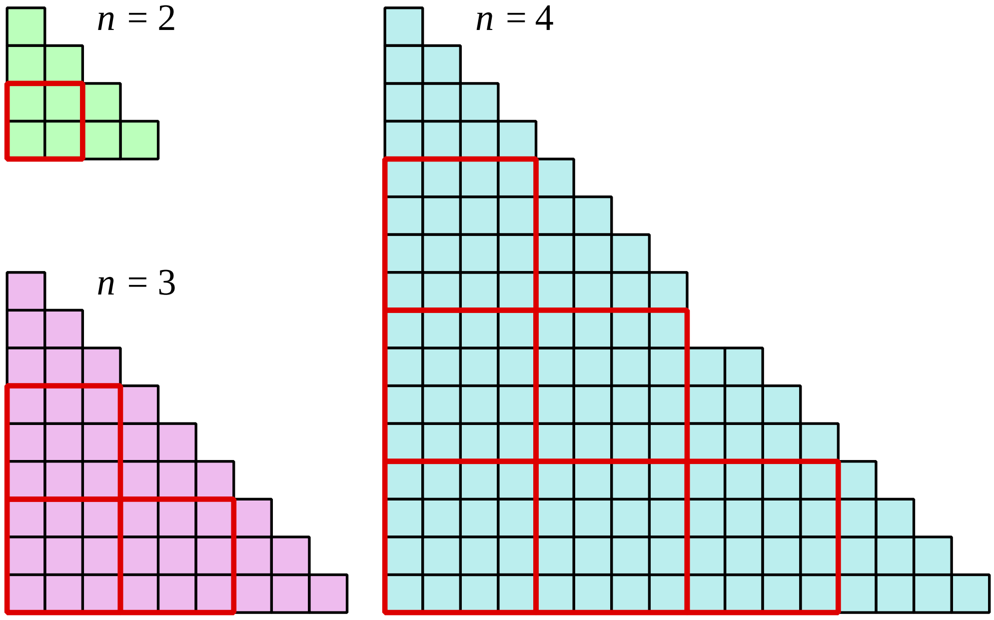
(i) Find an expression for $\sum_{r=1}^{2n-1} r$

[3 marks]

(ii) Hence show that $\sum_{r=n+1}^{2n-1} r = \frac{3}{2} n (n - 1)$, for $n \geqslant 2$

[3 marks]

Question 5



The diagrams suggest the following relationship between triangular numbers,

$$T_{n^2} = n^2 T_{n-1} + n T_n \qquad n \geqslant 2$$

The first two lines of a proof are given below. Complete the proof.

$$\begin{aligned} \text{RHS} &= n^2 T_{n-1} + n T_n \\ &= n^2 \sum_{1}^{n-1} r + n \sum_{1}^{n} r \\ &= \end{aligned}$$

[5 marks]

Question 6

Arithmetic Progressions

Algebraically, an Arithmetic Progression is a number sequence of the form;

$$a, \quad a + d, \quad a + 2d, \quad a + 3d, \quad \ldots, \quad a + (n - 1)\, d$$

where a is the initial term,
d is the common difference,
and n is number of terms.

The n^{th} term, L_n, is given by $L_n = a + (n - 1)\, d \qquad n \geqslant 1$

The sum of a AP is given by, $S_n = \dfrac{n}{2}\{ a + L \} \qquad n \geqslant 1$

In words this can be remembered as :

"n times the average of the first and last terms"

From substituting the first formula into the second, another formula is obtained for the sum of an AP. It is, $S_n = \dfrac{n}{2}\{ 2a + (n - 1)\, d \}, \qquad n \geqslant 1$

Observe that $\sum_{r=1}^{n} r$ is an Arithmetic Progression.

For this arithmetic progression,

(i) What is the the first term ?

[1 mark]

(ii) What is the common difference

[1 mark]

(iii) Hence show how the formula for the sum of an Arithmetic Progression can be used to derive a formula for $\sum_{r=1}^{n} r$

[3 marks]

Question 7

Given that $\sum_{r=n}^{3n} (80 - 3r) = 54$, find n

[8 marks]

2.4 Answers

2.4.1 Solutions to 2.2 Teaching Video Example

$$\sum_{r=n}^{2n} (15 - 2r) = 0$$

$$\sum_{1}^{2n} (15 - 2r) - \sum_{1}^{n} (15 - 2r) = 0$$

$$15 \sum_{1}^{2n} 1 - 2 \sum_{1}^{2n} r - 15 \sum_{1}^{n-1} 1 + 2 \sum_{1}^{n-1} r = 0$$

$$15 \times 2n - 2 \times \frac{1}{2} (2n)(2n + 1) - 15(n - 1) + 2 \times \frac{1}{2} (n - 1)(n) = 0$$

$$30n - 4n^2 - 2n - 15n + 15 + n^2 - n = 0$$

$$-3n^2 + 12n + 15 = 0$$

divide through by (-3)

$$n^2 - 4n - 5 = 0$$

$$(n - 5)(n + 1) = 0$$

Either $n = 5$ or $n = -1$ (reject)

$$\therefore\ n = 5$$

Writing out the sequence helps to make sense of what has been achieved;

Position	1	2	3	4	5	6	7	8	9	10	11
Term	13	11	9	7	5	3	1	− 1	− 3	− 5	− 7

[6 marks]

2.4.2 Solutions to 2.3 Exercise

Answer 1

(i) $7 + 13 + 19 + 25 + 31 = 5 \times 19$

[1 mark]

(ii)

$$\begin{aligned}
\text{LHS} &= \sum_{r=1}^{n} (6r + 1) \\
&= 6 \sum_{1}^{n} r + \sum_{1}^{n} 1 \\
&= 6 \times \frac{1}{2} \times n \times (n + 1) + n \\
&= 3n^2 + 3n + n \\
&= n(3n + 4) \\
&= \text{RHS} \qquad \square
\end{aligned}$$

[4 marks]

Answer 2

(i) $1 + 5 + 9 + 13 + 17 = 5 \times 9$

[1 mark]

(ii)

$$\sum_{1}^{n} (4r - 3) = n(2n - 1)$$

[2 marks]

(iii)

$$\begin{aligned}
\text{LHS} &= \sum_{1}^{n} (4r - 3) \\
&= 4 \sum_{1}^{n} r - 3 \sum_{1}^{n} 1 \\
&= 4 \times \frac{1}{2} \times n(n + 1) - 3n \\
&= 2n^2 + 2n - 3n \\
&= 2n^2 - n \\
&= n(2n - 1) \\
&= \text{RHS} \qquad \square
\end{aligned}$$

[4 marks]

Answer 3

$$\begin{aligned}
\text{LHS} &= \sum_{r=1}^{2n} (5r - 4) \\
&= 5\sum_{r=1}^{2n} r - 4\sum_{r=1}^{2n} 1 \\
&= 5 \times \frac{1}{2} \times 2n \times (2n + 1) - 4 \times 2n \\
&= 10n^2 + 5n - 8n \\
&= 10n^2 - 3n \\
&= n(10n - 3) \\
&= \text{RHS} \qquad \square
\end{aligned}$$

[4 marks]

Answer 4

(i)

$$\begin{aligned}
\sum_{r=1}^{2n-1} r &= \frac{1}{2} \times (2n - 1)\, 2n \\
&= n\,(2n - 1)
\end{aligned}$$

[3 marks]

(ii)

$$\begin{aligned}
\text{LHS} &= \sum_{r=n+1}^{2n-1} r \\
&= \sum_{1}^{2n-1} r - \sum_{1}^{n} r \\
&= n(2n - 1) - \frac{1}{2} \times n \times (n + 1) \\
&= \left(\frac{1}{2} n\right)(4n - 2 - n - 1) \\
&= \frac{1}{2} n (3n - 3) \\
&= \frac{3}{2} n (n - 1) \\
&= \text{RHS} \qquad \square
\end{aligned}$$

[3 marks]

Answer 5

$$\begin{aligned}
\text{RHS} &= n^2\, T_{n-1} + n\, T_n \\
&= n^2 \sum_{1}^{n-1} r + n \sum_{1}^{n} r \\
&= n^2 \times \frac{1}{2} \times (n-1)\, n + n \times \frac{1}{2} \times n\,(n+1) \\
&= \left(\frac{1}{2} n^2\right) \big((n-1)\, n + (n+1) \big) \\
&= \left(\frac{1}{2} n^2\right) \left(n^2 - n + n + 1 \right) \\
&= \left(\frac{1}{2} n^2\right)\left(n^2 + 1 \right) \\
&= \frac{1}{2} n^2 \left(n^2 + 1 \right) \\
&= T_{n^2} \\
&= \text{LHS} \qquad \square
\end{aligned}$$

[5 marks]

Answer 6

(i) $a = 1$

[1 mark]

(ii) $d = 1$

[1 mark]

(iii)

$$\begin{aligned}
S_n &= \frac{n}{2} \{ 2a + (n-1)\, d \} \\
\therefore \quad \sum_{r=1}^{n} r &= \frac{n}{2} \{ 2 \times 1 + (n-1) \times 1 \} \\
&= \frac{n}{2} \{ 2 + n - 1 \} \\
&= \frac{n}{2} \{ n + 1 \} \\
&= \frac{1}{2} n\,(n+1)
\end{aligned}$$

[3 marks]

Answer 7

$$\sum_{r=n}^{3n} (80 - 3r) = 54$$

$$\sum_{1}^{3n} (80 - 3r) - \sum_{1}^{n-1} (80 - 3r) = 54$$

$$80 \sum_{1}^{3n} 1 - 3 \sum_{1}^{3n} r - 80 \sum_{1}^{n-1} 1 + 3 \sum_{1}^{n-1} r = 54$$

$$80 \times 3n - \frac{3}{2} 3n (3n + 1) - 80 (n - 1) + \frac{3}{2} (n - 1) n = 54$$

