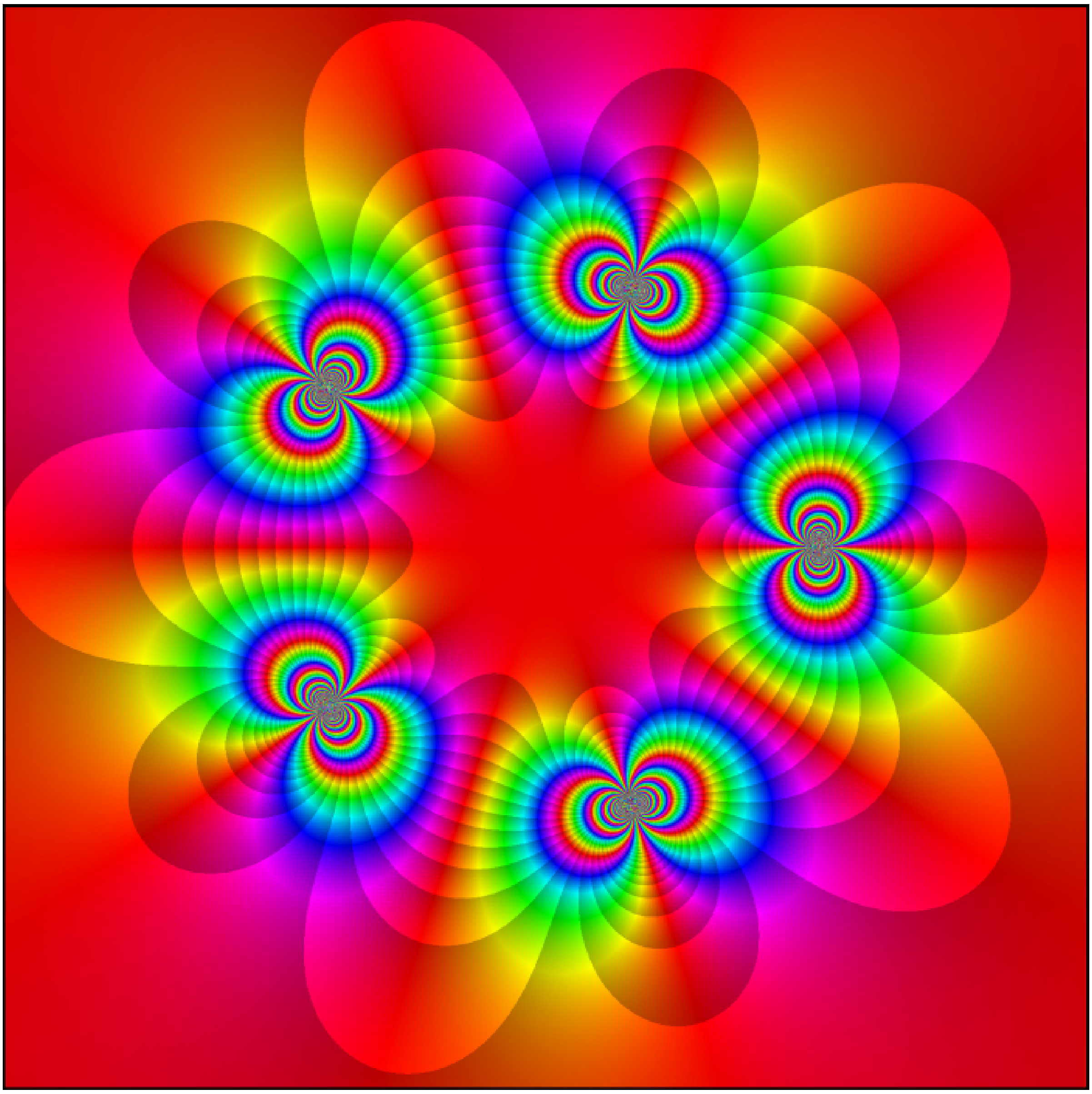
Further Pure A-Level Mathematics
Compulsory Course Component
Core 1

ROOTS ~ OF ~ POLYNOMIALS



The five roots in the complex plane of the polynomial equation $z^5 = 1$

ROOTS
OF
POLYNOMIALS

Lesson 1

Further A-Level Pure Mathematics : Core 1
Roots of Polynomials

1.1 Quadratic and Roots

The simple quadratic equation $a x^2 + bx + c = 0$ is a surprisingly rich source of mathematical ideas. It was the original motivation to develop the technique of completing the square, and a doorway into an understanding of complex numbers. Through iterating the simple quadratic $z^2 = c$ the world of Fractal Geometry was discovered in the 1980s. This topic, *Roots of Polynomials,* also starts by looking at the quadratic equation from a new perspective.

1.2 Sum Of Roots

The Sum Of The Roots

If α and β are the roots of the equation $a x^2 + bx + c = 0 \qquad a,\ b,\ c \in \mathbb{C}$

then, $$\alpha + \beta = -\frac{b}{a}$$

Proof

Without loss of generality, let $\alpha = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$

and $\beta = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$

then

$$\begin{aligned}
\alpha + \beta &= \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a} \\
&= -\frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} - \frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a} \\
&= -\frac{b}{2a} - \frac{b}{2a} \\
&= \frac{-2b}{2a} \\
&= -\frac{b}{a} \qquad \square
\end{aligned}$$

1.3 Product Of Roots

The Product Of The Roots

If α and β are the roots of the equation $a\,x^2 + bx + c = 0 \qquad a,\ b,\ c \in \mathbb{C}$

then, $$\alpha\beta = \frac{c}{a}$$

Proof

Without loss of generality, let $\alpha = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$

and $\beta = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$

then

$$\alpha\beta = \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right)$$

$$= \frac{(-b)^2 - \left(\sqrt{b^2 - 4ac}\right)^2}{4\,a^2} \qquad \text{Difference of two squares}$$

$$= \frac{b^2 - \left(b^2 - 4ac\right)}{4\,a^2}$$

$$= \frac{b^2 - b^2 + 4ac}{4\,a^2}$$

$$= \frac{4ac}{4\,a^2}$$

$$= \frac{c}{a} \qquad \square$$

1.4 Sum, Product Example

The roots of the quadratic $8\,x^2 + 2x - 15 = 0$ are α and β.
Without solving the equation, find the values of

