5.4 Solutions to 5.3 Exercise

Answer 1

(i) $\alpha + \beta + \gamma + \delta = -3$

(ii) $\alpha(\beta + \gamma + \delta) + \beta(\gamma + \delta) + \gamma(\delta) = 2$

(iii) $\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta = 1$

(iv) $\alpha\beta\gamma\delta = 4$

(v) $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma} + \dfrac{1}{\delta} = \dfrac{\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta}{\alpha\beta\gamma\delta} = \dfrac{1}{4}$

(vi) $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$

$$= (\alpha + \beta + \gamma + \delta)^2 - 2(\alpha(\beta + \gamma + \delta) + \beta(\gamma + \delta) + \gamma(\delta))$$

$$= (-3)^2 - 2(2)$$

$$= 5$$

[1, 1, 1, 1, 2, 2 marks]

Answer 2

$$a x^4 + b x^3 + c x^2 + dx + e = 0$$

Divide through by a

$$x^4 + \frac{b}{a}x^3 + \frac{c}{a}x^2 + \frac{d}{a}x + \frac{e}{a} = 0$$

$$\alpha + \beta + \gamma + \delta = (-3) + (-1) + 2 + 5$$

$$= 3$$

$$\alpha(\beta + \gamma + \delta) + \beta(\gamma + \delta) + \gamma(\delta)$$

$$= (-3)(-1 + 2 + 5) + (-1)(2 + 5) + (2)(5)$$

$$= -18 - 7 + 10$$

$$= -15$$

$$\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta$$

$$= (-3)(-1)(2) + (-1)(2)(5) + (2)(5)(-3) + (5)(-3)(-1)$$

$$= 6 - 10 - 30 + 15$$

$$= -19$$

$$\alpha\beta\gamma\delta = (-3)(-1)(2)(5)$$

$$= 30$$

$$x^4 - 3x^3 - 15x^2 + 19x + 30 = 0$$

[4 marks]

Answer 3

(i) $$2x^4 - 34x^3 + 202x^2 + dx + e = 0$$

Divide through by 2

$$x^4 - 17x^3 + 101x^2 + \frac{d}{2}x + \frac{e}{2} = 0$$

$$\text{Sum of roots} = 17$$

$$\therefore \alpha + \alpha + 1 + 2\alpha + 1 + 3\alpha + 1 = 17$$

$$7\alpha + 3 = 17$$

$$7\alpha = 14$$

$$\alpha = 2$$

So the roots are 2, 3, 5 and 7

(ii) $$\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta = -\frac{d}{2}$$

$$\begin{aligned}\therefore d &= -2(2 \times 3 \times 5 + 3 \times 5 \times 7 + 5 \times 7 \times 2 + 7 \times 2 \times 3) \\ &= -2(30 + 105 + 70 + 42) \\ &= -2 \times 247 \\ &= -494\end{aligned}$$

$$\alpha\beta\gamma\delta = \frac{e}{2}$$

$$\begin{aligned}e &= 2 \times 2 \times 3 \times 5 \times 7 \\ &= 420\end{aligned}$$

Answer 4

(a)

$$
\begin{aligned}
\text{LHS} &= \alpha^4 + \beta^4 \\
&= \left(\alpha^2\right)^2 + \left(\beta^2\right)^2 \\
&= \left(\alpha^2 + \beta^2\right)^2 - 2\,\alpha^2\beta^2 \\
&= \left(\alpha^2 + \beta^2\right)^2 - 2\left(\alpha\beta\ \right)^2 \\
&= \text{RHS} \qquad\qquad \square
\end{aligned}
$$

[2 marks]

(b) (i)

$$
\begin{aligned}
\alpha^2 + \beta^2 &= (\alpha + \beta)^2 - 2\alpha\beta \\
&= \left(-\frac{13}{7}\right)^2 - 2 \times \left(-\frac{8}{7}\right) \\
&= \frac{169}{49} + \frac{16}{7} \times \frac{7}{7} \\
&= \frac{281}{49}
\end{aligned}
$$

[2 marks]