Multiply through by 2

$$160 \times 3n - 9n (3n + 1) - 160 (n - 1) + 3 (n - 1) n = 108$$

$$480n - 27n^2 - 9n - 160n + 160 + 3n^2 - 3n - 108 = 0$$

$$-24n^2 + 308n + 52 = 0$$

Divide through by (-4)

$$6n^2 - 77n - 13 = 0$$

$$(6n + 1)(n - 13) = 0$$

Either $n = -\frac{1}{6}$ (reject) or $n = 13$

Position	1	2	3	4	5	6	7	8	9	10
Term	77	74	71	68	65	62	59	56	53	50

Position	11	12	13	14	15	16	17	18	19	20
Term	47	44	41	38	35	32	29	26	23	20

Position	21	22	23	24	25	26	27	28	29	30
Term	17	14	11	8	5	2	− 1	− 4	− 7	− 10

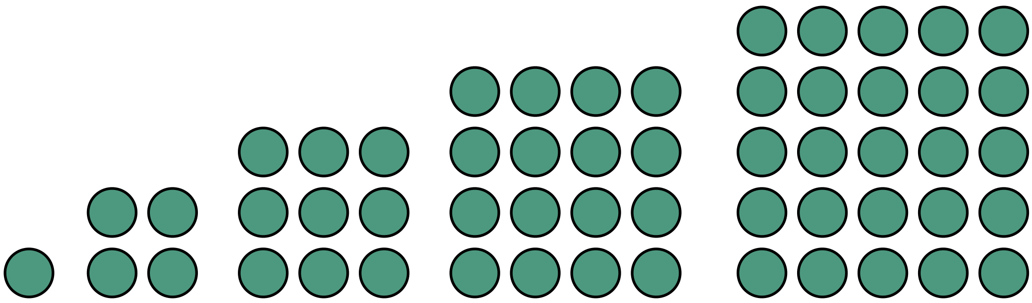
Position	31	32	33	34	35	36	37	38	39	40
Term	− 13	− 16	− 19	− 22	− 25	− 28	− 31	− 34	− 37	− 40

Between $n = 13$ and $n = 39$ (positives − negatives) $= 301 - 247 = 54$

[8 marks]

3.1 Square Numbers

The square numbers are so called because the start of the sequence, which begins 1, 4, 9, 16, 25, ... can be visualised as dots arranged in squares like so;

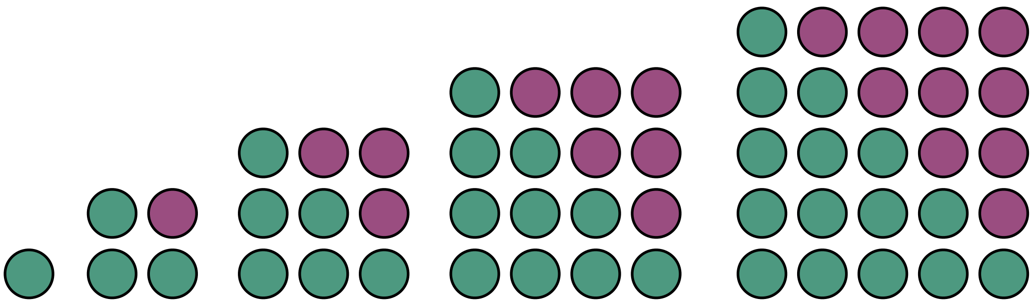


The sixth square number, $\square_6$, can be described in sigma notation as $\left(\sum_1^6 r + \sum_1^5 r\right)$

Here is the calculation,

$$\begin{aligned}
\square_6 &= T_6 + T_5 \\
&= \sum_1^6 r + \sum_1^5 r \\
&= 1 + 2 + 3 + 4 + 5 + 6 + 1 + 2 + 3 + 4 + 5 \\
&= 36
\end{aligned}$$

As a method of working out that 6^2 is 36, it's unnecessarily complicated, although the real interest is that it suggests that any square number, $\square_n$ is the sum of the two triangular numbers, $T_n + T_{n-1}$



The diagram shows visually how any given square number can be constructed from two triangular numbers.

$$\square_n = T_n + T_{n-1}$$

To absolutely confirm the relationship, most mathematicians would also like an algebraic proof.

Theorem

$$\square_n = T_n + T_{n-1}$$

Proof

$$\begin{aligned}
\text{RHS} &= T_n + T_{n-1} \\
&= \sum_1^n r + \sum_1^{n-1} r \\
&= \frac{1}{2} n (n + 1) + \frac{1}{2} (n - 1) n \\
&= \left(\frac{1}{2} n \right) \left(n + 1 + n - 1 \right) \\
&= \frac{1}{2} n (2n) \\
&= n^2 \\
&= \text{LHS} \qquad \square
\end{aligned}$$

3.2 Sum of Squares

Of more interest than the squares themselves, is the idea of summing the squares.
In other words, of considering the series,

$$\sum_1^n r^2 = 1 + 4 + 9 + 16 + 25 + \ldots + n^2$$

This has a formula which is worth memorising.
It can be used as a quotable result in examinations.

The Sum of Squares Formula

$$\sum_1^n r^2 = \frac{1}{6} n (n + 1) (2n + 1)$$

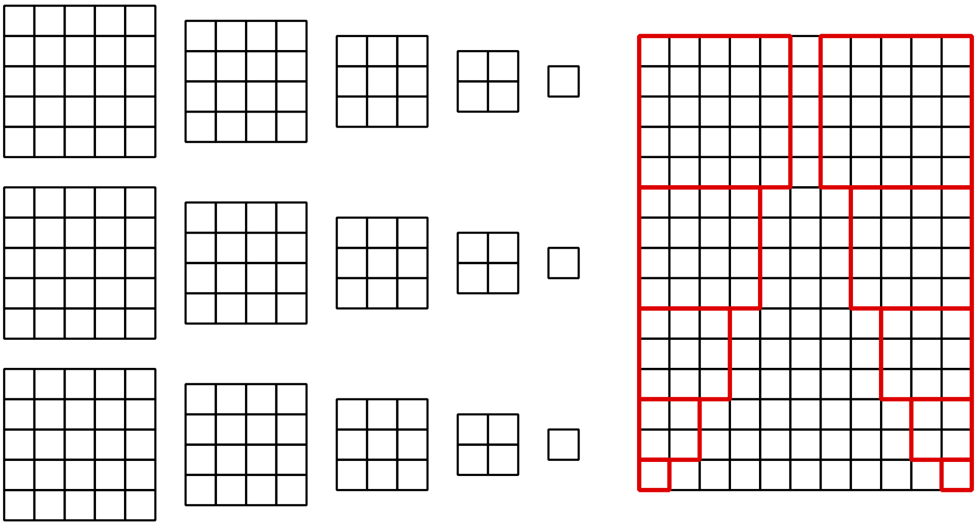
The formula is has a beautiful and clever visual proof.

Teaching video : http://www.NumberWonder.co.uk/v9092/3a.mp4
http://www.NumberWonder.co.uk/v9092/3b.mp4

<= Part 1 Part 2 =>

Watch the Part 1 video, which sets up the system of partitioning one of the three sets of “sum of squares” and transfers the parts into the right hand rectangle. Colour the diagrams below to show the partitioning and transfer.

n = 5



The Part 1 video invites you to complete the proof before watching Part 2.

☞

[6 marks]

3.3 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable.

Marks available : 40

Question 1

Evaluate,

(i) $\sum_{1}^{10} r^2$ **(ii)** $\sum_{r=25}^{50} r^2$

[4 marks]

Question 2

(i) Show that $\sum_{r=n+1}^{2n} r^2 = \frac{1}{6} n\ (2n + 1)\ (7n + 1)$

[4 marks]

(ii) Hence evaluate $\sum_{r=11}^{20} r^2$

[3 marks]

Question 3

(i) Show that $\sum_{r=1}^{n} \left(r^2 + r - 2 \right) = \frac{1}{3} n (n + 4) (n - 1)$

[4 marks]

(ii) Hence find the sum of the series $4 + 10 + 18 + 28 + 40 + ... + 418$

[3 marks]

Question 4

Further A-Level Examination Question from January 2014, F1, Q5 (Edexcel)

(a) Use the standard results for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$ to show that,

$$\sum_{r=1}^{n} \left(9r^2 - 4r \right) = \frac{1}{2} n (n + 1)(6n - 1)$$

for all positive integers n

[4 marks]

Given that $\sum_{r=1}^{12} \left(9r^2 - 4r + k(2^r) \right) = 6630$

(b) find the exact value of the constant k

[4 marks]

Question 5

Further A-Level Examination Question from January 2019, F1, Q3 (Edexcel)

(a) Use the standard results for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$ to show that, for all positive integers n,

$$\sum_{r=1}^{n} (2r+5)^2 = \frac{n}{3}\left[(an+b)^2 + c\right]$$

where a, b and c are integers to be found.

[5 marks]

(b) Use the answer to part (a) to evaluate $\sum_{r=0}^{100} (2r+5)^2$

[2 marks]

Question 6

(i) Find and prove a formula for the sum of the first n triangular numbers.

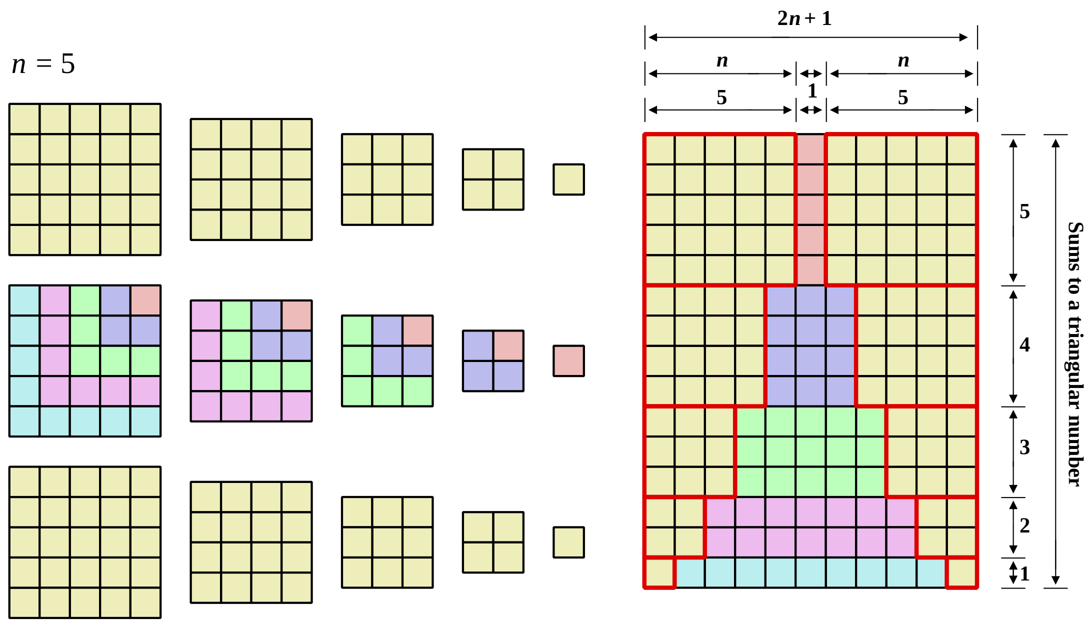
[5 marks]