(i) $\alpha + \beta$ **(ii)** $\alpha\beta$

(iii) $\dfrac{1}{\alpha} + \dfrac{1}{\beta}$ **(iv)** $\alpha^2 + \beta^2$

Teaching Video : http://www.NumberWonder.co.uk/v9093/1.mp4

1.5 After Watching the Teaching Video

(a) Having watched the teaching video complete the following,

(i) In general $a\,x^2 + bx + c = 0$ divided throughout by a gives,

☞

[1 mark]

(ii) This is is useful because,

☞

[1 mark]

(iii) For the particular example, $8\,x^2 + 2x - 15 = 0$ is divided by 8 to get,

☞

[1 mark]

(b) Without solving the equation, find the values of

(i) $\alpha + \beta$

☞

(ii) $\alpha\beta$

☞

[1, 1 mark]

(iii) $\dfrac{1}{\alpha} + \dfrac{1}{\beta}$

☞

(iv) $\alpha^2 + \beta^2$

☞

[2, 2 marks]

1.6 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable.
Make the method used clear.
Marks available : 40

Question 1

The roots of the quadratic equation $3x^2 + 7x - 2 = 0$ are α and β.

Without solving the equation, find the values of

(i) $\alpha + \beta$ **(ii)** $\alpha\beta$

(iii) $\dfrac{1}{\alpha} + \dfrac{1}{\beta}$ **(iv)** $\alpha^2 + \beta^2$

[1, 1, 2, 2 marks]

Question 2

The roots of the quadratic equation $4x^2 - 3x + 1 = 0$ are α and β.

Without solving the equation, find the values of,

(i) $\alpha + \beta$ **(ii)** $\alpha\beta$

(iii) $\alpha^2 + \beta^2$ **(iv)** $\dfrac{1}{\alpha^2} + \dfrac{1}{\beta^2}$

[1, 1, 2, 2 marks]

Question 3

(i) Prove that,

$$(\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta) = \alpha^3 + \beta^3$$

[2 marks]

(ii) The roots of the quadratic equation $5x^2 + 6x + 2 = 0$ are α and β.

Without solving the equation, find the exact value of $\alpha^3 + \beta^3$

[4 marks]

Question 4

The roots of a quadratic equation $ax^2 + bx + c = 0$ are $\alpha = \frac{1}{2}$ and $\beta = -\frac{2}{3}$

Find integer values for *a*, *b* and *c*

[3 marks]

Question 5

The complex roots of a quadratic equation $a x^2 + bx + c = 0$ are

$$\alpha = \frac{1 + 2\mathrm{i}}{3} \quad \text{and} \quad \beta = \frac{1 - 2\mathrm{i}}{3}$$

Find integer values for *a*, *b* and *c*

[4 marks]

Question 6

The roots of the equation $6 x^2 + 36x + k = 0$ are reciprocals of each other.

Find the value of the constant, *k*

[4 marks]

Question 7

Gerolamo Cardano (1501-1576) is credited with the first formula for solving cubic equations.

Depressed Cubic Formation Rule

Faced with a general cubic,

$$a x^3 + b x^2 + cx + d = 0 \qquad a, b, c, d \in \mathbb{C}$$

initiate a change of variable by replacing x with $t - \dfrac{b}{3a}$

This will always result in what is termed a depressed cubic, one of the form,

$$t^3 + pt + q = 0 \qquad p, q \in \mathbb{C}$$

(i) Make the appropriate change of variable for the cubic,

$$x^3 - 3x^2 + 12x + 16 = 0$$

and show that the resulting depressed cubic is,

$$t^3 + 9t + 26 = 0$$

[4 marks]

Root of a Cubic

Given a depressed cubic of the form

$$t^3 + pt + q = 0 \qquad p, q \in \mathbb{C}$$

where p and q are not both zero, and $4p^3 + 27q^2 \neq 0$, calculate,

$$C = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

A root of the cubic is then given by,

$$\alpha = C - \frac{p}{3C}$$

(ii) Determine a root of the depressed cubic,

$$t^3 + 9t + 26 = 0$$

[3 marks]

(iii) Using polynomial division, find all three roots, two of which are a complex conjugate pair, of the depressed cubic,

$$t^3 + 9t + 26 = 0$$

[2 marks]

(iv) List the three roots of the original cubic equation,

$$x^3 - 3x^2 + 12x + 16 = 0$$

[2 marks]

1.7 Answers

1.7.1 Solution to 1.5 After Watching the Teaching Video

(a) **(i)** $x^2 - \left(-\frac{b}{a}\right)x + \left(\frac{c}{a}\right) = 0$

[1 mark]

(ii) $x^2 - (\text{Sum of Roots}) + (\text{Product of Roots}) = 0$

[1 mark]

(iii) $x^2 + \frac{1}{4}x - \frac{15}{8} = 0$

[1 mark]

(b) **(i)** $\alpha + \beta = -\frac{1}{4}$ **(ii)** $\alpha\beta = -\frac{15}{8}$

[1, 1 mark]

(iii)

$$\begin{aligned}\frac{1}{\alpha} + \frac{1}{\beta} &= \frac{\beta}{\alpha\beta} + \frac{\alpha}{\alpha\beta} \\ &= \frac{\alpha + \beta}{\alpha\beta} \\ &= \frac{(-1) \times 8}{4 \times (-15)} \\ &= \frac{2}{15}\end{aligned}$$

[2 marks]

(iv)

$$\begin{aligned}\alpha^2 + \beta^2 &= \alpha^2 + 2\alpha\beta + \beta^2 - 2\alpha\beta \\ &= (\alpha + \beta)^2 - 2\alpha\beta \\ &= \left(-\frac{1}{4}\right)^2 - 2\left(-\frac{15}{8}\right) \\ &= \frac{1}{16} + \frac{60}{16} \\ &= \frac{61}{16} \qquad \text{or } 3.75\end{aligned}$$

[2 marks]

1.7.2 Solutions to 1.6 Exercise

Answer 1

$$3x^2 + 7x - 2 = 0$$

$$x^2 + \frac{7}{3}x - \frac{2}{3} = 0$$

(i) $\alpha + \beta = -\frac{7}{3}$ **(ii)** $\alpha\beta = -\frac{2}{3}$

[1, 1 marks]

