7.2 Answers

Answer 1

$$\begin{aligned}
(\alpha - \beta)^2 &= \alpha^2 - 2\alpha\beta + \beta^2 \\
&= \alpha^2 + 2\alpha\beta + \beta^2 - 4\alpha\beta \\
&= (\alpha + \beta)^2 - 4\alpha\beta \\
&= \left(-\frac{b}{a}\right)^2 - 4 \times \left(\frac{c}{a}\right) \\
&= \frac{b^2 - 4ac}{a^2} \\
\therefore\ \alpha - \beta &= \frac{\pm\sqrt{b^2 - 4ac}}{a}
\end{aligned}$$

[4 marks]

Answer 2

(i) As there is no term in x^2, $\alpha + \beta + \gamma = 0$

As α and β are real, γ must also be real to satisfy the above constraint.

[1 mark]

(ii) As there is no term in x^2, $\alpha + \beta + \gamma = 0$

$$\begin{aligned}
\therefore\quad 1 + (-4) + \gamma &= 0 \\
\gamma &= 3
\end{aligned}$$

Furthermore, $\alpha\beta + \beta\gamma + \gamma\alpha = m$

$$\begin{aligned}
m &= 1 \times (-4) + (-4) \times 3 + 3 \times 1 \\
&= -13
\end{aligned}$$

Finally, $\alpha\beta\gamma = -n$

$$\begin{aligned}
n &= -(1 \times (-4) \times 3) \\
&= 12
\end{aligned}$$

So, the equation was,

$$x^3 - 13x + 12 = 0$$

[4 marks]

Answer 3

(i)

$$3x^3 - px + 11 = 0$$

Divide through by 3

$$x^3 - \frac{p}{3}x + \frac{11}{3} = 0$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = -\frac{p}{3}$$

$$4 = -\frac{p}{3}$$

$$p = -12$$

[1 mark]

(ii) The cubic is thus, $x^3 + 4x + \frac{11}{3} = 0$

$$\alpha + \beta + \gamma = 0 \quad \alpha\beta + \beta\gamma + \gamma\alpha = 4 \quad \text{and} \quad \alpha\beta\gamma = -\frac{11}{3}$$

[2 marks]

(iii) $(3 - \alpha)(3 - \beta)(3 - \gamma)$

$$= (3 - \alpha)(9 - 3\gamma - 3\beta + \beta\gamma)$$

$$= 27 - 9\gamma - 9\beta + 3\beta\gamma - 9\alpha + 3\alpha\gamma + 3\alpha\beta - \alpha\beta\gamma$$

$$= 27 - 9(\alpha + \beta + \gamma) + 3(\alpha\beta + \beta\gamma + \gamma\alpha) - \alpha\beta\gamma$$

$$= 27 - 9(0) + 3(4) - \left(-\frac{11}{3}\right)$$

$$= \frac{128}{3}$$

[3 marks]

Answer 4
Clever Method

$$3x^2 + 2x + 1 = 0$$

$$x^2 + \frac{2}{3}x + \frac{1}{3} = 0$$

$$\alpha + \beta = -\frac{2}{3} \quad \text{and} \quad \alpha\beta = \frac{1}{3}$$

As α is a root, $3\alpha^2 + 2\alpha + 1 = 0$

$$\Rightarrow 3\alpha^2 + 3\alpha + 1 = a$$

$$1 - \alpha = -3\alpha(\alpha + 1)$$

As β is a root, $3\beta^2 + 2\beta + 1 = 0$

$$\Rightarrow 3\beta^2 + 3\beta + 1 = \beta$$

$$1 - \beta = -3\beta(\beta + 1)$$

So,

$$\left(\frac{1 - \alpha}{1 + \alpha}\right)^3 + \left(\frac{1 - \beta}{1 + \beta}\right)^3 = \left(\frac{-3\alpha(\alpha + 1)}{(1 + \alpha)}\right)^3 + \left(\frac{-3\beta(\beta + 1)}{(1 + \beta)}\right)^3$$

$$= -27\left(\alpha^3 + \beta^3\right)$$

$$= -27\left((\alpha + \beta)^3 - 3\alpha^2\beta - 3\alpha\beta^2\right)$$

$$= -27\left((\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)\right)$$

$$= -27(\alpha + \beta)^3 + 81\alpha\beta(\alpha + \beta)$$

$$= -27 \times \left(-\frac{2}{3}\right)^3 + 81 \times \left(\frac{1}{3}\right) \times \left(-\frac{2}{3}\right)$$

$$= 8 - 18$$

$$= -10$$

[8 marks]