(ii) Explain the connection between part (i) and the following photograph,

[2 marks]

3.4 Answers

3.4.1 Solution to 3.2 Teaching Video Proof



$$3\sum_{r=1}^{n} r^2 = \frac{1}{2}n(n+1)(2n+1)$$

$$\therefore \sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$$

[6 marks]

3.4.2 Solution to 3.3 Exercise

Answer 1

(i)

$$\sum_{1}^{n} r^2 = \frac{1}{6} n (n + 1)(2n + 1)$$

$$\sum_{1}^{10} r^2 = \frac{1}{6} \times 10 \times 11 \times 21$$

$$= 385$$

[2 marks]

(ii)

$$\sum_{1}^{n} r^2 = \frac{1}{6} n (n + 1)(2n + 1)$$

$$\sum_{r=25}^{50} r^2 = \sum_{1}^{50} r^2 - \sum_{1}^{24} r^2$$

$$= \frac{1}{6} \times 50 \times 51 \times 101 - \frac{1}{6} \times 24 \times 25 \times 49$$

$$= 42925 - 4900$$

$$= 38025$$

[2 marks]

Answer 2

(i)

$$\text{LHS} = \sum_{r=n+1}^{2n} r^2$$

$$= \sum_{1}^{2n} r^2 - \sum_{1}^{n} r^2$$

$$= \frac{1}{6} \times 2n (2n + 1)(4n + 1) - \frac{1}{6} \times n (n + 1)(2n + 1)$$

$$= \left(\frac{n (2n + 1)}{6} \right) \left(2 (4n + 1) - (n + 1) \right)$$

$$= \frac{1}{6} n (2n + 1)(7n + 1)$$

$$= \text{RHS} \qquad \square$$

[4 marks]

(ii) Use the part (i) result with $n = 10$

$$\sum_{r=11}^{20} r^2 = \frac{1}{6} \times 10 \times 21 \times 71$$

$$= 2485$$

[3 marks]

Answer 3

(i)

$$\begin{aligned}
\text{LHS} &= \sum_{r=1}^{n}\left(r^2 + r - 2\right) \\
&= \sum_{1}^{n} r^2 + \sum_{1}^{n} r - 2\sum_{1}^{n} 1 \\
&= \frac{1}{6} \times n\,(n+1)\,(2n+1) + \frac{1}{2} \times n\,(n+1) - 2n \\
&= \left(\frac{n}{6}\right)\left((n+1)\,(2n+1) + 3(n+1) - 12\right) \\
&= \left(\frac{n}{6}\right)\left(2n^2 + 3n + 1 + 3n + 3 - 12\right) \\
&= \left(\frac{n}{6}\right)\left(2n^2 + 6n - 8\right) \\
&= \left(\frac{n}{3}\right)\left(n^2 + 3n - 4\right) \\
&= \frac{1}{3}\, n\,(n+4)\,(n-1) \\
&= \text{RHS} \qquad \square
\end{aligned}$$

[4 marks]

(ii) To find the number of terms,

$$n^2 + n - 2 = 418$$

$$n^2 + n - 420 = 0$$

$$\left[\begin{array}{c} FACT\{420\} = 2 \times 2 \times 3 \times 5 \times 7 \\ \sqrt{420} = 20.5 \end{array}\right]$$

$$(n + 21)\,(n - 20) = 0$$

Either $n = -21$ (reject) or $n = 20$

Use the part (i) formula with $n = 20$

$$\begin{aligned}
0 + 4 + 10 + 18 + \ldots + 418 &= \sum_{1}^{20}\left(r^2 + r - 2\right) \\
&= \frac{1}{3} \times 20 \times 24 \times 19 \\
&= 3040
\end{aligned}$$

[3 marks]

Answer 4

(a)

$$\begin{aligned}
\text{LHS} &= \sum_{r=1}^{n}\left(9r^2 - 4r\right) \\
&= 9\sum_{1}^{n} r^2 - 4\sum_{1}^{n} r \\
&= 9 \times \frac{1}{6} \times n(n+1)(2n+1) - 4 \times \frac{1}{2} n(n+1) \\
&= \left(\frac{n(n+1)}{6}\right)\left(9(2n+1) - 12\right) \\
&= \left(\frac{n(n+1)}{6}\right)\left(18n - 3\right) \\
&= \frac{1}{2} n(n+1)(6n-1) \\
&= \text{RHS} \qquad \square
\end{aligned}$$

[4 marks]

(b)

$$\begin{aligned}
\sum_{r=1}^{12}\left(9r^2 - 4r + k\left(2^r\right)\right) &= 6630 \\
\sum_{1}^{12}\left(9r^2 - 4r\right) + k\sum_{1}^{12}\left(2^r\right) &= 6630 \\
\frac{1}{2} \times 12 \times 13 \times 71 + k\left(2^1 + 2^2 + \ldots + 2^{12}\right) &= 6630 \\
5538 + k \times \frac{2\left(2^{12} - 1\right)}{2 - 1} &= 6630 \\
k \times \frac{2 \times 4095}{1} &= 1092 \\
k &= \frac{2}{15}
\end{aligned}$$

[4 marks]

Answer 5

(a)

$$\text{LHS} = \sum_{r=1}^{n} (2r + 5)^2$$

$$= \sum_{1}^{n} \left(4r^2 + 20r + 25\right)$$

$$= 4 \sum_{1}^{n} r^2 + 20 \sum_{1}^{n} r + 25 \sum_{1}^{n} 1$$

$$= 4 \times \frac{1}{6} \times n(n+1)(2n+1) + 20 \times \frac{1}{2} n(n+1) + 25n$$

$$= \left(\frac{n}{6}\right)\left(4(n+1)(2n+1) + 60(n+1) + 150\right)$$

$$= \left(\frac{n}{6}\right)\left(8n^2 + 12n + 4 + 60n + 60 + 150\right)$$

$$= \left(\frac{n}{6}\right)\left(8n^2 + 72n + 214\right)$$

$$= \frac{n}{3}\left(4n^2 + 36n + 107\right)$$

$$= \frac{n}{3}\left(4\left[n^2 + 9n\right] + 107\right)$$

$$= \frac{n}{3}\left(4\left[\left(n + \frac{9}{2}\right)^2 - \frac{81}{4}\right] + 107\right)$$

$$= \frac{n}{3}\left(4\left(n + \frac{9}{2}\right)^2 - 81 + 107\right)$$

$$= \frac{n}{3}\left((2n + 9)^2 + 26\right)$$

$$\therefore\ a = 2,\ \ b = 9 \ \text{ and } \ c = 26$$

[5 marks]

(b)

$$\sum_{r=0}^{100} (2r + 5)^2 = \sum_{1}^{100} (2r + 5)^2 + 5^2$$

$$= \frac{100}{3}\left((2 \times 100 + 9)^2 + 26\right) + 5^2$$

$$= 1456925$$

[2 marks]

Answer 6

(i) The series of triangular numbers to be summed is,

$$1 + 3 + 6 + 10 + 15 + 21 + ... + \frac{1}{2} n (n + 1)$$

$$= \sum_{1}^{n} \frac{1}{2} r (r + 1)$$

$$= \frac{1}{2} \sum_{1}^{n} r^2 + \frac{1}{2} \sum_{1}^{n} r$$

$$= \frac{1}{2} \times \frac{1}{6} n (n + 1) (2n + 1) + \frac{1}{2} \times \frac{1}{2} n (n + 1)$$

$$= \left(\frac{n (n + 1)}{12} \right) \left((2n + 1) + 3 \right)$$

$$= \frac{n (n + 1) (2n + 4)}{12}$$

$$= \frac{n (n + 1) (n + 2)}{6}$$

[5 marks]

(ii) The photograph corresponds to the case $n = 4$

$$\frac{4 \times 5 \times 6}{6} = 20 \text{ tins}$$

[2 marks]

Lesson 4

4.1 Cubic Numbers

The cubic numbers are so called because the start of the sequence, which begins 1, 8, 27, 64, 125, ... can be visualised as unit cubes arranged in large cubes like so;

4.2 Sum of Cubes

The obvious question is "What is the formula for summing the first *n* cubes" ?
In other words, of considering the series,

$$\sum_{1}^{n} r^3 = 1 + 8 + 27 + 64 + 125 + \ldots + n^3$$

This has a formula which is worth memorising.
It can be used as a quotable result in examinations.

The Sum of Cubes Formula

$$\sum_{1}^{n} r^3 = \frac{1}{4} n^2 (n + 1)^2$$

Notice that it is the formula for a triangular number, squared.
That is,

$$\sum_{1}^{n} r^3 = \left(\sum_{1}^{n} r \right)^2$$

4.3 Proof

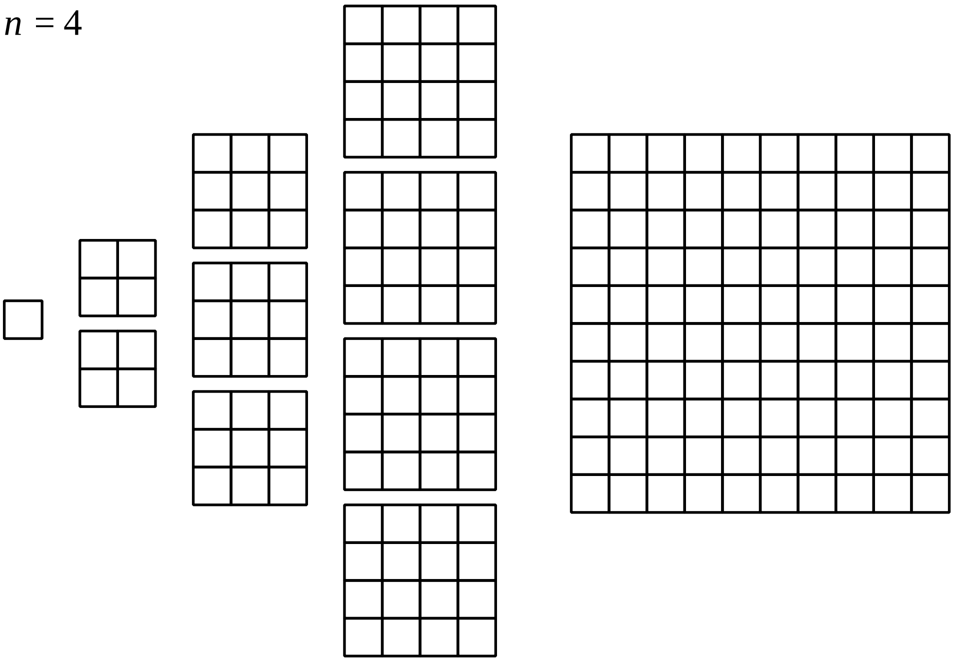
The idea behind the visual proof is to take the cubes and slice them up.
The slices are then arranged into a square.