(iii)

$$\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\beta}{\alpha\beta} + \frac{\alpha}{\alpha\beta}$$

$$= \frac{\alpha + \beta}{\alpha\beta}$$

$$= \frac{(-7) \times 3}{3 \times (-2)}$$

$$= \frac{7}{2}$$

[2 marks]

(iv)

$$\alpha^2 + \beta^2 = \alpha^2 + 2\alpha\beta + \beta^2 - 2\alpha\beta$$

$$= (\alpha + \beta)^2 - 2\alpha\beta$$

$$= \left(-\frac{7}{3}\right)^2 - 2\left(-\frac{2}{3}\right)$$

$$= \frac{49}{9} + \frac{12}{9}$$

$$= \frac{61}{9} \qquad \text{or } 6\frac{7}{9}$$

[2 marks]

Answer 2

$$4x^2 - 3x + 1 = 0$$

$$x^2 - \frac{3}{4}x + \frac{1}{4} = 0$$

(i) $\alpha + \beta = \dfrac{3}{4}$ **(ii)** $\alpha\beta = \dfrac{1}{4}$

[1, 1 marks]

(iii)

$$\alpha^2 + \beta^2 = \alpha^2 + 2\alpha\beta + \beta^2 - 2\alpha\beta$$

$$= (\alpha + \beta)^2 - 2\alpha\beta$$

$$= \left(\frac{3}{4}\right)^2 - 2\left(\frac{1}{4}\right)$$

$$= \frac{9}{16} - \frac{8}{16}$$

$$= \frac{1}{16} \quad \text{or } 0.0625$$

[2 marks]

(iv)

$$\frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{\beta^2}{\alpha^2\beta^2} + \frac{\alpha^2}{\alpha^2\beta^2}$$

$$= \frac{\alpha^2 + \beta^2}{(\alpha\beta)^2}$$

$$= \frac{1 \times 4^2}{16 \times 1}$$

$$= 1$$

[2 marks]

Answer 3

(i)

$$\begin{aligned} \text{LHS} &= (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta) \\ &= \alpha^3 + 3\alpha^2\beta + 3\alpha\beta^2 + \beta^3 - 3\alpha^2\beta - 3\alpha\beta^2 \\ &= \alpha^3 + \beta^3 \\ &= \text{RHS} \qquad \square \end{aligned}$$

[2 marks]

(ii)

$$5x^2 + 6x + 2 = 0$$

$$x^2 + \frac{6}{5}x + \frac{2}{5} = 0$$

$$\therefore \ \alpha + \beta = -\frac{6}{5} \ \text{ and } \ \alpha\beta = \frac{2}{5}$$

$$\begin{aligned} \alpha^3 + \beta^3 &= (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta) \\ &= \left(-\frac{6}{5}\right)^3 - 3 \times \frac{2}{5} \times \left(-\frac{6}{5}\right) \\ &= -\frac{216}{125} + \frac{36}{25} \times \frac{5}{5} \\ &= \frac{-216+180}{125} \\ &= -\frac{36}{125} \ \text{ or } \ 0.288 \end{aligned}$$

[4 marks]

Answer 4

$$\begin{aligned} x^2 - \left(\frac{1}{2} - \frac{2}{3}\right)x + \left(\frac{1}{2} \times \frac{-2}{3}\right) &= 0 \\ x^2 - \left(-\frac{1}{6}\right)x - \frac{1}{3} &= 0 \\ 6x^2 + x - 2 &= 0 \end{aligned}$$

[3 marks]

Answer 5

$$\begin{aligned} x^2 - \left(\frac{1+2i}{3} + \frac{1-2i}{3}\right)x + \left(\frac{1+2i}{3}\right)\left(\frac{1-2i}{3}\right) &= 0 \\ 9x^2 - (3+6i+3-6i)x + (1+2i)(1-2i) &= 0 \\ 9x^2 - 6x + 5 &= 0 \end{aligned}$$

[4 marks]

Answer 6

$$6x^2 + 36x + k = 0$$

$$x^2 + 6x + \frac{k}{6} = 0$$

Let the two roots be α and $\frac{1}{\alpha}$

Look at the product of the roots:

$$\alpha \times \frac{1}{\alpha} = \frac{k}{6}$$

$$1 = \frac{k}{6}$$

$$k = 6$$

[4 marks]

Answer 7
(i)

$$x^3 - 3x^2 + 12x + 16 = 0$$

The change of variable is, $x = t - \frac{(-3)}{3 \times 1}$

$$x = t + 1$$

$$(t + 1)^3 - 3(t + 1)^2 + 12(t + 1) + 16 = 0$$

$$t^3 + 3t^2 + 3t + 1 - 3t^2 - 6t - 3 + 12t + 12 + 16 = 0$$

$$t^3 + 9t + 26 = 0 \qquad \square$$

[4 marks]

(ii)

$$C = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} \text{ with } p = 9 \text{ and } q = 26$$

$$= \sqrt[3]{-\frac{26}{2} + \sqrt{\frac{26^2}{4} + \frac{9^3}{27}}}$$

$$= \sqrt[3]{-13 + \sqrt{169 + 27}}$$

$$= \sqrt[3]{-13 + 14}$$

$$= 1$$

$$\alpha = 1 - \frac{9}{3 \times 1}$$

$$= -2$$

[3 marks]

(iii) Using the technique of polynomial division from the year 1 course,

$$\begin{array}{r|l}
 & \quad\;\; t^2 - 2t + 13 \\ \hline
t + 2 & t^3 + 0t^2 + 9t + 26 \\
 & t^3 + 2t^2 \\ \cline{2-2}
 & \quad - 2t^2 + 9t \\
 & \quad - 2t^2 - 4t \\ \cline{2-2}
 & \qquad\quad 13t + 26 \\
 & \qquad\quad 13t + 26 \\ \cline{2-2}
 & \qquad\qquad\quad 0 \quad \leftarrow \text{ as expected}
\end{array}$$

