1.7 Answers

1.7.1 Solution to 1.5 After Watching the Teaching Video

(a) **(i)** $x^2 - \left(-\frac{b}{a}\right)x + \left(\frac{c}{a}\right) = 0$

[1 mark]

(ii) $x^2 - (\text{Sum of Roots}) + (\text{Product of Roots}) = 0$

[1 mark]

(iii) $x^2 + \frac{1}{4}x - \frac{15}{8} = 0$

[1 mark]

(b) **(i)** $\alpha + \beta = -\frac{1}{4}$ **(ii)** $\alpha\beta = -\frac{15}{8}$

[1, 1 mark]

(iii)

$$\begin{aligned}\frac{1}{\alpha} + \frac{1}{\beta} &= \frac{\beta}{\alpha\beta} + \frac{\alpha}{\alpha\beta} \\ &= \frac{\alpha + \beta}{\alpha\beta} \\ &= \frac{(-1) \times 8}{4 \times (-15)} \\ &= \frac{2}{15}\end{aligned}$$

[2 marks]

(iv)

$$\begin{aligned}\alpha^2 + \beta^2 &= \alpha^2 + 2\alpha\beta + \beta^2 - 2\alpha\beta \\ &= (\alpha + \beta)^2 - 2\alpha\beta \\ &= \left(-\frac{1}{4}\right)^2 - 2\left(-\frac{15}{8}\right) \\ &= \frac{1}{16} + \frac{60}{16} \\ &= \frac{61}{16} \qquad \text{or } 3.75\end{aligned}$$

[2 marks]

1.7.2 Solutions to 1.6 Exercise

Answer 1

$$3x^2 + 7x - 2 = 0$$

$$x^2 + \frac{7}{3}x - \frac{2}{3} = 0$$

(i) $\alpha + \beta = -\frac{7}{3}$ **(ii)** $\alpha\beta = -\frac{2}{3}$

[1, 1 marks]

(iii)

$$\begin{aligned} \frac{1}{\alpha} + \frac{1}{\beta} &= \frac{\beta}{\alpha\beta} + \frac{\alpha}{\alpha\beta} \\ &= \frac{\alpha + \beta}{\alpha\beta} \\ &= \frac{(-7) \times 3}{3 \times (-2)} \\ &= \frac{7}{2} \end{aligned}$$

[2 marks]

(iv)

$$\begin{aligned} \alpha^2 + \beta^2 &= \alpha^2 + 2\alpha\beta + \beta^2 - 2\alpha\beta \\ &= (\alpha + \beta)^2 - 2\alpha\beta \\ &= \left(-\frac{7}{3}\right)^2 - 2\left(-\frac{2}{3}\right) \\ &= \frac{49}{9} + \frac{12}{9} \\ &= \frac{61}{9} \qquad \text{or } 6\frac{7}{9} \end{aligned}$$

[2 marks]

Answer 2

$$4x^2 - 3x + 1 = 0$$

$$x^2 - \frac{3}{4}x + \frac{1}{4} = 0$$

(i) $\alpha + \beta = \dfrac{3}{4}$ **(ii)** $\alpha\beta = \dfrac{1}{4}$

[1, 1 marks]

(iii)

$$\begin{aligned} \alpha^2 + \beta^2 &= \alpha^2 + 2\alpha\beta + \beta^2 - 2\alpha\beta \\ &= (\alpha + \beta)^2 - 2\alpha\beta \\ &= \left(\frac{3}{4}\right)^2 - 2\left(\frac{1}{4}\right) \\ &= \frac{9}{16} - \frac{8}{16} \\ &= \frac{1}{16} \qquad \text{or } 0.0625 \end{aligned}$$

[2 marks]

(iv)

$$\begin{aligned} \frac{1}{\alpha^2} + \frac{1}{\beta^2} &= \frac{\beta^2}{\alpha^2\beta^2} + \frac{\alpha^2}{\alpha^2\beta^2} \\ &= \frac{\alpha^2 + \beta^2}{(\alpha\beta)^2} \\ &= \frac{1 \times 4^2}{16 \times 1} \\ &= 1 \end{aligned}$$

[2 marks]

Answer 3

(i)

$$\begin{aligned} \text{LHS} &= (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta) \\ &= \alpha^3 + 3\alpha^2\beta + 3\alpha\beta^2 + \beta^3 - 3\alpha^2\beta - 3\alpha\beta^2 \\ &= \alpha^3 + \beta^3 \\ &= \text{RHS} \qquad \square \end{aligned}$$