Teaching Video : http://www.NumberWonder.co.uk/v9092/4a.mp4
http://www.NumberWonder.co.uk/v9092/4b.mp4

<= Part 1 Part 2 =>

Watch the Part 1 video, which takes the sliced up cubes and starts to transfer the slices into the right hand rectangle. Colour the diagrams below to show the transfer. You are invited to try to complete the proof before watching Part 2.

$n = 4$



[6 marks]

4.4 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable.

Marks available : 40

Question 1

Evaluate,

(i) $\sum_{1}^{12} r^3$ **(ii)** $\sum_{r=50}^{75} r^3$

[4 marks]

Question 2

(i) Show that $\sum_{r=n+1}^{3n} r^3 = n^2 (4n + 1)(5n + 2)$

[4 marks]

(ii) Hence evaluate $\sum_{r=11}^{30} r^3$

[3 marks]

Question 3

(i) Show that $\sum_{r=1}^{n} r^2 (r - 1) = \frac{1}{12} n (n + 1)(3n + 2)(n - 1)$

[4 marks]

(ii) Hence find the sum of the series $4 + 18 + 48 + 100 + ... + 3840$

[3 marks]

Question 4

Further A-Level Examination Question from June 2012. IAL, FP1, Q4 (Edexcel)

(a) Use the standard results for $\sum_{r=1}^{n} r^3$ and $\sum_{r=1}^{n} r$ to show that,

$$\sum_{r=1}^{n} \left(r^3 + 6r - 3 \right) = \frac{1}{4} n^2 \left(n^2 + 2n + 13 \right)$$

for all positive integers *n*.

[5 marks]

(b) Hence find the exact value of $\sum_{r=16}^{30} \left(r^3 + 6r - 3 \right)$

[2 marks]

Question 5

Further A-Level Examination Question from June 2014. IAL, FP1(R), Q5

(a) Use the standard results for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^3$ to show that,

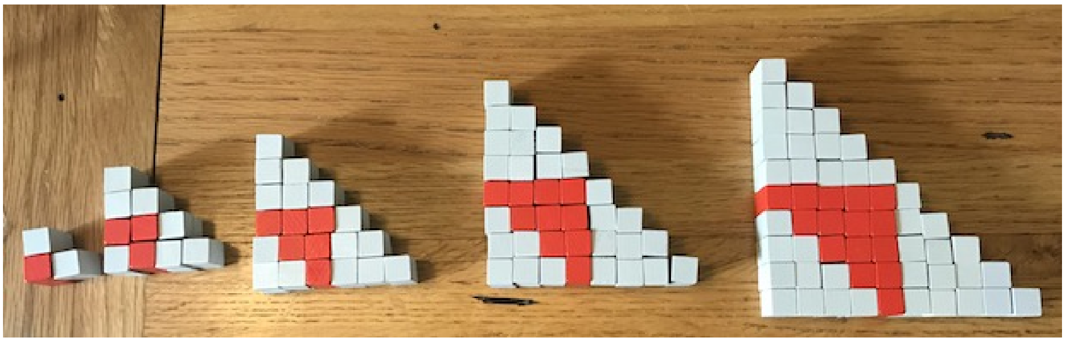
$$\sum_{r=1}^{n} r\left(r^2 - 3\right) = \frac{1}{4} n(n+1)(n+3)(n-2)$$

[5 marks]

(b) Calculate the value of $\sum_{r=10}^{50} r\left(r^2 - 3\right)$

[3 marks]

Question 6



The pattern suggests an iterative relationship between triangular numbers.

(i) Which one of the following is the relationship suggested ?

(a) $4T_n = T_{2n}$

(b) $T_{n+1} + 3T_n = T_{2n}$

(c) $3T_n + T_{n-1} = T_{2n}$

(d) $4T_n - T_{n-1} = T_{2n}$

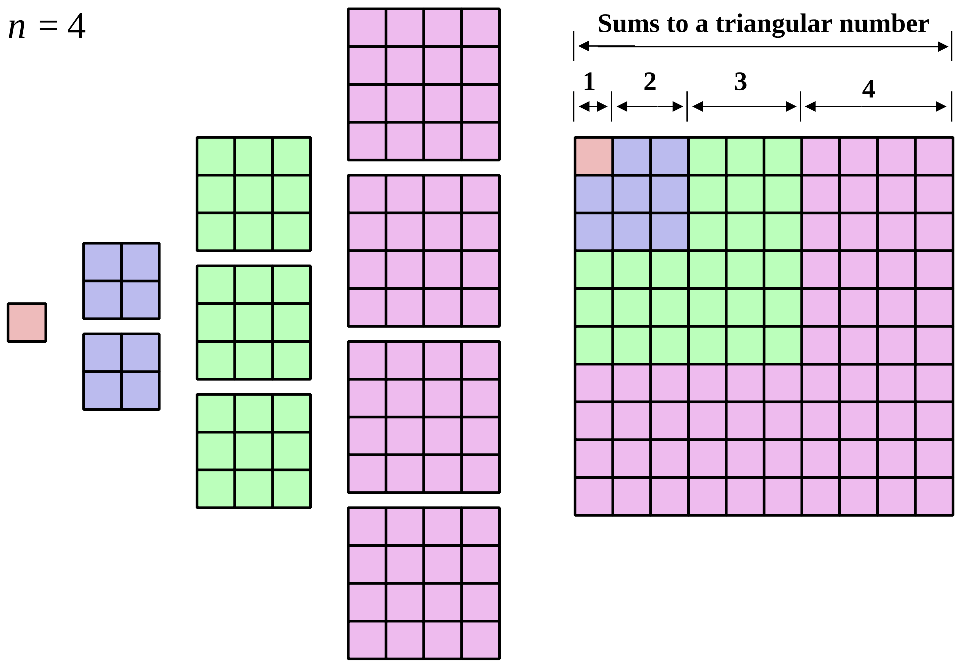
[2 marks]

(ii) Prove the relationship using algebra and sigma notation

[5 marks]

4.5 Answers

4.5.1 Solution to the Teaching Video Proof



$$\sum_{1}^{n} r^3 = \left(\sum_{1}^{n} r \right)^2$$

$$= \frac{1}{4} n^2 \, (n + 1)^2$$

[6 marks]

4.5.2 Solutions to 4.4 Exercise

Answer 1

(i)

$$\sum_{1}^{n} r^3 = \frac{1}{4} n^2 (n + 1)^2$$

$$\sum_{1}^{12} r^3 = \frac{1}{4} \times 12^2 \times 13^2$$

$$= 6084$$

[2 marks]

(ii)

$$\sum_{r=50}^{75} r^3 = \sum_{1}^{75} r^3 - \sum_{1}^{49} r^3$$

$$= \frac{1}{4} \times 75^2 \times 76^2 - \frac{1}{4} \times 49^2 \times 50^2$$

$$= 8122500 - 1500625$$

$$= 6621875$$

[2 marks]

Answer 2

(i)

$$\begin{aligned}
\text{LHS} &= \sum_{r=n+1}^{3n} r^3 \\
&= \sum_{1}^{3n} r^3 - \sum_{1}^{n} r^3 \\
&= \frac{1}{4}(3n)^2(3n+1)^2 - \frac{1}{4}n^2(n+1)^2 \\
&= \left(\frac{n^2}{4}\right)\left(9(3n+1)^2 - (n+1)^2\right) \\
&= \left(\frac{n^2}{4}\right)\left(9\left(9n^2+6n+1\right) - \left(n^2+2n+1\right)\right) \\
&= \left(\frac{n^2}{4}\right)\left(81n^2+54n+9-n^2-2n-1\right) \\
&= \left(\frac{n^2}{4}\right)\left(80n^2+52n+8\right) \\
&= n^2\left(20n^2+13n+2\right) \\
&= n^2(4n+1)(5n+2) \\
&= \text{RHS} \qquad \square
\end{aligned}$$

[4 marks]

(ii) This is part (a) with $n = 10$

$$\begin{aligned}
\therefore \sum_{r=11}^{30} r^3 &= 10^2 \times 41 \times 52 \\
&= 213200
\end{aligned}$$

[3 marks]

Answer 3
(i)

$$\begin{aligned}
\text{LHS} &= \sum_{r=1}^{n} r^2 (r-1) \\
&= \sum_{1}^{n} r^3 - \sum_{1}^{n} r^2 \\
&= \frac{1}{4} n^2 (n+1)^2 - \frac{1}{6} n (n+1)(2n+1) \\
&= \left(\frac{n(n+1)}{12} \right) \left(3n(n+1) - 2(2n+1) \right) \\
&= \left(\frac{n(n+1)}{12} \right) \left(3n^2 + 3n - 4n - 2 \right) \\
&= \left(\frac{n(n+1)}{12} \right) \left(3n^2 - n - 2 \right) \\
&= \frac{1}{12} n(n+1) \left(3n^2 - n - 2 \right) \\
&= \frac{1}{12} n(n+1)(3n+2)(n-1) \\
&= \text{RHS} \qquad \square
\end{aligned}$$

[4 marks]

(ii) To work out how many terms there are,

$$\begin{aligned}
n^2 (n-1) &= 3840 \\
FACT\{3840\} &= 2^8 \times 3 \times 5 \\
\therefore \quad 3840 &= \left(2^4\right)^2 \times 3 \times 5 \\
&= 16^2 \times 15
\end{aligned}$$

