2.5 Answers

2.5.1 Solutions to 2.3 Teaching Video

$$2x^2 + 7x + 1 = 0$$

Divide through by 2

$$x^2 + \frac{7}{2}x + \frac{1}{2} = 0$$

$$\therefore\ \alpha + \beta = -\frac{7}{2} \quad \text{and} \quad \alpha\beta = \frac{1}{2}$$

Sum of new roots : $(2\alpha + 1) + (2\beta + 1)$

$$= 2(\alpha + \beta) + 2$$

$$= 2 \times \left(-\frac{7}{2}\right) + 2$$

$$= -5$$

Product of new roots : $(2\alpha + 1)(2\beta + 1)$

$$= 4\alpha\beta + 2(\alpha + \beta) + 1$$

$$= 4 \times \frac{1}{2} + 2 \times \left(-\frac{7}{2}\right) + 1$$

$$= 2 - 7 + 1$$

$$= -4$$

The sought after polynomial is thus,

$$w^2 + 5w - 4 = 0$$

[5 marks]

2.5.2 Solutions to 2.4 Exercise

Answer 1

$$3x^2 - 4x + 6 = 0$$

Divide through by 3

$$x^2 - \frac{4}{3}x + 2 = 0$$

$$\therefore\ \alpha + \beta = \frac{4}{3} \quad \text{and} \quad \alpha\beta = 2$$

$$\begin{aligned} \text{Sum of new roots : } (3\alpha + 1) + (3\beta + 1) &= 3(\alpha + \beta) + 2 \\ &= 3 \times \left(\frac{4}{3}\right) + 2 \\ &= 6 \end{aligned}$$

$$\begin{aligned} \text{Product of new roots : } (3\alpha + 1)(3\beta + 1) &= 9\alpha\beta + 3(\alpha + \beta) + 1 \\ &= 9 \times 2 + 3 \times \left(\frac{4}{3}\right) + 1 \\ &= 18 + 4 + 1 \\ &= 23 \end{aligned}$$

The sought after polynomial is thus,

$$w^2 - 6w + 23 = 0$$

[5 marks]

Answer 2

$$3x^2 + 6x + 1 = 0$$

Divide through by 3

$$x^2 + 2x + \frac{1}{3} = 0$$

$$\therefore\ \alpha + \beta = -2 \text{ and } \alpha\beta = \frac{1}{3}$$

Sum of new roots : $(1 - \alpha) + (1 - \beta)$

$$= 2 - (\alpha + \beta)$$
$$= 2 - (-2)$$
$$= 4$$

Product of new roots : $(1 - \alpha)(1 - \beta)$

$$= 1 - (\alpha + \beta) + \alpha\beta$$
$$= 1 - (-2) + \frac{1}{3}$$
$$= \frac{10}{3}$$

The sought after polynomial is thus,

$$w^2 - 4w + \frac{10}{3} = 0$$
$$3w^2 - 12w + 10 = 0$$

[5 marks]

Answer 3

$$2x^2 + 3x + 7 = 0$$

Divide through by 2

$$x^2 + \frac{3}{2}x + \frac{7}{2} = 0$$

$$\therefore \ \alpha + \beta = -\frac{3}{2} \text{ and } \alpha\beta = \frac{7}{2}$$

Sum of new roots : $\left(\alpha^2 + 1\right) + \left(\beta^2 + 1\right)$

$$= \alpha^2 + \beta^2 + 2$$

$$= (\alpha + \beta)^2 - 2\alpha\beta + 2$$

$$= \left(-\frac{3}{2}\right)^2 - 2 \times \frac{7}{2} + 2$$

$$= \frac{9}{4} - 7 + 2$$

$$= -\frac{11}{4}$$

Product of new roots : $\left(\alpha^2 + 1\right)\left(\beta^2 + 1\right)$

$$= \alpha^2\beta^2 + \alpha^2 + \beta^2 + 1$$

$$= (\alpha\beta)^2 + (\alpha + \beta)^2 - 2\alpha\beta + 1$$

$$= \left(\frac{7}{2}\right)^2 + \left(-\frac{3}{2}\right)^2 - 2 \times \frac{7}{2} + 1$$

$$= \frac{49}{4} + \frac{9}{4} - 7 + 1$$

$$= \frac{17}{2}$$

The sought after polynomial is thus,

$$w^2 + \frac{11}{4}w + \frac{17}{2} = 0$$

$$4w^2 + 11w + 34 = 0$$

[6 marks]