[2 marks]

(ii)

$$5x^2 + 6x + 2 = 0$$

$$x^2 + \frac{6}{5}x + \frac{2}{5} = 0$$

$$\therefore \ \alpha + \beta = -\frac{6}{5} \ \text{ and } \ \alpha\beta = \frac{2}{5}$$

$$\begin{aligned} \alpha^3 + \beta^3 &= (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta) \\ &= \left(-\frac{6}{5}\right)^3 - 3 \times \frac{2}{5} \times \left(-\frac{6}{5}\right) \\ &= -\frac{216}{125} + \frac{36}{25} \times \frac{5}{5} \\ &= \frac{-216 + 180}{125} \\ &= -\frac{36}{125} \ \text{ or } \ 0.288 \end{aligned}$$

[4 marks]

Answer 4

$$x^2 - \left(\frac{1}{2} - \frac{2}{3}\right)x + \left(\frac{1}{2} \times \frac{-2}{3}\right) = 0$$

$$x^2 - \left(-\frac{1}{6}\right)x - \frac{1}{3} = 0$$

$$6x^2 + x - 2 = 0$$

[3 marks]

Answer 5

$$x^2 - \left(\frac{1 + 2i}{3} + \frac{1 - 2i}{3}\right)x + \left(\frac{1 + 2i}{3}\right)\left(\frac{1 - 2i}{3}\right) = 0$$

$$9x^2 - (3 + 6i + 3 - 6i)x + (1 + 2i)(1 - 2i) = 0$$

$$9x^2 - 6x + 5 = 0$$

[4 marks]

Answer 6

$$6x^2 + 36x + k = 0$$

$$x^2 + 6x + \frac{k}{6} = 0$$

Let the two roots be α and $\frac{1}{\alpha}$

Look at the product of the roots:

$$\alpha \times \frac{1}{\alpha} = \frac{k}{6}$$

$$1 = \frac{k}{6}$$

$$k = 6$$

[4 marks]

Answer 7
(i)

$$x^3 - 3x^2 + 12x + 16 = 0$$

The change of variable is, $x = t - \frac{(-3)}{3 \times 1}$

$$x = t + 1$$

$$(t + 1)^3 - 3(t + 1)^2 + 12(t + 1) + 16 = 0$$

$$t^3 + 3t^2 + 3t + 1 - 3t^2 - 6t - 3 + 12t + 12 + 16 = 0$$

$$t^3 + 9t + 26 = 0 \qquad \square$$

[4 marks]

(ii)

$$C = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} \text{ with } p = 9 \text{ and } q = 26$$

$$= \sqrt[3]{-\frac{26}{2} + \sqrt{\frac{26^2}{4} + \frac{9^3}{27}}}$$

$$= \sqrt[3]{-13 + \sqrt{169 + 27}}$$

$$= \sqrt[3]{-13 + 14}$$

$$= 1$$

$$\alpha = 1 - \frac{9}{3 \times 1}$$

$$= -2$$

[3 marks]

(iii) Using the technique of polynomial division from the year 1 course,

$$\begin{array}{r|llllllll}
\multicolumn{1}{r}{} & & & t^2 & - & 2t & + & 13 \\ \cline{2-8}
t + 2 & t^3 & + & 0t^2 & + & 9t & + & 26 \\
 & t^3 & + & 2t^2 & & & & \\ \cline{2-4}
 & & - & 2t^2 & + & 9t & & \\
 & & - & 2t^2 & - & 4t & & \\ \cline{3-6}
 & & & & & 13t & + & 26 \\
 & & & & & 13t & + & 26 \\ \cline{5-8}
 & & & & & & & 0 \quad \leftarrow \text{ as expected}
\end{array}$$

For the quadratic, $t = \dfrac{2 \pm \sqrt{2^2 - 4 \times 1 \times 13}}{2 \times 1}$

$$= \frac{2 \pm \sqrt{-48}}{2}$$

$$= 1 \pm 2\sqrt{3}\,\text{i}$$

$$-2, \qquad 1 - 2\sqrt{3}\,\text{i} \qquad 1 + 2\sqrt{3}\,\text{i}$$

[2 marks]

(iv)

Using $x = t + 1$

$$-1, \qquad 2 - 2\sqrt{3}\,\text{i}, \qquad 2 + 2\sqrt{3}\,\text{i}$$

[2 marks]