That is, $n = 16$

$$\begin{aligned}
0 + 4 + 18 + 48 + 100 + \ldots + 3840 &= \sum_{1}^{16} r^2 (r-1) \\
&= \frac{1}{12} \times 16 \times 17 \times 50 \times 15 \\
&= 17000
\end{aligned}$$

[3 marks]

Answer 4

(a)

$$\text{LHS} = \sum_{r=1}^{n} \left(r^3 + 6r - 3 \right)$$

$$= \sum_{1}^{n} r^3 + 6 \sum_{1}^{n} r - 3 \sum_{1}^{n} 1$$

$$= \frac{1}{4} n^2 (n + 1)^2 + 6 \times \frac{1}{2} n (n + 1) - 3n$$

$$= \left(\frac{n}{4} \right)\left(n (n + 1)^2 + 12 (n + 1) - 12 \right)$$

$$= \left(\frac{n}{4} \right)\left(n \left(n^2 + 2n + 1 \right) + 12n + 12 - 12 \right)$$

$$= \left(\frac{n^2}{4} \right)\left(n^2 + 2n + 1 + 12 \right)$$

$$= \left(\frac{n^2}{4} \right)\left(n^2 + 2n + 13 \right)$$

$$= \frac{1}{4} n^2 \left(n^2 + 2n + 13 \right)$$

$$= \text{RHS} \qquad \square$$

[5 marks]

(b)

$$\sum_{r=16}^{30} \left(r^3 + 6r - 3 \right) = \sum_{1}^{30} \left(r^3 + 6r - 3 \right) - \sum_{1}^{15} \left(r^3 + 6r - 3 \right)$$

$$= \frac{1}{4} \times 30^2 \times 973 - \frac{1}{4} \times 15^2 \times 268$$

$$= 218925 - 15075$$

$$= 203850$$

[**2 marks]**

Answer 5

(a)

$$
\begin{aligned}
\text{LHS} &= \sum_{r=1}^{n} r\left(r^2 - 3\right) \\
&= \sum_{1}^{n} r^3 - 3\sum_{1}^{n} r \\
&= \frac{1}{4} n^2 (n+1)^2 - 3 \times \frac{1}{2} n(n+1) \\
&= \left(\frac{n(n+1)}{4}\right)\left(n(n+1) - 6\right) \\
&= \left(\frac{n(n+1)}{4}\right)\left(n^2 + n - 6\right) \\
&= \frac{1}{4} n(n+1)(n+3)(n-2) \\
&= \text{RHS} \qquad \square
\end{aligned}
$$

[5 marks]

(b)

$$
\begin{aligned}
\sum_{10}^{50} r\left(r^2 - 3\right) &= \sum_{1}^{50} r\left(r^2 - 3\right) - \sum_{1}^{9} r\left(r^2 - 3\right) \\
&= \frac{1}{4} \times 50 \times 51 \times 53 \times 48 - \frac{1}{4} \times 9 \times 10 \times 12 \times 7 \\
&= 1621800 - 1890 \\
&= 1619910
\end{aligned}
$$

[3 marks]

Answer 6

(i) **(c)** $3T_n + T_{n-1} = T_{2n}$

[2 marks]

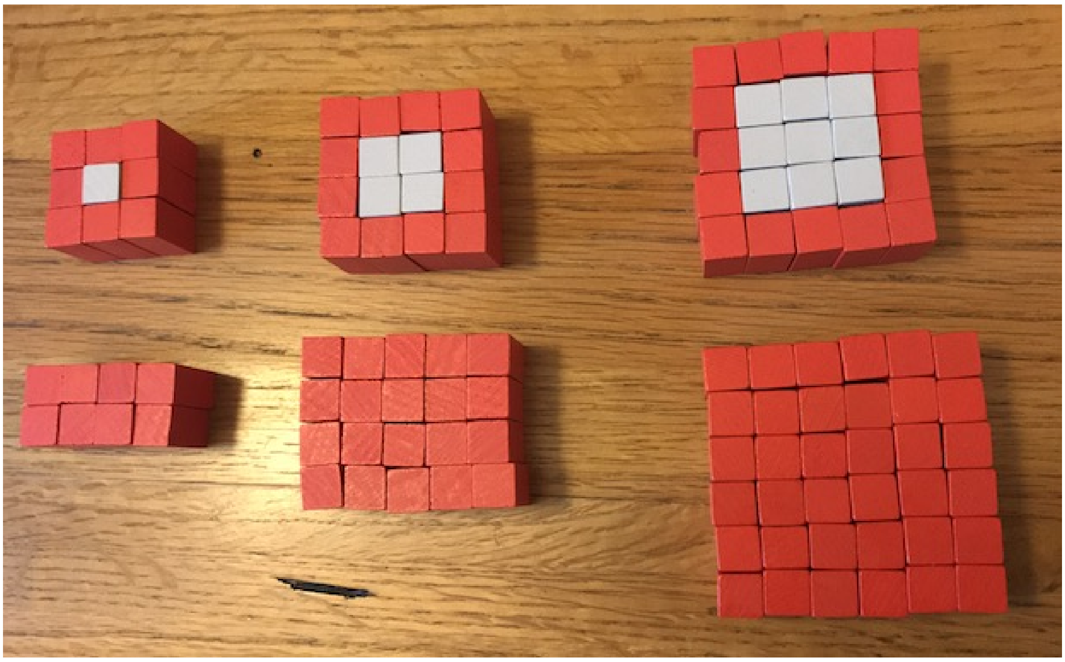
(ii)

$$
\begin{aligned}
\text{LHS} &= 3T_n + T_{n-1} \\
&= 3\sum_{1}^{n} r + \sum_{1}^{n-1} r \\
&= 3 \times \frac{1}{2} \times n\,(n+1) + \frac{1}{2} \times (n-1)\,n \\
&= \left(\frac{n}{2}\right)\left(3\,(n+1) + (n-1)\right) \\
&= \left(\frac{n}{2}\right)\left(3n + 3 + n - 1\right) \\
&= \frac{1}{2}\,n\,(4n+2) \\
&= \frac{1}{2}\,(2n)\,(2n+1) \\
&= T_{2n} \\
&= \text{RHS} \qquad \square
\end{aligned}
$$

[5 marks]

5.1 Pattern Spotting

Knowledge of the standard results for $\sum_{1}^{n} r^3$, $\sum_{1}^{n} r^2$, $\sum_{1}^{n} r$ and $\sum_{1}^{n} 1$ allows an efficient means of proving suspected results involving series.



For example, in the above photograph the framed square sequence is depicted.
The upper portion of the photograph shows 3×3, 4×4 and 5×5 framed squares.
The individual squares are actually small unit cubes because it's not possible in the physical world to have an area without depth. As the interested is in the area of the uppermost face of each cube, the cubes will frequently be called squares !
The squares forming the frames are red, those not forming the frame, white.

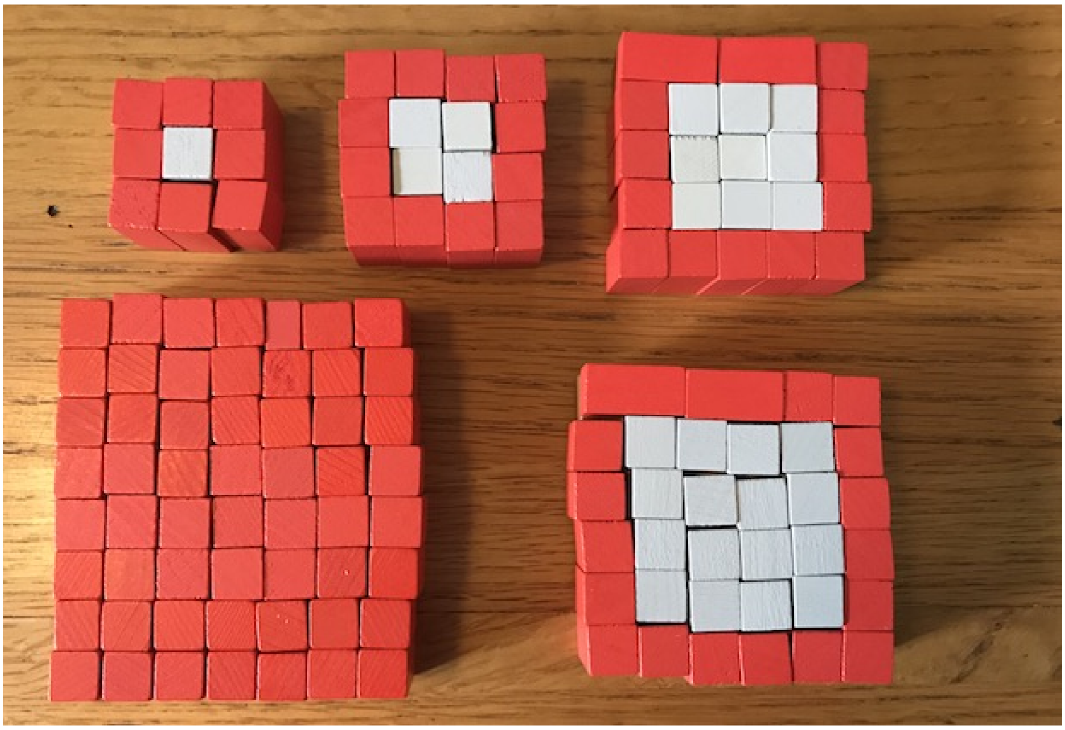
The upper left 3×3 framed square has a frame of eight squares.
These eight squares have been arranged into a 2×4 rectangle, lower left.

The upper left 3×3 plus the upper middle 4×4 frame squares contain a total of twenty red framing squares, these arranged as a 4×5 rectangle, lower middle.

Finally, the frames of all three framed squares provides thirty-six red squares and the lower right arrangement shows these as a 6×6 square.

Two questions; **(a)** What is the suspected pattern ?
(b) How might that pattern's existence be confirmed ?

5.2 Enter the Mathematics



It's often wise to obtain one or two more terms in the emerging pattern to check that what is suspected continues to hold. The above photograph shows the next "all red" rectangle in the sequence and the four framed squares it came from.

Tables helps clarify observations. This first records the number of red squares in each term. The formula for the n^{th} term arises from noticing that the number of red squares in any given term is given by subtracting a (large white) square number from the corresponding (large red plus white) square number. The red+white large square has a side length two more than the corresponding large white square.

