8.3 Answers to 8.2 Exercise

Answer 1
METHOD 1

$$x^3 + x + 1 = 0$$

$$\alpha + \beta + \gamma = 0 \quad \alpha\beta + \beta\gamma + \gamma\alpha = 1 \quad \alpha\beta\gamma = -1$$

Standard result :

$$\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$$
$$= 0^2 - 2 \times 1$$
$$= -2$$

Standard result :

$$\alpha^3 + \beta^3 + \gamma^3 = (\alpha + \beta + \gamma)^3 - 3(\alpha + \beta + \gamma)(\alpha\beta + \beta\gamma + \gamma\alpha) + 3\alpha\beta\gamma$$
$$= 0^3 - 3 \times 0 \times 1 + 3 \times (-1)$$
$$= -3$$

Standard result for a depressed cubic: (proven above)

$$\alpha^4 + \beta^4 + \gamma^4 = 2(\alpha\beta + \beta\gamma + \gamma\alpha)^2$$
$$= 2 \times 1^2$$
$$= 2$$

Standard result : (proven above)

$$\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 = (\alpha\beta + \beta\gamma + \gamma\alpha)^2 - 2\alpha\beta\gamma(\alpha + \beta + \gamma)$$
$$= 1^2 - 2 \times (-1) \times 0$$
$$= 1$$

Standard result : (proven above)

$$\alpha^3\beta^3 + \beta^3\gamma^3 + \gamma^3\alpha^3$$
$$= (\alpha\beta + \beta\gamma + \gamma\alpha)^3 - 3\alpha\beta\gamma(\alpha\beta + \beta\gamma + \gamma\alpha)(\alpha + \beta + \gamma) + 3(\alpha\beta\gamma)^2$$
$$= 1^3 - 3 \times (-1) \times 1 \times 0 + 3 \times (-1)^2$$
$$= 4$$

Also as α is a root, $\alpha^3 + \alpha + 1 = 0$ likewise for β and γ.

Finally, as $\alpha + \beta + \gamma = 0$

$$\alpha + \beta = -\gamma$$
$$(\alpha + \beta)^2 = \gamma^2$$
$$\alpha^2 + \beta^2 = \gamma^2 - 2\alpha\beta$$

Similarly, $\beta^2 + \gamma^2 = \alpha^2 - 2\beta\gamma$ and $\gamma^2 + \alpha^2 = \beta^2 - 2\gamma\alpha$

Sum of the new roots :

$$(\alpha-\beta)^2+(\beta-\gamma)^2+(\gamma-\alpha)^2$$

$$=\alpha^2-2\alpha\beta+\beta^2+\beta^2-2\beta\gamma+\gamma^2+\gamma^2-2\gamma\alpha+\alpha^2$$

$$=2\left(\alpha^2+\beta^2+\gamma^2\right)-2\left(\alpha\beta+\beta\gamma+\gamma\alpha\right)$$

$$=2\times(-2)-2\times 1$$

$$=-6$$

Sum of product pairs :

$$(\alpha-\beta)^2(\beta-\gamma)^2+(\beta-\gamma)^2(\gamma-\alpha)^2+(\gamma-\alpha)^2(\alpha-\beta)^2$$

$$\begin{aligned}&=\left(\alpha^2-2\alpha\beta+\beta^2\right)\left(\beta^2-2\beta\gamma+\gamma^2\right)\\&+\left(\beta^2-2\beta\gamma+\gamma^2\right)\left(\gamma^2-2\gamma\alpha+\alpha^2\right)\\&+\left(\gamma^2-2\gamma\alpha+\alpha^2\right)\left(\alpha^2-2\alpha\beta+\beta^2\right)\end{aligned}$$

$$\begin{aligned}&=\alpha^2\beta^2-2\alpha^2\beta\gamma+\alpha^2\gamma^2-2\alpha\beta^3+4\alpha\beta^2\gamma-2\alpha\beta\gamma^2+\beta^4-2\beta^3\gamma+\beta^2\gamma^2\\&+\beta^2\gamma^2-2\beta^2\gamma\alpha+\beta^2\alpha^2-2\beta\gamma^3+4\beta\gamma^2\alpha-2\beta\gamma\alpha^2+\gamma^4-2\gamma^3\alpha+\gamma^2\alpha^2\\&+\gamma^2\alpha^2-2\gamma^2\alpha\beta+\gamma^2\beta^2-2\gamma\alpha^3+4\gamma\alpha^2\beta-2\gamma\alpha\beta^2+\alpha^4-2\alpha^3\beta+\alpha^2\beta^2\end{aligned}$$