Answer 4

$$5x^2 + 15x + 8 = 0$$

Divide through by 5

$$x^2 + 3x + \frac{8}{5} = 0$$

$$\therefore \ \alpha + \beta = -3 \text{ and } \ \alpha\beta = \frac{8}{5}$$

Sum of new roots : $\left(1 + \frac{1}{\alpha}\right) + \left(1 + \frac{1}{\beta}\right)$

$$= 2 + \frac{1}{\alpha} + \frac{1}{\beta}$$

$$= 2 + \frac{\beta}{\beta\alpha} + \frac{\alpha}{\alpha\beta}$$

$$= 2 + \frac{\alpha + \beta}{\alpha\beta}$$

$$= 2 + \frac{(-3) \times 5}{8}$$

$$= \frac{1}{8}$$

Product of new roots : $\left(1 + \frac{1}{\alpha}\right)\left(1 + \frac{1}{\beta}\right)$

$$= 1 + \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\alpha\beta}$$

$$= 1 + \frac{\beta}{\beta\alpha} + \frac{\alpha}{\alpha\beta} + \frac{1}{\alpha\beta}$$

$$= 1 + \frac{\alpha + \beta + 1}{\alpha\beta}$$

$$= 1 + \frac{(-3 + 1) \times 5}{8}$$

$$= -\frac{1}{4}$$

The sought after polynomial is thus,

$$w^2 - \frac{1}{8}w - \frac{1}{4} = 0$$

$$8w^2 - w - 2 = 0$$

[8 marks]

Answer 5

(a)

$$3x^2 + 2x + 5 = 0$$

Divide through by 3

$$x^2 + \frac{2}{3}x + \frac{5}{3} = 0$$

$$\therefore \alpha + \beta = -\frac{2}{3} \text{ and } \alpha\beta = \frac{5}{3}$$

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$$

$$= \left(-\frac{2}{3}\right)^2 - 2 \times \frac{5}{3}$$

$$= \frac{4}{9} - \frac{30}{9}$$

$$= -\frac{26}{9}$$

[2 marks]

(b)

$$(\alpha + \beta)^3 = \alpha^3 + 3\alpha^2\beta + 3\alpha\beta^2 + \beta^3$$

$$\therefore \alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$$

$$= \left(-\frac{2}{3}\right)^3 - 3 \times \frac{5}{3} \times \left(-\frac{2}{3}\right)$$

$$= -\frac{8}{27} + \frac{90}{27}$$

$$= \frac{82}{27} \qquad \square$$

[2 marks]

(c)

$$\text{Sum of new roots : } \left(\alpha + \frac{\alpha}{\beta^2}\right) + \left(\beta + \frac{\beta}{\alpha^2}\right)$$

$$= (\alpha + \beta) + \frac{\alpha^3 + \beta^3}{\alpha^2\beta^2}$$

$$= \left(-\frac{2}{3}\right) + \frac{82 \times 9}{27 \times 25}$$

$$= -\frac{2}{3} + \frac{82}{75}$$

$$= \frac{32}{75}$$

Product of new roots : $\left(\alpha + \frac{\alpha}{\beta^2}\right)\left(\beta + \frac{\beta}{\alpha^2}\right)$

$$= \alpha\beta\left(1 + \frac{1}{\beta^2}\right)\left(1 + \frac{1}{\alpha^2}\right)$$