Term	1	2	3	4	...	n
Boundary	8	12	16	20	...	$(n+2)^2 - n^2$

This "cumulative" table gives the dimensions of the "all red" rectangles,

Term	1	2	3	4	...	n
Cumulative	8	20	36	56	...	
Rectangle	2×4	4×5	6×6	8×7	...	$(2n)(n+3)$

After this considerable amount of setting up, a conjecture can be put forward,

$$\sum_{1}^{n} \left((r+2)^2 - r^2 \right) = 2n(n+3)$$

And now, at last, a proof can be worked on.

5.3 The Proof

$$\sum_{1}^{n}\left((r+2)^2 - r^2 \right) = 2n(n+3)$$

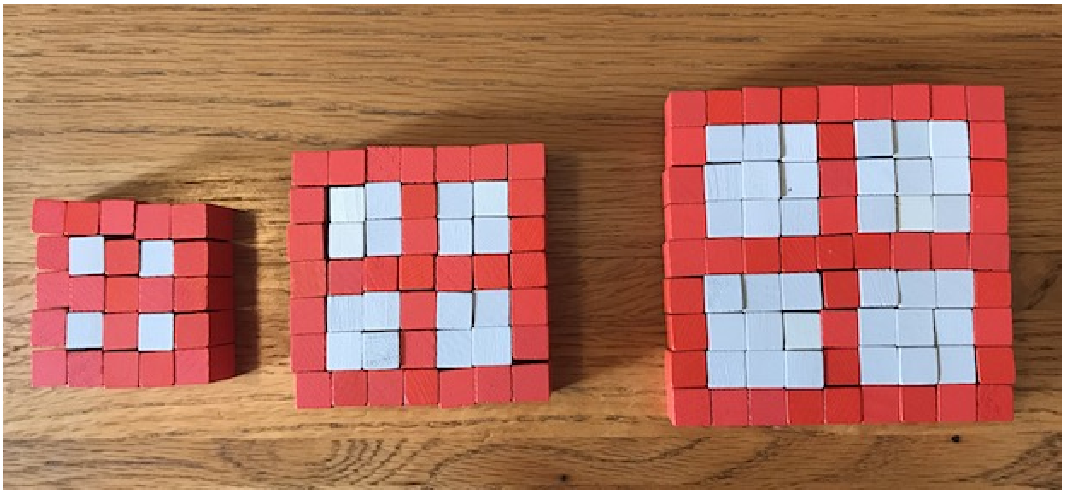
$$\begin{aligned}
\text{LHS} &= \sum_{1}^{n}\left((r+2)^2 - r^2 \right) \\
&= \sum_{1}^{n}\left(r^2 + 4r + 4 - r^2 \right) \\
&= \sum_{1}^{n}\left(4r + 4 \right) \\
&= 4\sum_{1}^{n} r + 4\sum_{1}^{n} 1 \\
&= 4 \times \frac{1}{2} n(n+1) + 4n \\
&= \left(2n \right)\left(n + 1 + 2 \right) \\
&= 2n(n+3) \\
&= \text{RHS} \qquad \square
\end{aligned}$$

This shows that an “all red” rectangle can always be formed measuring (2n) by (n + 3).

5.4 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable.
Marks Available : 40

Question 1



The first three members of the “windows” sequence are shown in the photograph.

(i) Complete the following table so show the number of red squares in each diagram. (The squares are actually unit cubes)

(ii) Add a formula for the n^{th} term.

Term	1	2	3	4	...	n
Red Frame		33		57	...	

[6 marks]

(iii) In the manner of the introductory example, complete the following “Cumulative Reds” table for the sequence of “all red” rectangles that could be formed along with the dimensions of those rectangles, chosen to form a sequence.

(iv) Complete the formula for the dimensions of the n^{th} “all red ”rectangle

Term	1	2	3	4	...	n
Cumulative		54			...	
Rectangle		6×9			...	$(3n)$ ( )

[6 marks]

(v) Write down a conjecture for the relationship between the two tables.

[3 marks]

(vi) Prove your conjecture.

[4 marks]

Question 2

(i) Show that,

$$\sum_{1}^{n} (4r - 3)^2 = \frac{1}{3} n \left(16n^2 - 12n - 1 \right)$$

[4 marks]

(ii) Show that,

$$\sum_{2}^{n} (4r - 5)^2 = \frac{1}{3} n \left(16n^2 - 36n + 23 \right) - 1$$

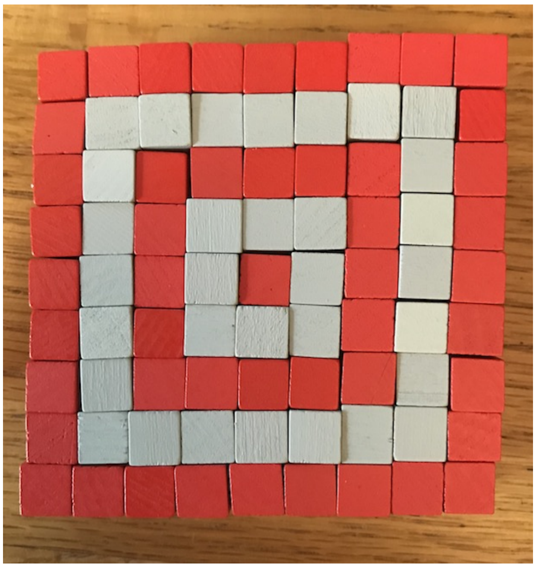
[4 marks]

(iii) Evaluate.

$$\sum_{1}^{3} (4r - 3)^2 - \sum_{2}^{3} (4r - 5)^2$$

[2 marks]

(iv)



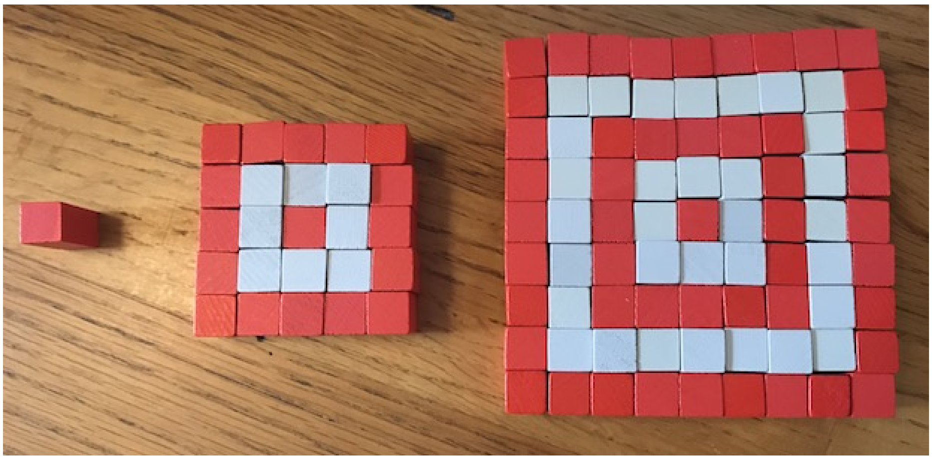
Explain why your part (iii) answer gives the number of red squares in the “concentric square frames” photograph.

[3 marks]

(v) Use your part **(i)** and **(ii)** answers to show that,

$$\sum_{1}^{n} (4r - 3)^2 - \sum_{2}^{n} (4r - 5)^2 = 8n^2 - 8n + 1$$

[3 marks]



(vi) Find an expression for the cumulative number of red squares in all the terms in the "concentric square frames" sequence up to term n and hence show that when $n = 5$ the red squares, actually unit cubes, could build a cuboid measuring $5 \times 5 \times 13$.

[5 marks]

5.5 Answers

Answer 1

(i) and **(ii)**

The n^{th} red+white square will have a side of length $2n + 3$

From this is subtracted 4 white squares of side length n

$$(2n + 3)^2 - 4n^2 = 4n^2 + 12n + 9 - 4n^2$$
$$= 12n + 9$$

Term	1	2	3	4	...	n
Red Frame	21	33	45	57	...	$12n + 9$

[6 marks]

(iii) and **(iv)**

Term	1	2	3	4	...	n
Cumulative	21	54	99	156	...	
Rectangle	3×7	6×9	9×11	12×13	...	$(3n)(2n + 5)$

[6 marks]

(v) $\displaystyle\sum_{1}^{n} (12r + 9) = 3n(2n + 5)$

[3 marks]

(vi)

$$\text{LHS} = \sum_{1}^{n} (12n + 9)$$
$$= 12\sum_{1}^{n} r + 9\sum_{1}^{n} 1$$
$$= 12 \times \frac{1}{2} \times n(n + 1) + 9n$$
$$= 6n^2 + 6n + 9n$$
$$= 6n^2 + 15n$$
$$= (3n)(2n + 5)$$
$$= \text{RHS} \qquad \square$$

[4 marks]

Answer 2

(i)

$$\text{LHS} = \sum_{1}^{n} (4r - 3)^2$$

$$= \sum_{1}^{n} \left(16r^2 - 24r + 9\right)$$

$$= 16 \sum_{1}^{n} r^2 - 24 \sum_{1}^{n} r + 9 \sum_{1}^{n} 1$$

$$= 16 \times \frac{1}{6} n(n+1)(2n+1) - 24 \times \frac{1}{2} n(n+1) + 9n$$

$$= \left(\frac{n}{3}\right)\left(8(n+1)(2n+1) - 36(n+1) + 27\right)$$

$$= \left(\frac{n}{3}\right)\left(16n^2 + 24n + 8 - 36n - 36 + 27\right)$$

$$= \frac{1}{3} n \left(16n^2 - 12n - 1\right)$$

$$= \text{RHS} \qquad \square$$

[4 marks]

(ii)

$$\text{LHS} = \sum_{2}^{n} (4r - 5)^2$$

$$= \sum_{1}^{n} (4r - 5)^2 - 1$$

$$= \sum_{1}^{n} \left(16r^2 - 40r + 25\right) - 1$$

$$= 16 \sum_{1}^{n} r^2 - 40 \sum_{1}^{n} r + 25 \sum_{1}^{n} 1 - 1$$

$$= 16 \times \frac{1}{6} n(n+1)(2n+1) - 40 \times \frac{1}{2} n(n+1) + 25n - 1$$

$$= \left(\frac{n}{3}\right)\left(8(n+1)(2n+1) - 60(n+1) + 75\right) - 1$$

$$= \left(\frac{n}{3}\right)\left(16n^2 + 24n + 8 - 60n - 60 + 75\right) - 1$$

$$= \frac{1}{3} n \left(16n^2 - 36n + 23\right) - 1$$

$$= \text{RHS} \qquad \square$$

[4 marks]