For the quadratic, $t = \dfrac{2 \pm \sqrt{2^2 - 4 \times 1 \times 13}}{2 \times 1}$

$$= \frac{2 \pm \sqrt{-48}}{2}$$

$$= 1 \pm 2\sqrt{3}\,\mathrm{i}$$

$$-2, \qquad 1 - 2\sqrt{3}\,\mathrm{i} \qquad 1 + 2\sqrt{3}\,\mathrm{i}$$

[2 marks]

(iv)

Using $x = t + 1$

$$-1, \qquad 2 - 2\sqrt{3}\,\mathrm{i}, \qquad 2 + 2\sqrt{3}\,\mathrm{i}$$

[2 marks]

2.1 Root Relations I

Given a function, a standard mathematical technique is to move it to a place where it is more easily studied, before then moving the results of the studies back to the original location. With polynomials, it's often easiest to move the polynomial by moving it's roots rather than the polynomial itself. All the more so as the actual values of the roots need not be found. The "move" need not be a simple translation and, indeed, all sorts of interesting other transformations may come into play.

2.2 Cardano's Use of Root Relations

In the last question of last lesson's exercise it was shown that this process was used by Cardano with his formula for solving cubic equations.

Faced with a general cubic,

$$a x^3 + b x^2 + cx + d = 0$$

he would move it by replacing x with $t - \dfrac{b}{3a}$

This always resulted in what is termed a "depressed cubic", one of the form,

$$t^3 + pt + q = 0$$

Cardano knew how to find the roots of this, and by moving those roots back he then had the roots of the original cubic.

Gerolamo Cardano (1501-1576)

An Italian polymath who was one of the most influential mathematicians of the Renaissance.
Today, he is most remembered for his achievements in algebra.
He made the first systematic use of negative numbers in Europe, published with attribution the solutions of other mathematicians for the cubic and quartic equations and acknowledged the existence of imaginary numbers.

2.3 Example

Let the roots of the quadratic equation $2x^2 + 7x + 1 = 0$ are α and β.
Without solving the equation, find an all integer coefficient polynomial with roots

$$2\alpha + 1 \quad \text{and} \quad 2\beta + 1$$

Teaching Video : http://www.NumberWonder.co.uk/v9093/2.mp4

After watching the teaching video write out your own version of the solution.

☞

[5 marks]

2.4 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable.
Make the method used clear.

Marks available : 40

Question 1

The roots of the quadratic equation $3x^2 - 4x + 6 = 0$ are α and β.
Without solving the equation, find an all integer coefficient polynomial with roots

$$3\alpha + 1 \quad \text{and} \quad 3\beta + 1$$

[5 marks]

Question 2

The roots of the quadratic equation $3x^2 + 6x + 1 = 0$ are α and β.
Without solving the equation, find an all integer coefficient polynomial with roots

$$1 - \alpha \quad \text{and} \quad 1 - \beta$$

[5 marks]

Question 3

The roots of the quadratic equation $2x^2 + 3x + 7 = 0$ are α and β.
Without solving the equation, find an all integer coefficient polynomial with roots

$$\alpha^2 + 1 \quad \text{and} \quad \beta^2 + 1$$

[6 marks]

Question 4

The roots of the quadratic equation $5x^2 + 15x + 8 = 0$ are α and β.
Without solving the equation, find an all integer coefficient polynomial with roots

$$1 + \frac{1}{\alpha} \quad \text{and} \quad 1 + \frac{1}{\beta}$$

[8 marks]

Question 5

Further A-Level Examination Question from January 2018, IAL, F1, Q4

The quadratic equation $3x^2 + 2x + 5 = 0$ has roots α and β

Without solving the equation,

(a) find the value of $\alpha^2 + \beta^2$

[2 marks]

(b) show that $\alpha^3 + \beta^3 = \dfrac{82}{27}$

[2 marks]

(c) find a quadratic equation with roots $\left(\alpha + \dfrac{\alpha}{\beta^2} \right)$ and $\left(\beta + \dfrac{\beta}{\alpha^2} \right)$

Give the answer in the form $px^2 + qx + r = 0$, with $p, q, r \in \mathbb{Z}$

[4 marks]

Question 6
This question is about using Cardano's method to find the roots of the cubic,

$$x^3 - 66x + 340 = 0$$

It's a little more challenging than the question at the end of previous exercise although the cubic is already in depressed form so there is no need to change the variable. Here is a recap of the theory;

Roots of a Cubic
Given a depressed cubic of the form

$$t^3 + pt + q = 0 \qquad\qquad p,\ q \in \mathbb{C}$$

where p and q are not both zero, and $4p^3 + 27q^2 \neq 0$, calculate,

$$C = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

A root of the cubic is then given by,

$$\alpha = C - \frac{p}{3C}$$

Polynomial division can then be used to find the remaining roots.

To help with a tricky step, there is a part (i) which is useful when tackling the main problem.

(i) Expand the brackets,

$$\left(\sqrt{3} - 5\right)^3$$

giving your answer in the form $a\sqrt{3} + b$ for integer a and b

[2 marks]

(ii) Determine, using the method of Cardano, the real root of the cubic,

$$x^3 - 66x + 340 = 0$$

[4 marks]

(iii) Using polynomial division with your part (ii) answer, find all three roots, two of which are a complex conjugate pair, of the depressed cubic,

$$x^3 - 66x + 340 = 0$$

[2 marks]

2.5 Answers

2.5.1 Solutions to 2.3 Teaching Video

$$2x^2 + 7x + 1 = 0$$

Divide through by 2

$$x^2 + \frac{7}{2}x + \frac{1}{2} = 0$$

$$\therefore \ \alpha + \beta = -\frac{7}{2} \quad \text{and} \quad \alpha\beta = \frac{1}{2}$$

Sum of new roots : $(2\alpha + 1) + (2\beta + 1)$

$$= 2(\alpha + \beta) + 2$$

$$= 2 \times \left(-\frac{7}{2}\right) + 2$$

$$= -5$$

Product of new roots : $(2\alpha + 1)(2\beta + 1)$

$$= 4\alpha\beta + 2(\alpha + \beta) + 1$$

$$= 4 \times \frac{1}{2} + 2 \times \left(-\frac{7}{2}\right) + 1$$

$$= 2 - 7 + 1$$

$$= -4$$

The sought after polynomial is thus,

$$w^2 + 5w - 4 = 0$$

[5 marks]

