Answer 1

$$a x^2 + bx + c = 0$$

Divide through by a

$$x^2 + \frac{b}{a} x + \frac{c}{a} = 0$$

$$\text{Sum of the roots} = -\frac{b}{a}$$

$$-\frac{b}{a} = \frac{-3 + 4\text{i}}{2} + \frac{-3 - 4\text{i}}{2}$$

$$= -3$$

$$\text{Product of the roots} = \frac{c}{a}$$

$$\frac{c}{a} = \left(\frac{-3 + 4\text{i}}{2}\right)\left(\frac{-3 - 4\text{i}}{2}\right)$$

$$= \frac{9 + 12\text{i} - 12\text{i} - 16\text{i}^2}{4}$$

$$= \frac{25}{4}$$

$$\text{Quadratic : } x^2 + 3x + \frac{25}{4} = 0$$

Multiply through by 4

$$4 x^2 + 12x + 25 = 0$$

[3 marks]

Answer 2

(a)

$$a x^4 + 7x^3 + 5x^2 + 3x - 4 = 0$$

Divide through by *a*

$$x^4 + \frac{7}{a}x^3 + \frac{5}{a}x^2 + \frac{3}{a}x - \frac{4}{a} = 0$$

$$-\frac{4}{a} = \alpha\beta\gamma\delta$$

$$-\frac{4}{a} = -1$$

$$\therefore a = 4$$

[1 mark]

(b)

$$x^4 + \frac{7}{4}x^3 + \frac{5}{4}x^2 + \frac{3}{4}x - 1 = 0$$

$$\sum\alpha = \alpha + \beta + \gamma + \delta = -\frac{7}{4}$$

$$\sum\alpha\beta = \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \frac{5}{4}$$

$$\sum\alpha\beta\gamma = \alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta = -\frac{3}{4}$$

[3 marks]

(c)

$$\begin{aligned}
&\alpha^2 + \beta^2 + \gamma^2 + \delta^2 \\
&\quad = (\alpha + \beta + \gamma + \delta)^2 - 2(\alpha(\beta + \gamma + \delta) + \beta(\gamma + \delta) + \gamma(\delta)) \\
&\quad = \left(\frac{7}{4}\right)^2 - 2\left(\frac{5}{4}\right) \\
&\quad = \frac{49}{16} - \frac{10}{4} \\
&\quad = \frac{9}{16}
\end{aligned}$$

[3 marks]

Answer 3

(a)

$$5x^2 - 4x + 3 = 0$$

Divide through by 5

$$x^2 - \frac{4}{5}x + \frac{3}{5} = 0$$

$$\alpha + \beta = \frac{4}{5} \text{ and } \alpha\beta = \frac{3}{5}$$

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$$

$$= \left(\frac{4}{5}\right)^2 - 2 \times \frac{3}{5}$$

$$= \frac{16 - 30}{25}$$

$$= -\frac{14}{25}$$

$$\frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{\beta^2 + \alpha^2}{\alpha^2\beta^2}$$

$$= \frac{(-14) \times 5^2}{25 \times 3^2}$$

$$= -\frac{14}{9}$$

[5 marks]

(b) Sum of new roots : $\frac{3}{\alpha^2} + \frac{3}{\beta^2} = 3\left(\frac{1}{\alpha^2} + \frac{1}{\beta^2}\right)$

$$= 3\left(-\frac{14}{9}\right)$$

$$= -\frac{14}{3}$$

Product of new roots : $\left(\frac{3}{\alpha^2}\right)\left(\frac{3}{\beta^2}\right) = \frac{9}{(\alpha\beta)^2}$

$$= \frac{9 \times 5^2}{3^2}$$

$$= 25$$

The required polynomial is thus,

$$x^2 + \frac{14}{3}x + 25 = 0$$

$$3x^2 + 14x + 75 = 0$$

[4 marks]

Answer 4

(i)

$$\alpha^3 + \beta^3 + \gamma^3 = (\alpha + \beta + \gamma)^3 - 3(\alpha + \beta + \gamma)(\alpha\beta + \beta\gamma + \gamma\alpha) + 3\alpha\beta\gamma$$

[2 marks]

