4.4 Answers

4.4.1 Solution 4.2 Example (Teaching Video)

METHOD 1

$$3x^3 + x^2 + 2x + 6 = 0$$

Divide through by 3

$$x^3 + \frac{1}{3}x^2 + \frac{2}{3}x + 2 = 0$$

$$\alpha + \beta + \gamma = -\frac{1}{3} \qquad \alpha\beta + \beta\gamma + \gamma\alpha = \frac{2}{3} \qquad \alpha\beta\gamma = -2$$

Sum of new roots : $3\alpha + 3\beta + 3\gamma = 3(\alpha + \beta + \gamma)$

$$= 3 \times -\frac{1}{3}$$

$$= -1$$

Sum of product pairs of new roots :

$$3\alpha \times 3\beta + 3\beta \times 3\gamma + 3\gamma \times 3\alpha = 9(\alpha\beta + \beta\gamma + \gamma\alpha)$$

$$= 9 \times \frac{2}{3}$$

$$= 6$$

Sum of product triple of new roots :

$$3\alpha \times 3\beta \times 3\gamma = 27\alpha\beta\gamma$$

$$= 27 \times -2$$

$$= -54$$

The polynomial with roots 3α, 3β and 3γ is thus, $w^3 + w^2 + 6w + 54 = 0$

METHOD 2

$$w = 3x \Rightarrow x = \frac{w}{3}$$

$$3\left(\frac{w}{3}\right)^3 + \left(\frac{w}{3}\right)^2 + 2\left(\frac{w}{3}\right) + 6 = 0$$

$$\frac{w^3}{9} + \frac{w^2}{9} + \frac{2w}{3} + 6 = 0$$

Multiply through by 9

$w^3 + w^2 + 6w + 54 = 0$ as before with Method 1

[6 marks]

4.4.2 Solutions to 4.3 Exercise

Answer 1

(i) $x^3 - 7x^2 + 6x + 5 = 0 \Rightarrow m = -7,\ n = 6$

[2 marks]

(ii) **METHOD 1**

$$x^3 - 7x^2 + 6x + 5 = 0$$

$$\alpha + \beta + \gamma = 7 \qquad \alpha\beta + \beta\gamma + \gamma\alpha = 6 \qquad \alpha\beta\gamma = -5$$

Sum of new roots : $2\alpha + 2\beta + 2\gamma = 2(\alpha + \beta + \gamma)$

$$= 2 \times 7$$

$$= 14$$

Sum of product pairs of new roots :

$$2\alpha \times 2\beta + 2\beta \times 2\gamma + 2\gamma \times 2\alpha = 4(\alpha\beta + \beta\gamma + \gamma\alpha)$$

$$= 4 \times 6$$

$$= 24$$

Sum of product triple of new roots :

$$2\alpha \times 2\beta \times 2\gamma = 8\alpha\beta\gamma$$

$$= 8 \times -5$$

$$= -40$$

The polynomial with roots 2α, 2β and 2γ is thus, $w^3 - 14w^2 + 24w + 40 = 0$

METHOD 2

$$w = 2x \Rightarrow x = \frac{w}{2}$$

$$\left(\frac{w}{2}\right)^3 - 7\left(\frac{w}{2}\right)^2 + 6\left(\frac{w}{2}\right) + 5 = 0$$

$$\frac{w^3}{8} - \frac{7w^2}{4} + 3w + 5 = 0$$

Multiply through by 8

$w^3 - 14w^2 + 24w + 40 = 0$ as before with Method 1

[3 marks]

Answer 2

METHOD 1

$$x^3 + 10x^2 + 3x - 7 = 0$$

$$\alpha + \beta + \gamma = -10 \qquad \alpha\beta + \beta\gamma + \gamma\alpha = 3 \qquad \alpha\beta\gamma = 7$$

Sum of new roots : $\alpha - 1 + \beta - 1 + \gamma - 1 = \alpha + \beta + \gamma - 3$

$$= -10 - 3$$

$$= -13$$

Sum of product pairs of new roots :

$$(\alpha - 1)(\beta - 1) + (\beta - 1)(\gamma - 1) + (\gamma - 1)(\alpha - 1)$$

$$= \alpha\beta - \alpha - \beta + 1 + \beta\gamma - \beta - \gamma + 1 + \gamma\alpha - \alpha - \gamma + 1$$

$$= (\alpha\beta + \beta\gamma + \gamma\alpha) - 2(\alpha + \beta + \gamma) + 3$$

$$= 3 - 2 \times (-10) + 3$$

$$= 3 + 20 + 3$$

$$= 26$$

Sum of product triple of new roots :

$$(\alpha - 1)(\beta - 1)(\gamma - 1)$$

$$= (\alpha - 1)(\beta\gamma - \beta - \gamma + 1)$$

$$= \alpha\beta\gamma - \alpha\beta - \alpha\gamma + \alpha - \beta\gamma + \beta + \gamma - 1$$

$$= \alpha\beta\gamma + (\alpha + \beta + \gamma) - (\alpha\beta + \beta\gamma + \gamma\alpha) - 1$$

$$= 7 - 10 - 3 - 1$$

$$= -7$$

The polynomial sought is thus, $w^3 + 13w^2 + 26w + 7 = 0$

METHOD 2

$$w = x - 1 \Rightarrow x = w + 1$$

$$(w + 1)^3 + 10(w + 1)^2 + 3(w + 1) - 7 = 0$$

$$w^3 + 3w^2 + 3w + 1 + 10w^2 + 20w + 10 + 3w + 3 - 7 = 0$$

$$w^3 + 13w^2 + 26w + 7 = 0$$

As before with Method 1

[5 marks]