2.5.2 Solutions to 2.4 Exercise

Answer 1

$$3x^2 - 4x + 6 = 0$$

Divide through by 3

$$x^2 - \frac{4}{3}x + 2 = 0$$

$$\therefore \ \alpha + \beta = \frac{4}{3} \text{ and } \alpha\beta = 2$$

Sum of new roots : $(3\alpha + 1) + (3\beta + 1)$

$$= 3(\alpha + \beta) + 2$$

$$= 3 \times \left(\frac{4}{3}\right) + 2$$

$$= 6$$

Product of new roots : $(3\alpha + 1)(3\beta + 1)$

$$= 9\alpha\beta + 3(\alpha + \beta) + 1$$

$$= 9 \times 2 + 3 \times \left(\frac{4}{3}\right) + 1$$

$$= 18 + 4 + 1$$

$$= 23$$

The sought after polynomial is thus,

$$w^2 - 6w + 23 = 0$$

[5 marks]

Answer 2

$$3x^2 + 6x + 1 = 0$$

Divide through by 3

$$x^2 + 2x + \frac{1}{3} = 0$$

$$\therefore \ \alpha + \beta = -2 \ \text{ and } \ \alpha\beta = \frac{1}{3}$$

Sum of new roots : $(1 - \alpha) + (1 - \beta)$

$$= 2 - (\alpha + \beta)$$
$$= 2 - (-2)$$
$$= 4$$

Product of new roots : $(1 - \alpha)(1 - \beta)$

$$= 1 - (\alpha + \beta) + \alpha\beta$$
$$= 1 - (-2) + \frac{1}{3}$$
$$= \frac{10}{3}$$

The sought after polynomial is thus,

$$w^2 - 4w + \frac{10}{3} = 0$$
$$3w^2 - 12w + 10 = 0$$

[5 marks]

Answer 3

$$2x^2 + 3x + 7 = 0$$

Divide through by 2

$$x^2 + \frac{3}{2}x + \frac{7}{2} = 0$$

$$\therefore\ \alpha + \beta = -\frac{3}{2} \text{ and } \alpha\beta = \frac{7}{2}$$

Sum of new roots : $\left(\alpha^2 + 1\right) + \left(\beta^2 + 1\right)$

$$= \alpha^2 + \beta^2 + 2$$

$$= (\alpha + \beta)^2 - 2\alpha\beta + 2$$

$$= \left(-\frac{3}{2}\right)^2 - 2 \times \frac{7}{2} + 2$$

$$= \frac{9}{4} - 7 + 2$$

$$= -\frac{11}{4}$$

Product of new roots : $\left(\alpha^2 + 1\right)\left(\beta^2 + 1\right)$

$$= \alpha^2\beta^2 + \alpha^2 + \beta^2 + 1$$

$$= (\alpha\beta)^2 + (\alpha + \beta)^2 - 2\alpha\beta + 1$$

$$= \left(\frac{7}{2}\right)^2 + \left(-\frac{3}{2}\right)^2 - 2 \times \frac{7}{2} + 1$$

$$= \frac{49}{4} + \frac{9}{4} - 7 + 1$$

$$= \frac{17}{2}$$

The sought after polynomial is thus,

$$w^2 + \frac{11}{4}w + \frac{17}{2} = 0$$

$$4w^2 + 11w + 34 = 0$$

[6 marks]

Answer 4

$$5x^2 + 15x + 8 = 0$$

Divide through by 5

$$x^2 + 3x + \frac{8}{5} = 0$$

$$\therefore \ \alpha + \beta = -3 \text{ and } \alpha\beta = \frac{8}{5}$$

Sum of new roots : $\left(1 + \frac{1}{\alpha}\right) + \left(1 + \frac{1}{\beta}\right)$

$$= 2 + \frac{1}{\alpha} + \frac{1}{\beta}$$

$$= 2 + \frac{\beta}{\beta\alpha} + \frac{\alpha}{\alpha\beta}$$

$$= 2 + \frac{\alpha + \beta}{\alpha\beta}$$

$$= 2 + \frac{(-3) \times 5}{8}$$

$$= \frac{1}{8}$$

Product of new roots : $\left(1 + \frac{1}{\alpha}\right)\left(1 + \frac{1}{\beta}\right)$

$$= 1 + \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\alpha\beta}$$

$$= 1 + \frac{\beta}{\beta\alpha} + \frac{\alpha}{\alpha\beta} + \frac{1}{\alpha\beta}$$

$$= 1 + \frac{\alpha + \beta + 1}{\alpha\beta}$$

$$= 1 + \frac{(-3 + 1) \times 5}{8}$$

$$= -\frac{1}{4}$$

The sought after polynomial is thus,

$$w^2 - \frac{1}{8}w - \frac{1}{4} = 0$$

$$8w^2 - w - 2 = 0$$

[8 marks]

Answer 5

(a)

$$3x^2 + 2x + 5 = 0$$

Divide through by 3

$$x^2 + \frac{2}{3}x + \frac{5}{3} = 0$$

$$\therefore \alpha + \beta = -\frac{2}{3} \text{ and } \alpha\beta = \frac{5}{3}$$

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$$

$$= \left(-\frac{2}{3}\right)^2 - 2 \times \frac{5}{3}$$

$$= \frac{4}{9} - \frac{30}{9}$$

$$= -\frac{26}{9}$$

[2 marks]

(b)

$$(\alpha + \beta)^3 = \alpha^3 + 3\alpha^2\beta + 3\alpha\beta^2 + \beta^3$$

$$\therefore \alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$$

$$= \left(-\frac{2}{3}\right)^3 - 3 \times \frac{5}{3} \times \left(-\frac{2}{3}\right)$$

$$= -\frac{8}{27} + \frac{90}{27}$$

$$= \frac{82}{27} \qquad \square$$

[2 marks]

(c)