Straight Forward Method

$$3x^2 + 2x + 1 = 0$$

$$x^2 + \frac{2}{3}x + \frac{1}{3} = 0$$

$$\alpha + \beta = -\frac{2}{3} \text{ and } \alpha\beta = \frac{1}{3}$$

Let $x = \left(\frac{1-\alpha}{1+\alpha}\right)$ and $y = \left(\frac{1-\beta}{1+\beta}\right)$ so $x^3 + y^3$ is sought

$$\text{Then, } x + y = \left(\frac{1-\alpha}{1+\alpha}\right) + \left(\frac{1-\beta}{1+\beta}\right)$$

$$= \frac{(1-\alpha)(1+\beta) + (1-\beta)(1+\alpha)}{(1+\alpha)(1+\beta)}$$

$$= \frac{2(1-\alpha\beta)}{1+(\alpha+\beta)+\alpha\beta}$$

$$= \frac{2\left(1-\left(\frac{1}{3}\right)\right)}{1+\left(-\frac{2}{3}\right)+\frac{1}{3}}$$

$$= 2$$

$$\text{and,} \quad xy = \left(\frac{1-\alpha}{1+\alpha}\right)\left(\frac{1-\beta}{1+\beta}\right)$$

$$= \frac{1-(\alpha+\beta)+\alpha\beta}{1+(\alpha+\beta)+\alpha\beta}$$

$$= \frac{1-\left(-\frac{2}{3}\right)+\frac{1}{3}}{1+\left(-\frac{2}{3}\right)+\frac{1}{3}}$$

$$= 3$$

From the Binomial theorem, $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

deduce that, $x^3 + y^3 = (x+y)^3 - 3xy(x+y)$

$$\therefore \left(\frac{1-\alpha}{1+\alpha}\right)^3 + \left(\frac{1-\beta}{1+\beta}\right)^3 = 2^3 - 3 \times 2 \times 3$$

$$= 8 - 18$$

$$= -10$$

[8 marks]

Solution was written and posted by MHH on Math.StackExchange.com, July 9th 2020

Answer 5

$$f(x) = kx^2 + 3x - 11 \qquad g(x) = mx^3 - 2x^2 + 3x - 9$$

Let f have roots α_f and β_f in which case $\alpha_f + \beta_f = -\dfrac{3}{k}$ and $\alpha_f\beta_f = -\dfrac{11}{k}$

Let g have roots α_g, β_g and γ_g in which case,

$$\alpha_g + \beta_g + \gamma_g = \frac{2}{m} \quad \alpha_g\beta_g + \beta_g\gamma_g + \gamma_g\alpha_g = \frac{3}{m} \text{ and } \alpha_g\beta_g\gamma_g = \frac{9}{m}$$

We are told that,

- the sum of the roots of f is equal to the product of the roots of g

$$\alpha_f + \beta_f = \alpha_g\beta_g\gamma_g$$
$$-\frac{3}{k} = \frac{9}{m}$$
$$\frac{k}{3} = -\frac{m}{9}$$
$$k = -\frac{m}{3}$$

- g has at least one root on the imaginary axis

Let α_g be that root, $\alpha_g = K\mathrm{i}$, and so $g(K\mathrm{i}) = 0$

$$g(K\mathrm{i}) = 0$$
$$m(K\mathrm{i})^3 - 2(K\mathrm{i})^2 + 3(K\mathrm{i}) - 9 = 0$$
$$-mK^3\mathrm{i} + 2K^2 + 3K\mathrm{i} - 9 = 0$$
$$\text{Equating real parts : } 2K^2 - 9 = 0 \Rightarrow K^2 = \frac{9}{2}$$
$$\text{Equating imaginary parts : } -mK^3 + 3K = 0$$
$$mK^3 = 3K$$
$$m = \frac{3}{K^2}$$
$$= \frac{3 \times 2}{9}$$
$$= \frac{2}{3} \qquad \therefore\ k = -\frac{2}{9}$$

(a) **(i)**

$$
\begin{aligned}
f(x) &= 0 \\
k x^2 + 3x - 11 &= 0 \\
-\frac{2}{9} x^2 + 3x - 11 &= 0 \\
2x^2 - 27x + 99 &= 0 \\
x &= \frac{27 \pm \sqrt{27^2 - 4 \times 2 \times 99}}{2 \times 2} \\
&= \frac{27 \pm \sqrt{-63}}{4} \\
&= \frac{27 \pm 3\sqrt{7}\,\mathrm{i}}{4}
\end{aligned}
$$

(ii)

$$
\begin{aligned}
g(x) &= 0 \\
m x^3 - 2x^2 + 3x - 9 &= 0 \\
\frac{2}{3} x^3 - 2x^2 + 3x - 9 &= 0 \\
2x^3 - 6x^2 + 9x - 27 &= 0
\end{aligned}
$$

As $K^2 = \dfrac{9}{2}$ and a root was $\alpha_g = K\mathrm{i}$, $\alpha_g = \pm \dfrac{3}{\sqrt{2}}\,\mathrm{i} = \pm \dfrac{3\sqrt{2}}{2}\,\mathrm{i}$