(iii)

$$\sum_{1}^{3}(4r-3)^2-\sum_{2}^{3}(4r-5)^2 = 1^2+5^2+9^2-3^2-7^2$$
$$= 1+25+81-9-49$$
$$= 49$$

[2 marks]

(iv) The outer frame, a square to side length 9, gives rise to a large square of area 9^2. From this could be subtracted a square of side length 7, and so the outer frame contains $9^2 - 7^2$ individual red squares.
Similarly, the central frame contains $5^2 - 3^2$ red squares.
Plus the single red square at the centre, 1^2

$$\text{Total area of red frame} = \left(9^2-7^2\right)+\left(5^2-3^2\right)+1^2$$
$$= 1^2+5^2+9^2-3^2-7^2$$
$$= \sum_{1}^{3}(4r-3)^2-\sum_{2}^{3}(4r-5)^2 \qquad \square$$

[4 marks]

(v)

$$\text{LHS} = \sum_{1}^{n}(4r-3)^2-\sum_{2}^{n}(4r-5)^2$$
$$= \frac{1}{3}n\left(16n^2-12n-1\right)-\left(\frac{1}{3}n\left(16n^2-36n+23\right)-1\right)$$
$$= \frac{1}{3}n\left(16n^2-12n-1-16n^2+36n-23\right)+1$$
$$= \frac{1}{3}n\left(24n-24\right)+1$$
$$= 8n(n-1)+1$$
$$= 8n^2-8n+1$$
$$= \text{RHS} \qquad \square$$

[2 marks]

(vi)

$$\sum_1^n \left(8r^2 - 8r + 1 \right)$$

$$= 8\sum_1^n r^2 - 8\sum_1^n r + \sum_1^n 1$$

$$= 8 \times \frac{1}{6} \times n\,(n+1)\,(2n+1) - 8 \times \frac{1}{2} \times n\,(n+1) + n$$

$$= \frac{4}{3}n\,(n+1)\,(2n+1) - 4n\,(n+1) + n$$

$$= \left(\frac{n}{3}\right)\left(4(n+1)(2n+1) - 12(n+1) + 3\right)$$

$$= \left(\frac{n}{3}\right)\left(8n^2 + 12n + 4 - 12n - 12 + 3\right)$$

$$= \left(\frac{n}{3}\right)\left(8n^2 - 5\right)$$

When $n = 5$,

$$\left(\frac{n}{3}\right)\left(8n^2 - 5\right) = \left(\frac{5}{3}\right)\left(8 \times 5^2 - 5\right)$$

$$= 325$$

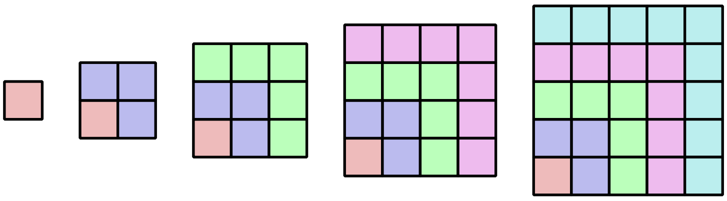
$$= 5 \times 5 \times 13 \qquad \square$$

[5 marks]

6.1 Test

Any solution based entirely on graphical or numerical methods is not acceptable
Marks Available : 40

Question 1



The partitioning of the squares from the "sum of squares" proof is an illustration that the sum of the first n odd numbers is the square number, $\square_n$.

Prove this result using the standard results for $\sum_1^n 1$ and $\sum_1^n r$

[4 marks]

Question 2

Further A-Level Examination Question from June 2018, IAL, F1, Q1

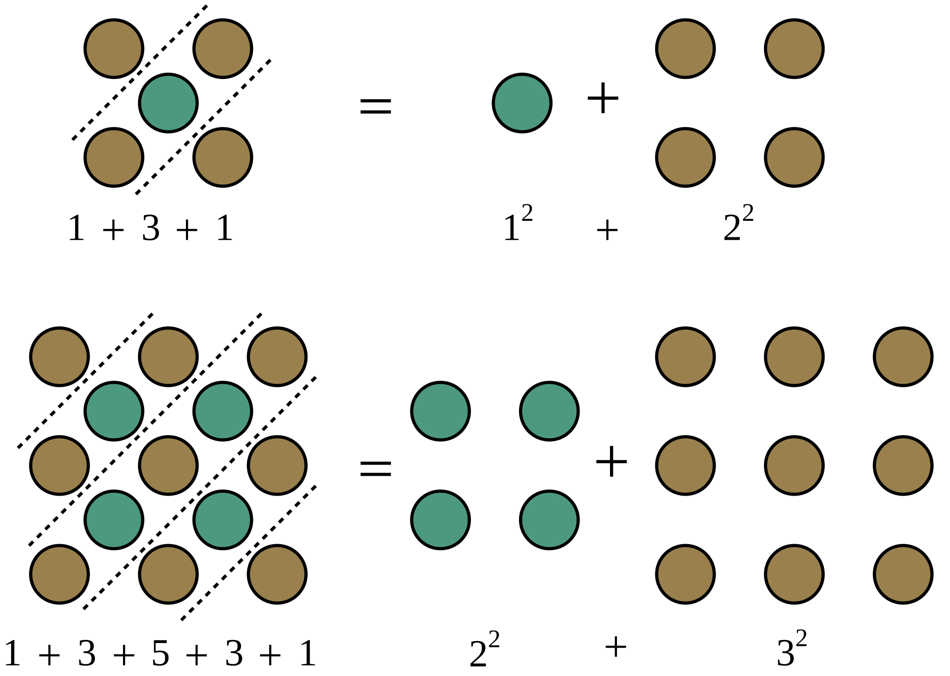
Use the standard results for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$ to show that, for all positive integers n,

$$\sum_{r=1}^{n} r(r+3) = \frac{n}{a}(n+1)(n+b)$$

where a and b are integers to be found.

[4 marks]

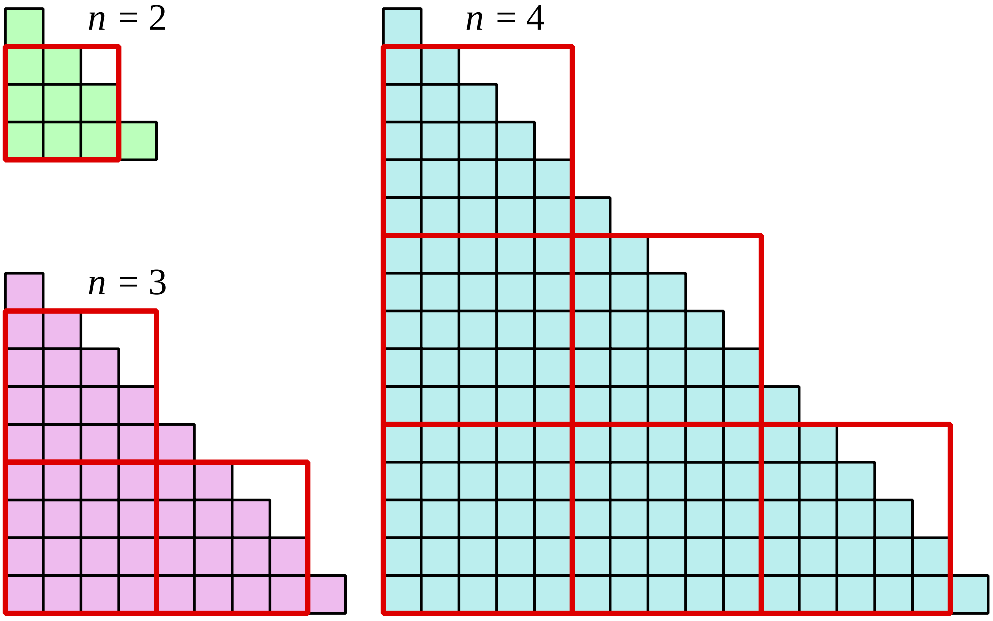
Question 3



State and prove the result suggested by the diagram

[5 marks]

Question 4



The diagrams suggest the following relationship between triangular numbers,

$$T_{n^2} = (n+1)^2\, T_{n-1} - (n-1)\, T_{n-1} + n \qquad\qquad n \geqslant 2$$

Prove this relationship by starting with the RHS and manipulating this into the LHS

[5 marks]

Question 5

Further A-Level Examination Question from May 2017. IAL, FP1, Q8

(a) Use the standard results for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$ to show that,

$$\sum_{r=1}^{n} \left(3r^2 + 8r + 3 \right) = \frac{1}{2} n\,(2n + 5)\,(n + 3)$$

for all positive integers *n*

[5 marks]

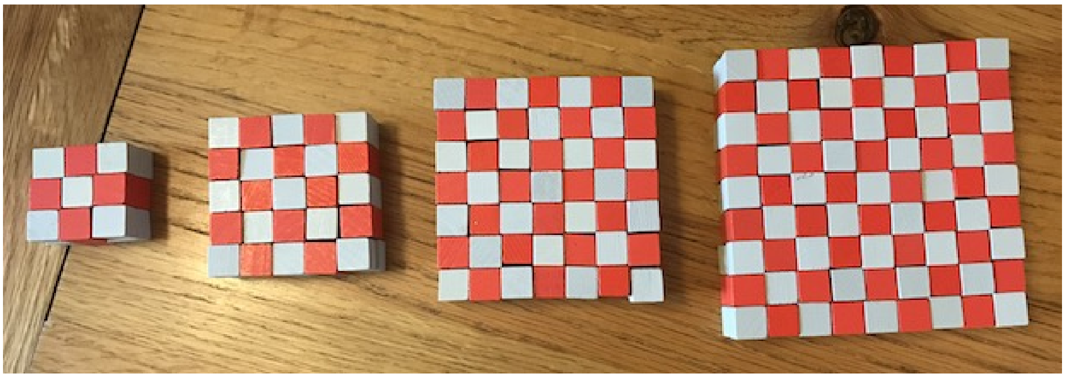
Given that $\sum_{r=1}^{12} \left(3r^2 + 8r + 3 + k\left(2^{r-1}\right) \right) = 3520$

(b) find the exact value of the constant *k*

[4 marks]

Question 6

The first few terms of the chequered board sequence are shown in the photograph.