(ii)

$$
\begin{aligned}
\alpha^4 + \beta^4 &= \left(\alpha^2 + \beta^2\right)^2 - 2(\alpha\beta)^2 \\
&= \left(\frac{281}{49}\right)^2 - 2\left(-\frac{8}{7}\right)^2 \\
&= \frac{78961}{2401} - \frac{128}{49} \times \frac{49}{49} \\
&= \frac{72689}{2401}
\end{aligned}
$$

[4 marks]

Answer 5

(a)

$$4x^2 + 3x + 1 = 0$$

Divide through by 4

$$x^2 + \frac{3}{4}x + \frac{1}{4} = 0$$

$$\therefore \left(\alpha + \beta\right) = -\frac{3}{4} \quad \text{and} \quad \alpha\beta = \frac{1}{4}$$

[2 marks]

(b)

$$\left(\alpha^2 + \beta^2\right) = (\alpha + \beta)^2 - 2\alpha\beta$$

$$= \left(-\frac{3}{4}\right) - 2 \times \frac{1}{4}$$

$$= \frac{1}{16} \quad \text{or} \quad 0.0625$$

[2 marks]

METHOD 1

(c) Sum of new roots,

$$4\alpha - \beta + 4\beta - \alpha = 3(\alpha + \beta)$$

$$= 3 \times \left(-\frac{3}{4}\right)$$

$$= -\frac{9}{4}$$

Product of new roots,

$$\left(4\alpha - \beta\right)\left(4\beta - \alpha\right) = \left(16\alpha\beta - 4\alpha^2 - 4\beta^2 + \alpha\beta\right)$$

$$= 17\alpha\beta - 4\left(\alpha^2 + \beta^2\right)$$

$$= 17 \times \left(\frac{1}{4}\right) - 4 \times \left(\frac{1}{16}\right)$$

$$= 4$$

The new quadratic with roots $\left(4\alpha - \beta\right)$ and $\left(4\beta - \alpha\right)$ is,

$$w^2 + \frac{9}{4}w + 4 = 0$$

$$4w^2 + 9w + 16 = 0$$

[4 marks]

METHOD 2

(c) From $(\alpha + \beta) = -\dfrac{3}{4} \quad \Rightarrow \alpha = -\dfrac{3}{4} - \beta$ and $\beta = -\dfrac{3}{4} - \alpha$

So the new roots can be expressed like this;

$$4\alpha - \beta = 4\alpha + \frac{3}{4} + \alpha \quad \text{and} \quad 4\beta - \alpha = 4\beta + \frac{3}{4} + \beta$$

$$= 5\alpha + \frac{3}{4} \qquad\qquad = 5\beta + \frac{3}{4}$$

The transformation sought is thus,

$$w = 5x + \frac{3}{4}$$

$$w = \frac{20x + 3}{4} \Leftrightarrow x = \frac{4w - 3}{20}$$

$$4x^2 + 3x + 1 = 0$$

$$4\left(\frac{4w - 3}{20}\right)^2 + 3\left(\frac{4w - 3}{20}\right) + 1 = 0$$

Multiply through by 100

$$(4w - 3)^2 + 15(4w - 3) + 100 = 0$$

$$16w^2 - 24w + 9 + 60w - 45 + 100 = 0$$

$$16w^2 + 36w + 64 = 0$$

Divide through by 4

$$4w^2 + 9w + 16 = 0$$

[4 marks]

Answer 6

$$\text{Product of the roots : } \alpha \times k\alpha \times k^2\alpha = 64$$

$$(k\alpha)^3 = 64$$

$$k\alpha = 4$$

$$k = \frac{4}{\alpha}$$

$$\text{Sum of the roots : } \alpha + k\alpha + k^2\alpha = 14$$

$$\alpha + \frac{4}{\alpha} \times \alpha + \left(\frac{4}{\alpha}\right)^2 \times \alpha = 14$$

$$\alpha + 4 + \frac{16}{\alpha} = 14$$

Multiply through by α

$$\alpha^2 - 10\alpha + 16 = 0$$

$$(\alpha - 2)(\alpha - 8) = 0$$

Either $\alpha = 2$ or $\alpha = 8$

When $\alpha = 2$, $k = 2$ and when $\alpha = 8$, $k = \frac{1}{2}$

[6 marks]