(ii) For a depressed cubic, there is no term in x^2, so $b = 0$ and therefore $\alpha + \beta + \gamma = 0$, in which case,

$$\begin{aligned}\alpha^3 + \beta^3 + \gamma^3 &= (0)^3 - 3(0)(\alpha\beta + \beta\gamma + \gamma\alpha) + 3\alpha\beta\gamma \\ &= 3\alpha\beta\gamma\end{aligned}$$

which is the same as section “6.2 Sum of Cubes for a depressed cubic” ☐

[2 marks]

(iii) The suitable transformation is,

$$x = t - \frac{b}{3a} \Rightarrow x = t - 1$$

$$\begin{aligned}x^3 + 3x^2 + 50 &= 0 \\ (t-1)^3 + 3(t-1)^2 + 50 &= 0 \\ t^3 - 3t^2 + 3t - 1 + 3t^2 - 6t + 3 + 50 &= 0 \\ t^3 - 3t + 52 &= 0\end{aligned}$$

[2 marks]

(iv)

$$\begin{aligned}&P^3 + Q^3 + R^3 \\ &= (P + Q + R)^3 - 3(P + Q + R)(PQ + QR + RP) + 3PQR \\ &= (\alpha\beta + \beta\gamma + \gamma\alpha)^3 - 3(\alpha\beta + \beta\gamma + \gamma\alpha)(\alpha\beta\beta\gamma + \beta\gamma\gamma\alpha + \gamma\alpha\alpha\beta) + 3\alpha\beta\beta\gamma\gamma\alpha \\ &= (\alpha\beta + \beta\gamma + \gamma\alpha)^3 - 3\alpha\beta\gamma(\alpha\beta + \beta\gamma + \gamma\alpha)(\beta + \gamma + \alpha) + 3(\alpha\beta\gamma)^2 \\ &= (\alpha\beta + \beta\gamma + \gamma\alpha)^3 - 3\alpha\beta\gamma(\alpha\beta + \beta\gamma + \gamma\alpha)(\alpha + \beta + \gamma) + 3(\alpha\beta\gamma)^2\end{aligned}$$

[4 marks]

(v) For the depressed cubic with roots α, β and γ,

$$t^3 - 3t + 52 = 0$$

$$\alpha + \beta + \gamma = 0, \quad \alpha\beta + \beta\gamma + \gamma\alpha = -3 \text{ and } \alpha\beta\gamma = -52$$

Sum of new roots : $\alpha^3 + \beta^3 + \gamma^3 = 3\alpha\beta\gamma$ as $\alpha + \beta + \gamma = 0$

$$= 3 \times (-52)$$

$$= -156$$

Sum of product pairs of new roots, from part (iv) :

$$\alpha^3\beta^3 + \beta^3\gamma^3 + \gamma^3\alpha^3$$

$$= (\alpha\beta + \beta\gamma + \gamma\alpha)^3 - 3\alpha\beta\gamma(\alpha\beta + \beta\gamma + \gamma\alpha)(\alpha + \beta + \gamma) + 3(\alpha\beta\gamma)^2$$

but for a depressed cubic $\alpha + \beta + \gamma = 0$ so this can be simplified to,

$$\Rightarrow \alpha^3\beta^3 + \beta^3\gamma^3 + \gamma^3\alpha^3 = (\alpha\beta + \beta\gamma + \gamma\alpha)^3 + 3(\alpha\beta\gamma)^2$$

$$= (-3)^3 + 3 \times (-52)^2$$

$$= -27 + 8112$$

$$= 8085$$

Sum of product triple of new roots : $\alpha^3\beta^3\gamma^3 = (\alpha\beta\gamma)^3$

$$= (-52)^3$$

$$= -140608$$

The cubic equation with roots α^3, β^3 and γ^3 is,

$$w^3 + 156w^2 + 8085w + 140608 = 0$$

[6 marks]

Answer 5

$$f(x) = mx^3 + 12x^2 + 4x + 16$$

Let f have roots α, β and γ in which case,

$$\alpha + \beta + \gamma = -\frac{12}{m} \quad \alpha\beta + \beta\gamma + \gamma\alpha = \frac{4}{m} \text{ and } \alpha\beta\gamma = -\frac{16}{m}$$