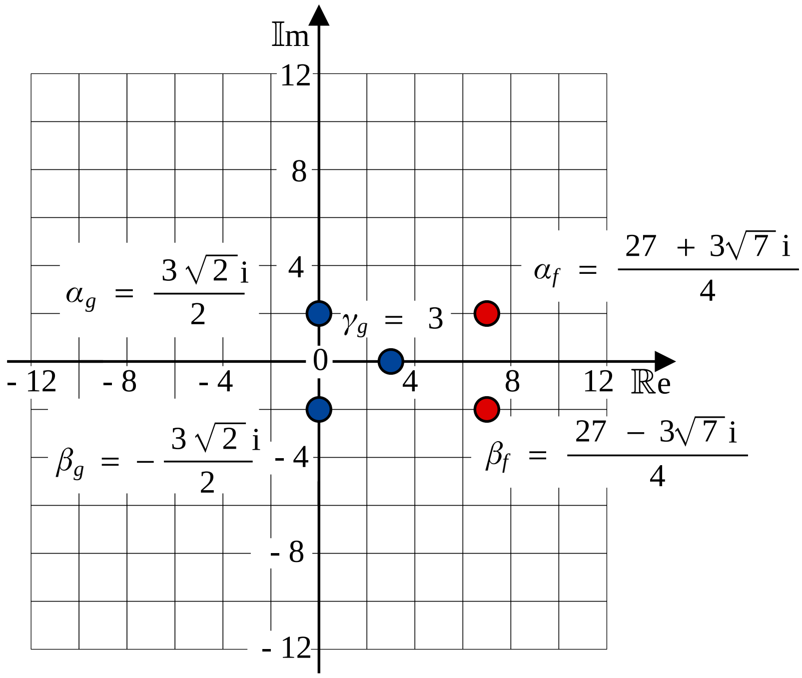
$g(x)$ had all real coefficients and so has a conjugate pair or roots

$$
\alpha_g = \frac{3\sqrt{2}}{2}\,\mathrm{i} \qquad \beta_g = -\frac{3\sqrt{2}}{2}\,\mathrm{i}
$$

$$
\begin{aligned}
\alpha_g \beta_g \gamma_g &= \frac{9}{m} \\
\frac{3\sqrt{2}}{2}\,\mathrm{i} \times \left(- \frac{3\sqrt{2}}{2}\,\mathrm{i} \right) \times \gamma_g &= \frac{9 \times 3}{2} \\
-\frac{9 \times 2 \times (-1)}{4} \times \gamma_g &= \frac{27}{2} \\
\gamma_g &= 3
\end{aligned}
$$

[7 marks]

(b)



[2 marks]

Answer 6

(i) Using the substitution method,

$$w = \frac{1}{x} \Rightarrow x = \frac{1}{w}$$

$$x^4 + x^3 + p\,x^2 + 4x - 2 = 0$$

$$\left(\frac{1}{w}\right)^4 + \left(\frac{1}{w}\right)^3 + p\left(\frac{1}{w}\right)^2 + 4\left(\frac{1}{w}\right) - 2 = 0$$

Multiply through by w^4

$$1 + w + p\,w^2 + 4\,w^3 - 2\,w^4 = 0$$

$$2\,w^4 - 4\,w^3 - p\,w^2 - w - 1 = 0$$

$$w^4 - 2\,w^3 - \frac{p}{2}\,w^2 - \frac{1}{2}\,w - \frac{1}{2} = 0$$

[3 marks]

(ii) For the original quartic,

$$\alpha + \beta + \gamma + \delta = -1 \qquad \alpha(\beta + \gamma + \delta) + \beta(\gamma + \delta) + \gamma(\delta) = p$$

$$\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta = -4 \qquad \alpha\beta\gamma\delta = 2$$

Standard result,

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2$$

$$= (\alpha + \beta + \gamma + \delta)^2 - 2(\alpha(\beta + \gamma + \delta) + \beta(\gamma + \delta) + \gamma(\delta))$$

$$\therefore\ \alpha^2 + \beta^2 + \gamma^2 + \delta^2 = (-1)^2 - 2p$$

$$= 1 - 2p$$

For the transformed quartic,

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta} = 2 \qquad \frac{1}{\alpha}\left(\frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta}\right) + \frac{1}{\beta}\left(\frac{1}{\gamma} + \frac{1}{\delta}\right) + \frac{1}{\gamma}\left(\frac{1}{\delta}\right) = -\frac{p}{2}$$

$$\frac{1}{\alpha\beta\gamma} + \frac{1}{\beta\gamma\delta} + \frac{1}{\gamma\delta\alpha} + \frac{1}{\delta\alpha\beta} = \frac{1}{2} \qquad \frac{1}{\alpha\beta\gamma\delta} = -\frac{1}{2}$$

So the standard result will become,

$$\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} + \frac{1}{\delta^2}$$

$$= \left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta}\right)^2 - 2\left(\frac{1}{\alpha}\left(\frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta}\right) + \frac{1}{\beta}\left(\frac{1}{\gamma} + \frac{1}{\delta}\right) + \frac{1}{\gamma}\left(\frac{1}{\delta}\right)\right)$$

$$\therefore\ \frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} + \frac{1}{\delta^2} = 2^2 - 2\left(-\frac{p}{2}\right)$$

$$= 4 + p$$

As $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = \dfrac{1}{\alpha^2} + \dfrac{1}{\beta^2} + \dfrac{1}{\gamma^2} + \dfrac{1}{\delta^2}$

$$1 - 2p = 4 + p$$

$$3p = -3$$

$$p = -1$$

[5 marks]