3.6 Answers

3.6.1 Solution to 3.4 Teaching Video Example

$$2x^3 + 5x^2 - 2x + 3 = 0$$

Divide through by 2

$$x^3 + \frac{5}{2}x^2 - x + \frac{3}{2} = 0$$

$$\therefore \ \alpha + \beta + \gamma = -\frac{5}{2} \qquad \alpha\beta + \beta\gamma + \gamma\alpha = -1 \quad \text{and} \quad \alpha\beta\gamma = -\frac{3}{2}$$

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma}{\alpha\beta\gamma} + \frac{\alpha\gamma}{\alpha\beta\gamma} + \frac{\alpha\beta}{\alpha\beta\gamma}$$

$$= \frac{\alpha\beta + \beta\gamma + \gamma\alpha}{\alpha\beta\gamma}$$

$$= \frac{(-1) \times 2}{(-3)}$$

$$= \frac{2}{3}$$

[4 marks]

3.6.2 Solution to 3.5 Exercise

Answer 1 $$4x^3 - 3x^2 - x + 6 = 0$$

Divide through by 4

$$x^3 - \frac{3}{4}x^2 - \frac{1}{4}x + \frac{3}{2} = 0$$

(i) $\alpha + \beta + \gamma = \frac{3}{4}$ **(ii)** $\alpha\beta + \beta\gamma + \gamma\alpha = -\frac{1}{4}$

(iii) $\alpha\beta\gamma = -\frac{3}{2} \Rightarrow \alpha^2\beta^2\gamma^2 = \frac{9}{4}$

(iv)
$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma}{\alpha\beta\gamma} + \frac{\alpha\gamma}{\alpha\beta\gamma} + \frac{\alpha\beta}{\alpha\beta\gamma}$$
$$= \frac{\alpha\beta + \beta\gamma + \gamma\alpha}{\alpha\beta\gamma}$$
$$= \frac{(-1) \times 2}{4 \times (-3)} = \frac{1}{6}$$

[1, 1, 1, 2 marks]

Answer 2

$$ax^3 + bx^2 + cx + d = 0 \qquad \Rightarrow x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} = 0$$

$$\alpha = 8, \quad \beta = 9 \text{ and } \gamma = -10$$

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$-\frac{b}{a} = 8 + 9 - 10 = 7$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$$

$$\frac{c}{a} = 8 \times 9 + 9 \times (-10) + (-10) \times 8$$
$$= 72 - 90 - 80 = -98$$

$$\alpha\beta\gamma = -\frac{d}{a}$$

$$-\frac{d}{a} = 8 \times 9 \times (-10) = -720$$

Thus the polynomial sought is,

$$x^3 - 7x^2 - 98x + 720 = 0$$

That is, $a = 1$, $b = -7$, $c = -98$, $d = 720$ (for example)

[2 marks]

Answer 3

$$2x^3 + 4x^2 + 7x + 1 = 0$$

Divide through by 2

$$x^3 + 2x^2 + \frac{7}{2}x + \frac{1}{2} = 0$$

(i) $(\alpha + \beta + \gamma)^2 = \alpha^2 + \beta^2 + \gamma^2 + 2(\alpha\beta + \beta\gamma + \gamma\alpha)$

(ii) $\alpha + \beta + \gamma = -2$

(iii) $\alpha\beta + \beta\gamma + \gamma\alpha = \frac{7}{2}$

(iv) $\alpha\beta\gamma = -\frac{1}{2}$

(v) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$

$$= (-2)^2 - 2 \times \frac{7}{2} = -3$$

[1, 1, 1, 1, 2 marks]

Answer 4

$$x^3 - 8x^2 + 28x - 32 = 0$$

$$\alpha + \beta + \gamma = 8 \qquad \alpha\beta + \beta\gamma + \gamma\alpha = 28 \qquad \alpha\beta\gamma = 32$$

(i)

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma}{\alpha\beta\gamma} + \frac{\alpha\gamma}{\alpha\beta\gamma} + \frac{\alpha\beta}{\alpha\beta\gamma}$$

$$= \frac{\alpha\beta + \beta\gamma + \gamma\alpha}{\alpha\beta\gamma}$$

$$= \frac{28}{32} = \frac{7}{8} \quad \text{or} \quad 0.875$$

(ii)

$$\begin{aligned}(\alpha + 2)(\beta + 2)(\gamma + 2) &= (\alpha + 2)(\beta\gamma + 2\beta + 2\gamma + 4) \\ &= \alpha\beta\gamma + 2\alpha\beta + 2\alpha\gamma + 4\alpha + 2\beta\gamma + 4\beta + 4\gamma + 8 \\ &= \alpha\beta\gamma + 2(\alpha\beta + \beta\gamma + \gamma\alpha) + 4(\alpha + \beta + \gamma) + 8 \\ &= 32 + 2 \times 28 + 4 \times 8 + 8 \\ &= 128\end{aligned}$$