From

- f has at least one root on the imaginary axis

Let α be that root, $\alpha = k\text{i}$, and so $f(k\text{i}) = 0$

$$\begin{aligned} f(k\text{i}) &= 0 \\ m(k\text{i})^3 + 12(k\text{i})^2 + 4(k\text{i}) + 16 &= 0 \\ -mk^3\text{i} - 12k^2 + 4k\text{i} + 16 &= 0 \end{aligned}$$

$$\text{Equating real parts : } -12k^2 + 16 = 0$$

$$\Rightarrow k^2 = \frac{16}{12} = \frac{4}{3}$$

$$\begin{aligned} \text{Equating imaginary parts : } -mk^3 + 4k &= 0 \\ mk^3 &= 4k \\ m &= \frac{4}{k^2} \\ &= \frac{4 \times 3}{4} \\ &= 3 \end{aligned}$$

$$3x^3 + 12x^2 + 4x + 16 = 0$$

As $k^2 = \dfrac{4}{3}$ and a root was $\alpha = k\text{i}$, $\alpha = \pm\dfrac{2}{\sqrt{3}}\text{i} = \pm\dfrac{2\sqrt{3}}{3}\text{i}$

$f(x)$ had all real coefficients and so has a conjugate pair or roots

$$\alpha = \frac{2\sqrt{3}}{3}\text{i} \qquad \beta = -\frac{2\sqrt{3}}{3}\text{i}$$

$$\begin{aligned} \alpha\beta\gamma &= -\frac{16}{m} \\ \frac{2\sqrt{3}}{3}\text{i} \times \left(-\frac{2\sqrt{3}}{3}\text{i}\right) \times \gamma_g &= -\frac{16}{3} \\ -\frac{4 \times 3 \times (-1)}{9} \times \gamma_g &= -\frac{16}{3} \\ \gamma_g &= -4 \end{aligned}$$

[7 marks]

Answer 6

Clever Method

Depressed Cubic Formation Rule
Faced with a general cubic,

$$a x^3 + b x^2 + cx + d = 0$$

initiate a change of variable by replacing x with $t - \dfrac{b}{3a}$

This will always result in what is termed a depressed cubic, one of the form,

$$t^3 + pt + q = 0$$

Let $f(x) = x^3 + p x^2 + qx + r = 0$

Let $g(x)$ be the related depressed cubic,

$$g(x) = f\left(x - \frac{p}{3}\right) = x^3 - \frac{p^2 - 3q}{3} x + \frac{2p^3 - 9pq + 27r}{r}$$

Note that the roots of $f(x)$ form an Arithmetic Progression if and only if the roots of $g(x)$ are of the form $(-d)$, 0, $(+d)$ for some complex number d which is equivalent to $g(x) = (x + d)\, x\, (x - d) = x^3 - d^2 x$. That is, the roots of $f(x)$ form an Arithmetic Progression if and only if the constant term of $g(x)$ is 0, which is the same as saying that $2p^3 - 9pq + 27r = 0$.

$$\therefore\ 2p^3 + 27r = 9pq \qquad \square$$

[8 marks]

Standard Method

$$x^3 + px + qx + r = 0$$

Without loss of generality, let the roots be

$$\alpha = m - d, \qquad \beta = m \text{ and } \gamma = m + d \qquad \text{for } m,\ d \in \mathbb{C}$$

$$\alpha + \beta + \gamma = p$$
$$m - d + m + m + d = p$$
$$3m = p \qquad (1)$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = q$$
$$(m - d)m + m(m + d) + (m + d)(m - d) = q$$
$$m^2 - md + m^2 + md + m^2 - d^2 = q$$
$$3m^2 - d^2 = q \qquad (2)$$

$$\alpha\beta\gamma = r$$
$$(m - d)\ m\ (m + d) = r$$
$$m(m^2 - d^2) = r$$
$$m\left(m^2 - \left(3m^2 - q\right)\right) = r \qquad \text{from (2)}$$
$$m\left(-2m^2 + q\right) = r$$
$$-2m^3 + mq = r$$
$$-2\left(\frac{p}{3}\right)^3 + \left(\frac{p}{3}\right)q = r$$

Multiply through by 27

$$-2p^3 + 9pq = 27r$$
$$\therefore 2p^3 + 27r = 9pq$$

As all of the steps used are reversible the theorem is proven □

[8 marks]