$$= \alpha\beta\left(1 + \frac{1}{\beta^2} + \frac{1}{\alpha^2} + \frac{1}{\alpha^2\beta^2}\right)$$

$$= \alpha\beta\left(1 + \frac{\alpha^2 + \beta^2}{\alpha^2\beta^2} + \frac{1}{\alpha^2\beta^2}\right)$$

$$= \alpha\beta + \frac{\alpha^2 + \beta^2}{\alpha\beta} + \frac{1}{\alpha\beta}$$

$$= \left(\frac{5}{3}\right) + \frac{-26 \times 3}{9 \times 5} + \frac{3}{5}$$

$$= \frac{25}{15} - \frac{26}{15} + \frac{9}{15}$$

$$= \frac{8}{15}$$

The sought after polynomial is thus,

$$w^2 - \frac{32}{75}w + \frac{8}{15} = 0$$

$$75w^2 - 32w + 40 = 0$$

[4 marks]

Answer 6

(i)

$$\left(\sqrt{3} - 5\right)^3 = \left(\sqrt{3}\right)^3 + 3\left(\sqrt{3}\right)^2(-5) + 3\left(\sqrt{3}\right)(-5)^2 + (-5)^3$$

$$= 3\sqrt{3} - 45 + 75\sqrt{3} - 125$$

$$= 78\sqrt{3} - 170$$

[2 marks]

(ii)

$$x^3 - 66x + 340 = 0$$

$$p = -66 \text{ and } q = 340$$

$$\text{Check } 4p^3 + 27q^2 = 4 \times (-66)^3 + 27 \times 340^2$$

$$= -1149984 + 3121200$$

$$= 1971216$$

$$\neq 0 \qquad \text{so OK to proceed...}$$

$$C = \sqrt[3]{-\frac{340}{2} + \sqrt{\frac{340^2}{4} + \frac{(-66)^3}{27}}}$$

$$= \sqrt[3]{-170 + \sqrt{28900 - 10648}}$$

$$= \sqrt[3]{-170 + 78\sqrt{3}}$$

$$= \sqrt{3} - 5 \qquad \text{from part (i)}$$

$$x = \sqrt{3} - 5 - \frac{-66}{3\left(\sqrt{3} - 5\right)}$$

$$= \sqrt{3} - 5 + \frac{22}{\left(\sqrt{3} - 5\right)} \times \frac{\left(\sqrt{3} + 5\right)}{\left(\sqrt{3} + 5\right)}$$

$$= \sqrt{3} - 5 + \frac{22\left(\sqrt{3} + 5\right)}{3 - 25}$$

$$= \sqrt{3} - 5 - \sqrt{3} - 5$$

$$= -10$$

[6 marks]

(iii) $\therefore\ (x + 10)$ is a factor of $x^3 - 66x + 340$

Using the technique of polynomial division from the year 1 course,

$$
\begin{array}{r|l}
 & \quad\; x^2 - 10x + 34 \\
\cline{2-2}
x + 10 & x^3 + 0x^2 - 66x + 340 \\
 & x^3 + 10x^2 \\
\cline{2-2}
 & \quad - 10x^2 - 66x \\
 & \quad - 10x^2 - 100x \\
\cline{2-2}
 & \qquad\qquad 34x + 340 \\
 & \qquad\qquad 34x + 340 \\
\cline{2-2}
 & \qquad\qquad\qquad 0 \;\leftarrow \text{ as expected}
\end{array}
$$

For the quadratic, $x = \dfrac{10 \pm \sqrt{(-10)^2 - 4 \times 1 \times 34}}{2 \times 1}$

$= \dfrac{10 \pm \sqrt{-36}}{2}$

$= 5 \pm 3\mathrm{i}$

The three roots of the cubic equation are,

$-10, \quad 5 - 3\mathrm{i} \quad 5 + 3\mathrm{i}$

[2 marks]