(iii)

$$(\alpha + \beta + \gamma)^2 = \alpha^2 + \beta^2 + \gamma^2 + 2(\alpha\beta + \beta\gamma + \gamma\alpha)$$ (see question 3)

$$\begin{aligned}\therefore\ \alpha^2 + \beta^2 + \gamma^2 &= (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha) \\ &= 8^2 - 2 \times 28 \\ &= 8\end{aligned}$$

[3, 3, 2 marks]

Answer 5

(i)

$$\left(6 + 4\sqrt{3}\,\mathrm{i}\right)^3 = \left(2\left(3 + 2\sqrt{3}\,\mathrm{i}\right)\right)^3$$
$$= 8\left(3 + 2\sqrt{3}\,\mathrm{i}\right)^3$$
$$= 8\left(3 + 2\sqrt{3}\,\mathrm{i}\right)\left(3 + 2\sqrt{3}\,\mathrm{i}\right)^2$$
$$= 8\left(3 + 2\sqrt{3}\,\mathrm{i}\right)\left(9 + 12\sqrt{3}\,\mathrm{i} + 12\,\mathrm{i}^2\right)$$
$$= 8\left(3 + 2\sqrt{3}\,\mathrm{i}\right)\left(-3 + 12\sqrt{3}\,\mathrm{i}\right)$$
$$= 24\left(3 + 2\sqrt{3}\,\mathrm{i}\right)\left(-1 + 4\sqrt{3}\,\mathrm{i}\right)$$
$$= 24\left(-3 + 12\sqrt{3}\,\mathrm{i} - 2\sqrt{3}\,\mathrm{i} + 24\,\mathrm{i}^2\right)$$
$$= 24\left(-27 + 10\sqrt{3}\,\mathrm{i}\right)$$
$$= -648 + 240\sqrt{3}\,\mathrm{i}$$

[3 marks]

(ii) $x^3 - 252x + 1296 = 0 \Rightarrow p = -252, \quad q = 1296$

$$4p^3 + 27q^2 = 4\,(-252)^3 + 27 \times 1296^2$$
$$= -64012032 + 45349632$$
$$= -18662400 \qquad \text{Which is } \neq 0 \text{ so OK to proceed}$$

$$C = \sqrt[3]{-\frac{1296}{2} + \sqrt{\frac{1296^2}{4} + \frac{(-252)^3}{27}}}$$
$$= \sqrt[3]{-648 + \sqrt{419904 - 592704}}$$
$$= \sqrt[3]{-648 + \sqrt{-172800}}$$
$$= \sqrt[3]{-648 + \sqrt{57600 \times 3 \times (-1)}}$$
$$= \sqrt[3]{-648 + 240\sqrt{3}\,\mathrm{i}}$$
$$= 6 + 4\sqrt{3}\,\mathrm{i} \qquad \text{from part (i)}$$

$$x = 6 + 4\sqrt{3}\,\mathrm{i} - \frac{(-252)}{3\left(6 + 4\sqrt{3}\,\mathrm{i}\right)}$$

$$= 6 + 4\sqrt{3}\,\mathrm{i} + \frac{84\left(6 - 4\sqrt{3}\,\mathrm{i}\right)}{\left(6 + 4\sqrt{3}\,\mathrm{i}\right)\left(6 - 4\sqrt{3}\,\mathrm{i}\right)}$$

$$= 6 + 4\sqrt{3}\,\mathrm{i} + \frac{84\left(6 - 4\sqrt{3}\,\mathrm{i}\right)}{36 + 48}$$

$$= 6 + 4\sqrt{3}\,\mathrm{i} + 6 - 4\sqrt{3}\,\mathrm{i}$$

$$= 12$$

[3 marks]

(iii) $\therefore\ \left(x - 12\right)$ is a factor of $x^3 - 252x + 1296$

Using the technique of polynomial division from the year 1 course,

$$\begin{array}{r|l}
 & x^2 + 12x - 108 \\
\hline
x - 12 & x^3 + 0x^2 - 252x + 1296 \\
 & \underline{x^3 - 12x^2} \\
 & \quad 12x^2 - 252x \\
 & \quad \underline{12x^2 - 144x} \\
 & \qquad -108x + 1296 \\
 & \qquad \underline{-108x + 1296} \\
 & \qquad\qquad 0 \leftarrow \text{as expected}
\end{array}$$