(i) Find an expression for the number of small red squares in the n^{th} term in this chequered board sequence.

[3 marks]

(ii) Find an expression for the cumulative number of small red squares in the first n terms of the chequered board sequence.

[3 marks]

(iii) Show that the cumulative number of small red squares in the first n terms of the chequered board sequence, when thought of as being unit cubes, can be assembled into a tower with a 2×2 base and a height that is the sum of the first n triangular numbers.

[4 marks]

(iv) Now that you know such a tower can always be built, explain how the result can be visually proven from the photograph at the start of this question.

[3 marks]

6.2 Answers

Answer 1

The result to be proved is that,

$$\sum_{1}^{n}(2r-1) = n^2$$

$$\begin{aligned}
\text{LHS} &= \sum_{1}^{n}(2r-1)\\
&= 2\sum_{1}^{n} r - \sum_{1}^{n} 1\\
&= 2 \times \frac{1}{2} n(n+1) - n\\
&= n(n+1) - n\\
&= n^2 + n - n\\
&= n^2\\
&= \text{RHS} \qquad \square
\end{aligned}$$

[4 marks]

Answer 2

$$\begin{aligned}
\text{LHS} &= \sum_{r=1}^{n} r(r+3)\\
&= \sum_{r=1}^{n}\left(r^2 + 3r\right)\\
&= \sum_{1}^{n} r^2 + 3\sum_{1}^{n} r\\
&= \frac{1}{6} \times n(n+1)(2n+1) + 3 \times \frac{1}{2} \times n(n+1)\\
&= \left(\frac{n(n+1)}{6}\right)(2n+1+9)\\
&= \left(\frac{n(n+1)}{6}\right)(2n+10)\\
&= \frac{n}{3}(n+1)(n+5)
\end{aligned}$$

$$\therefore\ a = 3, \quad b = 5$$

[4 marks]

Answer 3

$$\sum_{1}^{n+1}(2r-1)+\sum_{1}^{n}(2r-1)=n^2+(n+1)^2$$

$$\begin{aligned}
\text{LHS} &= \sum_{1}^{n+1}(2r-1)+\sum_{1}^{n}(2r-1)\\
&= 2\sum_{1}^{n+1}r-\sum_{1}^{n+1}1+2\sum_{1}^{n}r-\sum_{1}^{n}1\\
&= 2\times\frac{1}{2}\times(n+1)(n+2)-(n+1)+2\times\frac{1}{2}\times n(n+1)-n\\
&= (n+1)(n+2)-(n+1)+n(n+1)-n\\
&= n^2+3n+2-n-1+n^2+n-n\\
&= 2n^2+2n+1\\
&= n^2+\left(n^2+2n+1\right)\\
&= n^2+(n+1)^2\\
&= \text{RHS} \qquad \square
\end{aligned}$$

[5 marks]

Answer 4

$$\begin{aligned}
\text{RHS} &= (n+1)^2\,T_{n-1}-(n-1)\,T_{n-1}+n\\
&= \left[(n+1)^2-(n-1)\right]T_{n-1}+n\\
&= \left[n^2+2n+1-n+1\right]\times\frac{1}{2}(n-1)n+n\\
&= \left(\frac{1}{2}n\right)\left(\left[n^2+n+2\right]\times(n-1)+2\right)\\
&= \left(\frac{1}{2}n\right)\left(n^3+n^2+2n-n^2-n-2+2\right)\\
&= \left(\frac{1}{2}n\right)\left(n^3+n\right)\\
&= \frac{1}{2}n^2\left(n^2+1\right)\\
&= T_{n^2}\\
&= \text{RHS} \qquad \square
\end{aligned}$$

[5 marks]

Answer 5

(a)

$$\text{LHS} = \sum_{r=1}^{n} \left(3r^2 + 8r + 3 \right)$$

$$= 3\sum_{1}^{n} r^2 + 8\sum_{1}^{n} r + 3\sum_{1}^{n} 1$$

$$= 3 \times \frac{1}{6} \times n\,(n+1)\,(2n+1) + 8 \times \frac{1}{2} \times n\,(n+1) + 3n$$

$$= \left(\frac{n}{2}\right)\left((n+1)(2n+1) \; + 8(n+1) + 6 \right)$$

$$= \left(\frac{n}{2}\right)\left(2n^2 + 3n + 1 + 8n + 8 + 6 \right)$$

$$= \frac{1}{2} n \left(2n^2 + 11n + 15 \right)$$

$$= \frac{1}{2} n\,(2n+5)\,(n+3)$$

$$= \text{RHS} \qquad \square$$

[5 marks]

(b)

$$\sum_{r=1}^{12} \left(3r^2 + 8r + 3 + k\left(2^{r-1}\right) \right) = 3520$$

$$\sum_{1}^{12} \left(3r^2 + 8r + 3 \right) + k\sum_{1}^{12} \left(2^{r-1} \right) = 3520$$

$$\frac{1}{2} \times 12 \times 29 \times 15 + k\left(2^0 + 2^1 + \ldots + 2^{11}\right) = 3520$$

$$2610 + k \times \frac{2^0\,(2^{12} - 1)}{2 - 1} = 3520$$

$$k \times \frac{1 \times 4095}{1} = 910$$

$$k = \frac{2}{9}$$

[4 marks]

Answer 6

(i) The easiest way to obtain an expression for the number of red squares in any term in the sequence is to note that half of the “square less one” is red. This leads to $\frac{1}{2}\left(n^2 - 1 \right)$ but there remains the issue that the first square are is of side length 3, and the side length then increases by 2 with each subsequent term. This leads to the conclusion that the n^{th} square has side length of ($2n + 1$). Putting this all together gives,

$$\frac{1}{2}\left((2n + 1)^2 - 1 \right) = \frac{1}{2}\left(4n^2 + 4n + 1 - 1 \right)$$

$$= 2n^2 + 2n$$

$$= (2n)(n + 1)$$

This is four times the corresponding triangular number, an observation that will be useful in the last part of the question, although you may like to already see if you can spot the four lots of a triangular number in each term in the photograph.

[3 marks]

(ii)

$$\sum_1^n \left(2r^2 + 2r \right) = 2\sum_1^n r^2 + 2\sum_1^n r$$

$$= 2 \times \frac{1}{6} n(n + 1)(2n + 1) + 2 \times \frac{1}{2} n(n + 1)$$

$$= \left(\frac{n(n + 1)}{3} \right)\left(2n + 1 + 3 \right)$$

$$= \frac{2}{3} n(n + 1)(n + 2)$$

[3 marks]

(iii) The sum of the first n triangular numbers is,

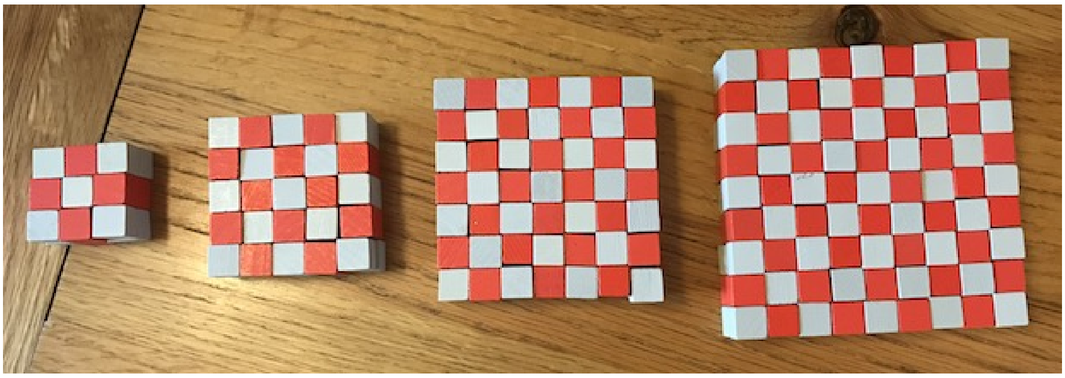
$$\sum_1^n \frac{1}{2} r(r + 1) = \frac{1}{2}\sum_1^n r^2 + \frac{1}{2}\sum_1^n r$$

$$\therefore \frac{2}{3} n(n + 1)(n + 2) = 4 \times \left(\frac{1}{2}\sum_1^n r^2 + \frac{1}{2}\sum_1^n r \right)$$

$$= 2 \times 2 \times \text{Sum of first } n \text{ triangular numbers} \qquad \square$$

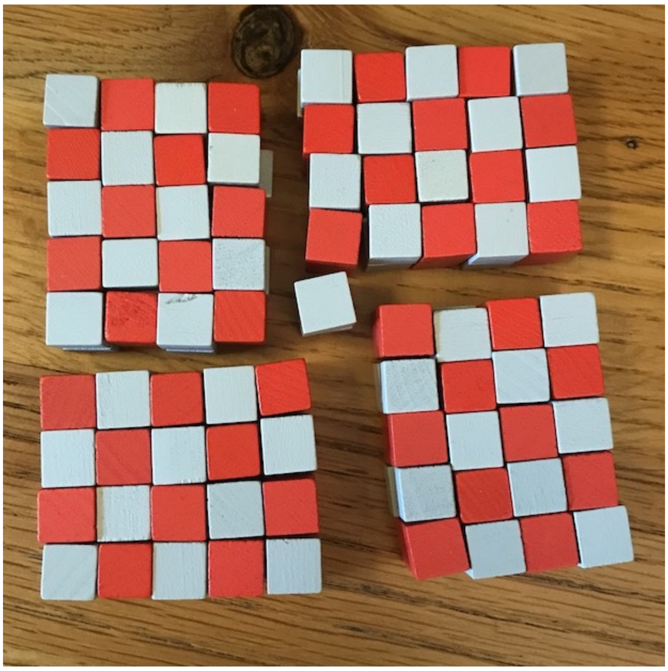
[4 marks]

(iv)



The very first term gives the 2×2 base.
Every subsequent term, n, adds on four times the n^{th} triangular number.
These can be seen in the terms in the photograph.

For example, the fourth term can be partitioned as shown, and by looking carefully at each of the four pieces (ignore the central 1×1 square which is always white) the fourth triangular number $1 + 2 + 3 + 4$ can be seen in the diagonals.



[3 marks]