Answer 3

METHOD 1

$$z^3 - 3z^2 + z + 5 = 0$$

$$\alpha + \beta + \gamma = 3 \qquad \alpha\beta + \beta\gamma + \gamma\alpha = 1 \qquad \alpha\beta\gamma = -5$$

Sum of new roots : $2\alpha + 1 + 2\beta + 1 + 2\gamma + 1$

$$= 2(\alpha + \beta + \gamma) + 3$$

$$= 2 \times 3 + 3$$

$$= 9$$

Sum of product pairs of new roots :

$$(2\alpha + 1)(2\beta + 1) + (2\beta + 1)(2\gamma + 1) + (2\gamma + 1)(2\alpha + 1)$$

$$= 4\alpha\beta + 2\alpha + 2\beta + 1 + 4\beta\gamma + 2\beta + 2\gamma + 1 + 4\gamma\alpha + 2\alpha + 2\gamma + 1$$

$$= 4(\alpha\beta + \beta\gamma + \gamma\alpha) + 4(\alpha + \beta + \gamma) + 3$$

$$= 4 \times 1 + 4 \times 3 + 3$$

$$= 19$$

Sum of product triple of new roots :

$$(2\alpha + 1)(2\beta + 1)(2\gamma + 1)$$

$$= (2\alpha + 1)(4\beta\gamma + 2\beta + 2\gamma + 1)$$

$$= 8\alpha\beta\gamma + 4\alpha\beta + 4\alpha\gamma + 2\alpha + 4\beta\gamma + 2\beta + 2\gamma + 1$$

$$= 8\alpha\beta\gamma + 2(\alpha + \beta + \gamma) + 4(\alpha\beta + \beta\gamma + \gamma\alpha) + 1$$

$$= 8 \times (-5) + 2 \times 3 + 4 \times 1 + 1$$

$$= -29$$

The polynomial sought is thus, $w^3 - 9w^2 + 19w + 29 = 0$

METHOD 2

$$w = 2x + 1 \Rightarrow x = \frac{w - 1}{2}$$

$$\left(\frac{w - 1}{2}\right)^3 - 3\left(\frac{w - 1}{2}\right)^2 + \left(\frac{w - 1}{2}\right) + 5 = 0$$

Multiply through by 8

$$(w - 1)^3 - 6(w - 1)^2 + 4(w - 1) + 40 = 0$$

$$w^3 - 3w^2 + 3w - 1 - 6w^2 + 12w - 6 + 4w - 4 + 40 = 0$$

$$w^3 - 9w^2 + 19w + 29 = 0$$

As before with Method 1

[5 marks]

Answer 4

A question in which Method 2 does not work !

METHOD 1

$$x^3 + 5x^2 - x + 3 = 0$$

$$\alpha + \beta + \gamma = -5 \qquad \alpha\beta + \beta\gamma + \gamma\alpha = -1 \qquad \alpha\beta\gamma = -3$$

Sum of new roots :

$$(\alpha + \beta + \gamma)^2 = \alpha^2 + \beta^2 + \gamma^2 + 2(\alpha\beta + \beta\gamma + \gamma\alpha)$$

$$\therefore\ \alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$$

$$= (-5)^2 - 2 \times (-1)$$

$$= 27$$

Sum of product pairs of new roots :

$$(\alpha\beta + \beta\gamma + \gamma\alpha)^2 = \alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 + 2\alpha\beta^2\gamma + 2\alpha^2\beta\gamma + 2\alpha\beta\gamma^2$$

$$= \alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 + 2\alpha\beta\gamma(\alpha + \beta + \gamma)$$

$$\therefore\ \alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 = (\alpha\beta + \beta\gamma + \gamma\alpha)^2 - 2\alpha\beta\gamma(\alpha + \beta + \gamma)$$

$$= (-1)^2 - 2 \times (-3) \times (-5)$$

$$= -29$$

Sum of product triple of new roots :

$$\alpha^2\beta^2\gamma^2 = (\alpha\beta\gamma)^2$$

$$= (-3)^2$$

$$= 9$$

The polynomial sought is thus, $w^3 - 27w^2 - 29w - 9 = 0$

[6 marks]

Answer 5

METHOD 2

$$w = \frac{1}{x + 1} \Rightarrow x + 1 = \frac{1}{w} \Rightarrow x = \frac{1 - w}{w}$$

$$\left(\frac{1 - w}{w}\right)^3 + 4\left(\frac{1 - w}{w}\right)^2 - 5\left(\frac{1 - w}{w}\right) - 7 = 0$$