Sum of new roots : $\left(\alpha + \dfrac{\alpha}{\beta^2}\right) + \left(\beta + \dfrac{\beta}{\alpha^2}\right)$

$$= (\alpha + \beta) + \frac{\alpha^3 + \beta^3}{\alpha^2\beta^2}$$

$$= \left(-\frac{2}{3}\right) + \frac{82 \times 9}{27 \times 25}$$

$$= -\frac{2}{3} + \frac{82}{75}$$

$$= \frac{32}{75}$$

Product of new roots : $\left(\alpha + \frac{\alpha}{\beta^2}\right)\left(\beta + \frac{\beta}{\alpha^2}\right)$

$$= \alpha\beta\left(1 + \frac{1}{\beta^2}\right)\left(1 + \frac{1}{\alpha^2}\right)$$

$$= \alpha\beta\left(1 + \frac{1}{\beta^2} + \frac{1}{\alpha^2} + \frac{1}{\alpha^2\beta^2}\right)$$

$$= \alpha\beta\left(1 + \frac{\alpha^2 + \beta^2}{\alpha^2\beta^2} + \frac{1}{\alpha^2\beta^2}\right)$$

$$= \alpha\beta + \frac{\alpha^2 + \beta^2}{\alpha\beta} + \frac{1}{\alpha\beta}$$

$$= \left(\frac{5}{3}\right) + \frac{-26 \times 3}{9 \times 5} + \frac{3}{5}$$

$$= \frac{25}{15} - \frac{26}{15} + \frac{9}{15}$$

$$= \frac{8}{15}$$

The sought after polynomial is thus,

$$w^2 - \frac{32}{75}w + \frac{8}{15} = 0$$

$$75w^2 - 32w + 40 = 0$$

[4 marks]

Answer 6

(i)

$$\left(\sqrt{3} - 5\right)^3 = \left(\sqrt{3}\right)^3 + 3\left(\sqrt{3}\right)^2(-5) + 3\left(\sqrt{3}\right)(-5)^2 + (-5)^3$$
$$= 3\sqrt{3} - 45 + 75\sqrt{3} - 125$$
$$= 78\sqrt{3} - 170$$

[2 marks]

(ii)

$$x^3 - 66x + 340 = 0$$
$$p = -66 \text{ and } q = 340$$

$$\text{Check } 4p^3 + 27q^2 = 4 \times (-66)^3 + 27 \times 340^2$$
$$= -1149984 + 3121200$$
$$= 1971216$$
$$\neq 0 \qquad \text{so OK to proceed...}$$

$$C = \sqrt[3]{-\frac{340}{2} + \sqrt{\frac{340^2}{4} + \frac{(-66)^3}{27}}}$$
$$= \sqrt[3]{-170 + \sqrt{28900 - 10648}}$$
$$= \sqrt[3]{-170 + 78\sqrt{3}}$$
$$= \sqrt{3} - 5 \qquad \text{from part (i)}$$

$$x = \sqrt{3} - 5 - \frac{-66}{3\left(\sqrt{3} - 5\right)}$$
$$= \sqrt{3} - 5 + \frac{22}{\left(\sqrt{3} - 5\right)} \times \frac{\left(\sqrt{3} + 5\right)}{\left(\sqrt{3} + 5\right)}$$
$$= \sqrt{3} - 5 + \frac{22\left(\sqrt{3} + 5\right)}{3 - 25}$$
$$= \sqrt{3} - 5 - \sqrt{3} - 5$$
$$= -10$$

[6 marks]

(iii) $\therefore \ (x + 10)$ is a factor of $x^3 - 66x + 340$

Using the technique of polynomial division from the year 1 course,

$$\begin{array}{r|rrrr} & x^2 & -\ 10x & +\ 34 & \\ \hline x + 10 & x^3 + 0x^2 & -\ 66x & +\ 340 & \\ & x^3 + 10x^2 & & & \\ \hline & -\ 10x^2 & -\ 66x & & \\ & -\ 10x^2 & -\ 100x & & \\ \hline & & 34x & +\ 340 & \\ & & 34x & +\ 340 & \\ \hline & & & 0 & \leftarrow \text{as expected} \end{array}$$

For the quadratic, $x = \dfrac{10 \pm \sqrt{(-10)^2 - 4 \times 1 \times 34}}{2 \times 1}$

$= \dfrac{10 \pm \sqrt{-36}}{2}$

$= 5 \pm 3\text{i}$

The three roots of the cubic equation are,

$-10, \quad 5 - 3\text{i} \quad 5 + 3\text{i}$

[2 marks]

3.1 Cubic and Roots

Having taken a fresh look at quadratic equations through a study of their roots, the focus can now shift to cubic equations. Thanks to Gerolamo Cardano it is known that there is, in principle, a formula to solve cubics. Alas, it has a square root nested inside a cube root which makes manipulating it intricate. Fortunately, the algebra can be skirted around by a more thoughtful approach. To illustrate the idea, it will first be applied it to the generalised quadratic equation.

3.2 A Proof Rewritten

The Roots of a Quadratic

If α and β are the roots of the equation $a x^2 + bx + c = 0 \qquad a, b, c \in \mathbb{C}$

then, $\qquad \alpha + \beta = -\dfrac{b}{a} \qquad \alpha\beta = \dfrac{c}{a}$

Proof

For a general quadratic, with roots α and β,

$$\begin{aligned} a x^2 + bx + c &= a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) \\ &= a\left((x-\alpha)(x-\beta)\right) \\ &= a\left(x^2 - \alpha x - \beta x + \alpha\beta\right) \\ &= a\left(x^2 - (\alpha+\beta)x + \alpha\beta\right) \end{aligned}$$

From which can be seen that,

$$\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = \left(x^2 - (\alpha+\beta)x + \alpha\beta\right)$$

And, by matching coefficients of x, the deduction made that,

$$\alpha + \beta = -\frac{b}{a} \quad \text{and} \quad \alpha\beta = \frac{c}{a} \qquad \square$$

The key feature of this proof, in comparison with that presented in Lesson 1, is that no use is made of the formula,

$$x = \frac{-b \pm \sqrt{b^2 + 4ac}}{2a}$$

3.3 A Similar Proof For Cubics

The Roots of a Cubic

If α, β and γ are roots of $a\,x^3 + b\,x^2 + cx + d = 0 \qquad a,\ b,\ c,\ d \in \mathbb{C}$

then, $\qquad \alpha + \beta + \gamma = -\dfrac{b}{a} \qquad \alpha\beta + \beta\gamma + \gamma\alpha = \dfrac{c}{a} \qquad \alpha\beta\gamma = -\dfrac{d}{a}$