For the quadratic, $x = \dfrac{-12 \pm \sqrt{(-12)^2 - 4 \times 1 \times (-108)}}{2 \times 1}$

$$= \frac{-12 \pm \sqrt{576}}{2}$$

$$= \frac{-12 \pm 24}{2}$$

$$= -18 \text{ or } 6$$

The three roots of the cubic equation are,

$12, \quad 6 \quad (-18)$

[2 marks]

Answer 6

(i) $(\alpha + \beta + \gamma)^3 =$

$$\alpha^3 + 3\alpha^2\beta + 3\alpha^2\gamma + 3\beta^2\alpha + \beta^3 + 3\beta^2\gamma + 3\gamma^2\alpha + 3\gamma^2\beta + \gamma^3 + 6\alpha\beta\gamma$$

[1 mark]

(ii)

$$3\alpha^3 + 3\alpha^2\beta + 3\alpha^2\gamma + 3\beta^2\alpha + 3\beta^3 + 3\beta^2\gamma + 3\gamma^2\alpha + 3\gamma^2\beta + 3\gamma^3$$
$$- 2(\alpha^3 + \beta^3 + \gamma^3) + 6\alpha\beta\gamma$$
$$= 3\alpha^2(\alpha + \beta + \gamma) + 3\beta^2(\alpha + \beta + \gamma) + 3\gamma^2(\alpha + \beta + \gamma)$$
$$- 2(\alpha^3 + \beta^3 + \gamma^3) + 6\alpha\beta\gamma$$
$$= 3(\alpha^2 + \beta^2 + \gamma^2)(\alpha + \beta + \gamma) - 2(\alpha^3 + \beta^3 + \gamma^3) + 6\alpha\beta\gamma$$

So,

$$2(\alpha^3 + \beta^3 + \gamma^3) = 3(\alpha^2 + \beta^2 + \gamma^2)(\alpha + \beta + \gamma) - (\alpha + \beta + \gamma)^3 + 6\alpha\beta\gamma$$

From question 3,

$$(\alpha + \beta + \gamma)^2 = \alpha^2 + \beta^2 + \gamma^2 + 2(\alpha\beta + \beta\gamma + \gamma\alpha)$$

That is, $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$

Thus,

$$2(\alpha^3 + \beta^3 + \gamma^3) =$$
$$3\left((\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)\right)(\alpha + \beta + \gamma) - (\alpha + \beta + \gamma)^3 + 6\alpha\beta\gamma$$

And finally,

$$\alpha^3 + \beta^3 + \gamma^3 =$$
$$\frac{3}{2}\left((\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)\right)(\alpha + \beta + \gamma) - \frac{1}{2}(\alpha + \beta + \gamma)^3 + 3\alpha\beta\gamma$$

Other expressions that can result include,

$$\alpha^3 + \beta^3 + \gamma^3 = (\alpha + \beta + \gamma)^3 - 3(\alpha + \beta + \gamma)(\alpha\beta + \beta\gamma + \gamma\alpha) + 3\alpha\beta\gamma$$

$$\alpha^3 + \beta^3 + \gamma^3 = (\alpha + \beta + \gamma)\left((\alpha + \beta + \gamma)^2 - 3(\alpha\beta + \beta\gamma + \gamma\alpha)\right) + 3\alpha\beta\gamma$$

[2 marks]

Answer 7

$$x^3 - 2x^2 + 4x - 5 = 0$$

$$p + q + r = 2 \qquad pq + qr + rp = 4 \qquad pqr = 5$$

(i)

$$\frac{2}{p} + \frac{2}{q} + \frac{2}{r} = \frac{2pr}{pqr} + \frac{2pr}{pqr} + \frac{2pq}{pqr}$$

$$= \frac{2(pq + qr + rq)}{pqr}$$

$$= \frac{2 \times 4}{5}$$

$$= \frac{8}{5}$$

[3 marks]

(ii)

$$(p - 4)(q - 4)(r - 4)$$

$$= (p - 4)(qr - 4q - 4r + 16)$$

$$= pqr - 4pq - 4pr + 16p - 4qr + 16q + 16r - 64$$

$$= pqr - 4(pq + qr + rp) + 16(p + q + r) - 64$$

$$= 5 - 4 \times 4 + 16 \times 2 - 64$$

$$= -43$$

[3 marks]

(iii) From question 6

$$p^3 + q^3 + r^3 =$$

$$\frac{3}{2}\left((p + q + r)^2 - 2(pq + qr + rp) \right)(p + q + r) - \frac{1}{2}(p + q + r)^3 + 3pqr$$

$$= \frac{3}{2}\left(2^2 - 2 \times 4\right) \times 2 - \frac{1}{2} \times 2^3 + 3 \times 5$$

$$= -12 - 4 + 15$$

$$= -16 + 120$$

$$= -1$$

[2 marks]