$$\begin{aligned}&=\alpha^4+\beta^4+\gamma^4+3\left(\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2\right)\\&\qquad-2\left(\alpha\beta^3+\beta\gamma^3+\gamma\alpha^3+\beta^3\gamma+\gamma^3\alpha+\alpha^3\beta\right)\end{aligned}$$

$$\begin{aligned}&=\alpha^4+\beta^4+\gamma^4+3\left(\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2\right)\\&\qquad-2\left(\alpha^3(\beta+\gamma)+\beta^3(\alpha+\gamma)+\gamma^3(\alpha+\beta)\right)\end{aligned}$$

$$\begin{aligned}&=\alpha^4+\beta^4+\gamma^4+3\left(\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2\right)\\&\qquad\qquad-2\left(\alpha^3(-\alpha)+\beta^3(-\beta)+\gamma^3(-\gamma)\right)\end{aligned}$$

$$=3\left(\alpha^4+\beta^4+\gamma^4+\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2\right)$$

$$=3\times\left(2+1\right)$$

$$=9$$

Sum of product triples :

$$
\begin{aligned}
(\alpha - \beta)^2 (\beta - \gamma)^2 (\gamma - \alpha)^2 \\
&= \left(\alpha^2 + \beta^2 - 2\alpha\beta\right)\left(\beta^2 + \gamma^2 - 2\beta\gamma\right)\left(\gamma^2 + \alpha^2 - 2\gamma\alpha\right) \\
&= \left(\gamma^2 - 4\alpha\beta\right)\left(\alpha^2 - 4\beta\gamma\right)\left(\beta^2 - 4\gamma\alpha\right) \\
&= \left(\gamma^2 - 4\alpha\beta\right)\left(\alpha^2\beta^2 - 4\gamma\alpha^3 - 4\beta^3\gamma + 16\alpha\beta\gamma^2\right) \\
&= \alpha^2\beta^2\gamma^2 - 4\gamma^3\alpha^3 - 4\beta^3\gamma^3 \\
&\qquad + 16\alpha\beta\gamma^4 - 4\alpha^3\beta^3 + 16\alpha^4\beta\gamma + 16\alpha\beta^4\gamma - 64\alpha^2\beta^2\gamma^2 \\
&= 16\alpha\beta\gamma\left(\alpha^3 + \beta^3 + \gamma^3\right) - 4\left(\alpha^3\beta^3 + \beta^3\gamma^3 + \gamma^3\alpha^3\right) - 63(\alpha\beta\gamma)^2 \\
&= 16 \times (-1) \times (-3) - 4 \times 4 - 63 \times (-1)^2 \\
&= 48 - 16 - 63 \\
&= -31
\end{aligned}
$$

Hence the equation is,

$$w^3 + 6w^2 + 9w + 31 = 0$$

[8 marks]

METHOD 2

Trying to find a transformation from *x* to *w*,

$$\text{let } w = (\alpha - \beta)^2$$
$$= \alpha^2 + \beta^2 - 2\alpha\beta$$
$$= \gamma^2 - 4\alpha\beta \quad \text{but } \alpha\beta\gamma = -1 \ \therefore\ \alpha\beta = -\frac{1}{\gamma}$$
$$= \gamma^2 + \frac{4}{\gamma}$$
$$\gamma w = \gamma^3 + 4$$
$$\text{but } \gamma^3 + \gamma + 1 = 0 \Rightarrow \gamma^3 = -\gamma - 1$$
$$\gamma w = -\gamma - 1 + 4$$
$$\gamma w + \gamma = 3$$
$$\gamma(w + 1) = 3$$
$$\gamma = \frac{3}{w + 1} \qquad \text{That is, } x = \frac{3}{w + 1}$$

$$x^3 + x + 1 = 0$$
$$\left(\frac{3}{w + 1}\right)^3 + \left(\frac{3}{w + 1}\right) + 1 = 0$$

Multiply through by $(w + 1)^3$

$$3^3 + 3(w + 1)^2 + (w + 1)^3 = 0$$
$$27 + 3w^2 + 6w + 3 + w^3 + 3w^2 + 3w + 1 = 0$$
$$w^3 + 6w^2 + 9w + 31 = 0$$

[8 marks]