Proof

For a general cubic, with roots α, β and γ,

$$\begin{aligned}
a\,x^3 + b\,x^2 + cx + d &= a\left(x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a}\right) \\
&= a\big(\,(x-\alpha)\,(x-\beta)\,(x-\gamma)\,\big) \\
&= a\big(\,(x^2 - \alpha x - \beta x + \alpha\beta)(x-\gamma\,)\,\big) \\
&= a\big(\,(x^2 - \alpha x - \beta x + \alpha\beta)\,x - (x^2 - \alpha x - \beta x + \alpha\beta)\,\gamma\,\big) \\
&= a\left(x^3 - \alpha\,x^2 - \beta\,x^2 + \alpha\beta x - \gamma x^2 + \alpha\gamma x + \beta\gamma x - \alpha\beta\gamma\right) \\
&= a\left(x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \beta\gamma + \gamma\alpha)\,x - \alpha\beta\gamma\right)
\end{aligned}$$

from which can be seen that,

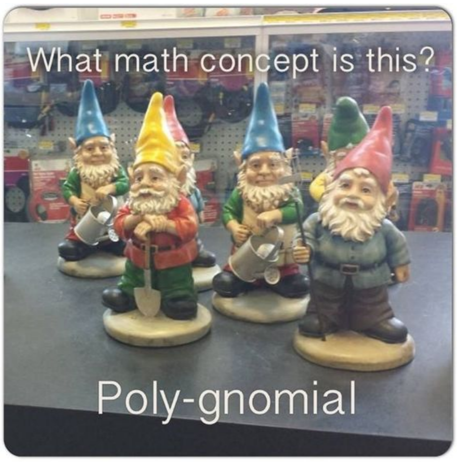
$$\left(x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a}\right) = \left(x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \beta\gamma + \gamma\alpha)\,x - \alpha\beta\gamma\right)$$

and, by matching coefficients of x, the deduction made that,

$$\alpha + \beta + \lambda = -\frac{b}{a}, \quad \alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a} \quad \text{and } \alpha\beta\gamma = -\frac{d}{a} \qquad \square$$

Notice that, as with the quadratic,

- The sum of the roots is $\left(-\dfrac{b}{a}\right)$
- The sum of all possible product pairs of roots is $\left(\dfrac{c}{a}\right)$



3.4 Example

The roots of the cubic equation $2x^3 + 5x^2 - 2x + 3 = 0$ are α, β and γ.

Without solving the equation, find the value of $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$

Teaching Video : http://www.NumberWonder.co.uk/v9093/3.mp4

Watch the teaching video and then write out a solution to the question.

☞

[4 marks]

3.5 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable.
Make the method used clear.
Marks available : 40

Question 1

The roots of the equation $4x^3 - 3x^2 - x + 6 = 0$ are α, β and γ
Without solving the equation, find the roots of

(i) $\alpha + \beta + \gamma$ **(ii)** $\alpha\beta + \beta\gamma + \gamma\alpha$

[1, 1 mark]

(iii) $\alpha^2 \beta^2 \gamma^2$ **(iv)** $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$

[1, 2 marks]

Question 2

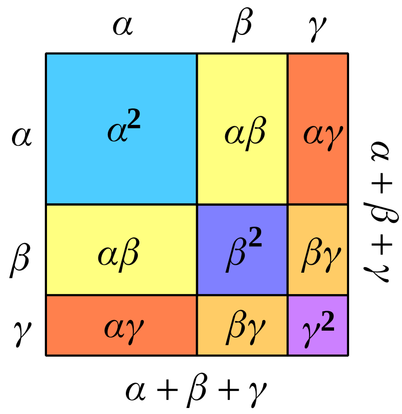
The roots of the cubic equation $a x^3 + b x^2 + cx + d = 0$ are,

$$\alpha = 8, \quad \beta = 9 \quad \text{and} \quad \gamma = -10$$

Find integer values for a, b, c and d

[2 marks]

Question 3



(i) With the aid of the diagram expand the brackets of $(\alpha + \beta + \gamma)^2$

[1 mark]

The roots of the equation $2x^3 + 4x^2 + 7x + 1 = 0$ are α, β and γ
Without solving the equation, write down the values of,

(ii) $\alpha + \beta + \gamma$

[1 mark]

(iii) $\alpha\beta + \beta\gamma + \gamma\alpha$

[1 mark]

(iv) $\alpha\beta\gamma$

[1 mark]

(v) $\alpha^2 + \beta^2 + \gamma^2$

[2 marks]

Question 4

Further A-Level Examination Question from June 2017 SAM, Core 2, Q1

The roots of the equation $x^3 - 8x^2 + 28x - 32 = 0$ are α, β and γ

Without solving the equation, find the value of

(i) $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$

[3 marks]

(ii) $(\alpha + 2)(\beta + 2)(\gamma + 2)$

[3 marks]

(iii) $\alpha^2 + \beta^2 + \gamma^2$

[2 marks]

Question 5

Our sequence of questions exploring Cardano's extraordinary achievement in finding and using a formula to solve cubic equations continues by looking at,

$$x^3 - 252x + 1296 = 0$$

As you will discover, this example involves complex numbers.
Here is a recap of the theory to be applied;

Roots of a Cubic

Given a depressed cubic of the form

$$t^3 + pt + q = 0 \qquad\qquad p,\ q \in \mathbb{C}$$

where p and q are not both zero, and $4p^3 + 27q^2 \neq 0$, calculate,

$$C = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

A root of the cubic is then given by,

$$\alpha = C - \frac{p}{3C}$$

Polynomial division can then be used to find the remaining roots.

To help with a tricky step, there is a part (i) which is useful when tackling the main problem.