Multiply through by w^3

$$(1 - w)^3 + 4w(1 - w)^2 - 5w^2(1 - w) - 7w^3 = 0$$

$$1 - 3w + 3w^2 + w^3 + 4w - 8w^2 + 4w^3 - 5w^2 + 5w^3 - 7w^3 = 0$$

$$w^3 - 10w^2 + w + 1 = 0$$

[5 marks]

Answer 6

$$w = x + 2 \Rightarrow x = w - 2$$

$$x^5 + x + 2 = 0$$

$$(w - 2)^5 + (w - 2) + 2 = 0$$

$$\begin{aligned} & 1 \times w^5 \times (-2)^0 \\ & + 5 \times w^4 \times (-2)^1 \\ & + 10 \times w^3 \times (-2)^2 \\ & + 10 \times w^2 \times (-2)^3 \\ & + 5 \times w^1 \times (-2)^4 \\ & + 1 \times w^0 \times (-2)^5 + w - 2 + 2 = 0 \end{aligned}$$

$$w^5 - 10w^4 + 40w^3 - 80w^2 + 81w - 32 = 0$$

[5 marks]

Answer 7

METHOD 1

$$x^3 + 2x^2 - 3x - 5 = 0$$

$$\alpha + \beta + \gamma = -2 \qquad \alpha\beta + \beta\gamma + \gamma\alpha = -3 \qquad \alpha\beta\gamma = 5$$

Sum of new roots :

$$\begin{aligned} \alpha + \beta + \beta + \gamma + \gamma + \alpha &= 2(\alpha + \beta + \gamma) \\ &= 2 \times (-2) \\ &= -4 \end{aligned}$$

Sum of product pairs of new roots :

$$\begin{aligned} &(\alpha + \beta)(\beta + \gamma) + (\beta + \gamma)(\gamma + \alpha) + (\gamma + \alpha)(\alpha + \beta) \\ &= \alpha\beta + \gamma a + \beta^2 + \beta\gamma + \beta\gamma + \alpha\beta + \gamma^2 + \gamma\alpha + \gamma\alpha + \beta\gamma + \alpha^2 + \alpha\beta \\ &= 3(\alpha\beta + \beta\gamma + \gamma\alpha) + \alpha^2 + \beta^2 + \gamma^2 \\ &= 3(\alpha\beta + \beta\gamma + \gamma\alpha) + (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha) \end{aligned}$$

Because...

$$(\alpha + \beta + \gamma)^2 = \alpha^2 + \beta^2 + \gamma^2 + 2(\alpha\beta + \beta\gamma + \gamma\alpha)$$

$$\therefore\ \alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$$

So we have that...

$$\begin{aligned} &(\alpha + \beta)(\beta + \gamma) + (\beta + \gamma)(\gamma + \alpha) + (\gamma + \alpha)(\alpha + \beta) \\ &= (\alpha\beta + \beta\gamma + \gamma\alpha) + (\alpha + \beta + \gamma)^2 \\ &= (-3) + (-2)^2 \\ &= 1 \end{aligned}$$

Sum of product triple of new roots :

$$\begin{aligned} (\alpha + \beta)(\beta + \gamma)(\gamma + \alpha) &= (-2 - \gamma)(-2 - \alpha)(-2 - \beta) \\ &= (-1)(2 + \gamma)(2 + \alpha)(2 + \beta) \\ &= (-1)(2 + \gamma)(4 + 2\beta + 2\alpha + \alpha\beta) \\ &= (-1)(8 + 4\beta + 4\alpha + 2\alpha\beta + 4\gamma + 2\beta\gamma + 2\alpha\gamma + \alpha\beta\gamma) \\ &= (-1)(8 + 4(\alpha + \beta + \gamma) + 2(\alpha\beta + \beta\gamma + \gamma\alpha) + \alpha\beta\gamma) \\ &= (-1)(8 + 4(-2) + 2(-3) + 5)) \\ &= 1 \end{aligned}$$

The polynomial sought is thus, $w^3 + 4w^2 + w - 1 = 0$

METHOD 2

$$x^3 + 2x^2 - 3x - 5 = 0$$

$$\alpha + \beta + \gamma = -2 \qquad \alpha\beta + \beta\gamma + \gamma\alpha = -3 \qquad \alpha\beta\gamma = 5$$

The new roots are : $\alpha + \beta, \ \ \beta + \gamma$ and $\gamma + \alpha$

which can be written as : $(-2 - \gamma), (-2 - \alpha)$ and $(-2 - \beta)$

$$w = -2 - x \Rightarrow x = -2 - w \Rightarrow x = (-1)(w + 2)$$

$$(-1)^3(w + 2)^3 + 2(-1)^2(w + 2)^2 - 3(-1)(w + 2) - 5 = 0$$

$$(-1)\left(w^3 + 6w^2 + 12w + 8\right) + 2w^2 + 8w + 8 + 3w + 6 - 5 = 0$$

multiply through by (-1)

$$w^3 + 6w^2 + 12w + 8 - 2w^2 - 8w - 8 - 3w - 6 + 5 = 0$$

$$w^3 + 4w^2 + w - 1 = 0$$

[8 marks]