(i) Expand the brackets,

$$\left(6 + 4\sqrt{3}\,\mathrm{i}\right)^3 \quad \text{where } \mathrm{i} = \sqrt{-1} \text{ such that } \mathrm{i}^2 = -1$$

giving your answer in the form $a\sqrt{3}\,\mathrm{i} + b$ for integer a and b

[3 marks]

(ii) Determine using the method of Cardano, the real root of the cubic,

$$x^3 - 252x + 1296 = 0$$

[3 marks]

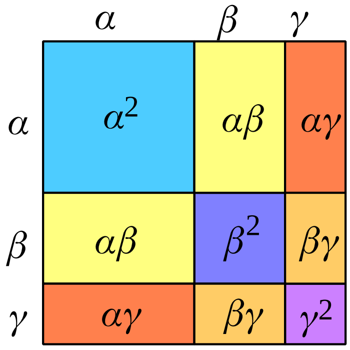
(iii) Using polynomial division with your part (ii) answer, find all three roots, which are all real, of the depressed cubic,

$$x^3 - 252x + 1296 = 0$$

[2 marks]

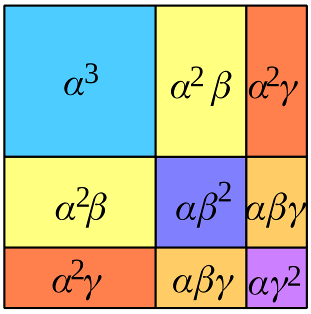
Question 6

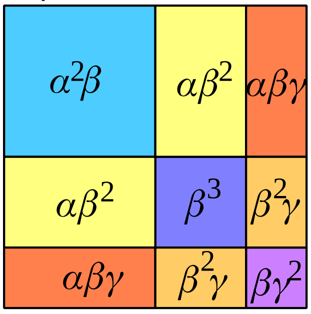
$$(\alpha+\beta+\gamma)^2 =$$

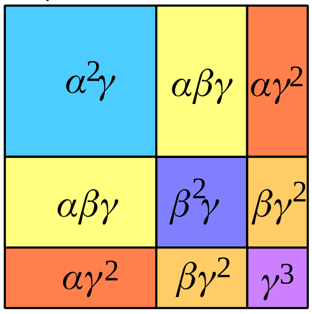


$$(\alpha+\beta+\gamma)^3 =$$

$$\alpha(\alpha+\beta+\gamma)^2 \quad + \quad \beta(\alpha+\beta+\gamma)^2 \quad + \quad \gamma(\alpha+\beta+\gamma)^2$$







(i) In question 3 a square with sides of length $\alpha + \beta + \gamma$ was of assistance in expanding the brackets of $(\alpha + \beta + \gamma)^2$

For $(\alpha + \beta + \gamma)^3$ a cube can be used in which case the three layers of the cube give cuboids with volumes as shown above.

By using the diagrams or otherwise, expand the brackets of $(\alpha + \beta + \gamma)^3$

[1 mark]

(ii) Manipulate your part (i) answer into a form that will allow you to express $\alpha^3 + \beta^3 + \gamma^3$ in terms of $\alpha + \beta + \gamma$, $\alpha\beta + \beta\gamma + \gamma\alpha$ and $\alpha\beta\gamma$

[2 marks]

Question 7

Further A-Level Examination Question from June 2019, Core 2, Q2 (Edexcel)

The roots of the equation $x^3 - 2x^2 + 4x - 5 = 0$ are *p*, *q* and *r*

Without solving the equation, find the value of

(i) $\dfrac{2}{p} + \dfrac{2}{q} + \dfrac{2}{r}$

[3 marks]

(ii) $(p - 4)(q - 4)(r - 4)$

[3 marks]

(iii) $p^3 + q^3 + r^3$

[2 marks]

3.6 Answers

3.6.1 Solution to 3.4 Teaching Video Example

$$2x^3 + 5x^2 - 2x + 3 = 0$$

Divide through by 2

$$x^3 + \frac{5}{2}x^2 - x + \frac{3}{2} = 0$$

$$\therefore \alpha + \beta + \gamma = -\frac{5}{2} \qquad \alpha\beta + \beta\gamma + \gamma\alpha = -1 \quad \text{and} \quad \alpha\beta\gamma = -\frac{3}{2}$$

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma}{\alpha\beta\gamma} + \frac{\alpha\gamma}{\alpha\beta\gamma} + \frac{\alpha\beta}{\alpha\beta\gamma}$$

$$= \frac{\alpha\beta + \beta\gamma + \gamma\alpha}{\alpha\beta\gamma}$$

$$= \frac{(-1) \times 2}{(-3)}$$

$$= \frac{2}{3}$$

[4 marks]

3.6.2 Solution to 3.5 Exercise

Answer 1 $$4x^3 - 3x^2 - x + 6 = 0$$

Divide through by 4

$$x^3 - \frac{3}{4}x^2 - \frac{1}{4}x + \frac{3}{2} = 0$$

(i) $\alpha + \beta + \gamma = \dfrac{3}{4}$ **(ii)** $\alpha\beta + \beta\gamma + \gamma\alpha = -\dfrac{1}{4}$

(iii) $\alpha\beta\gamma = -\dfrac{3}{2} \Rightarrow \alpha^2\beta^2\gamma^2 = \dfrac{9}{4}$

(iv)
$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma}{\alpha\beta\gamma} + \frac{\alpha\gamma}{\alpha\beta\gamma} + \frac{\alpha\beta}{\alpha\beta\gamma}$$
$$= \frac{\alpha\beta + \beta\gamma + \gamma\alpha}{\alpha\beta\gamma}$$
$$= \frac{(-1) \times 2}{4 \times (-3)} = \frac{1}{6}$$

[1, 1, 1, 2 marks]

Answer 2

$$ax^3 + bx^2 + cx + d = 0 \qquad \Rightarrow x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} = 0$$

$$\alpha = 8, \quad \beta = 9 \text{ and } \gamma = -10$$

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$-\frac{b}{a} = 8 + 9 - 10 = 7$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$$

$$\frac{c}{a} = 8 \times 9 + 9 \times (-10) + (-10) \times 8$$

$$= 72 - 90 - 80 = -98$$

$$\alpha\beta\gamma = -\frac{d}{a}$$

$$-\frac{d}{a} = 8 \times 9 \times (-10) = -720$$

Thus the polynomial sought is,

$$x^3 - 7x^2 - 98x + 720 = 0$$

That is, $a = 1, \quad b = -7, \quad c = -98, \quad d = 720$ (for example)

[2 marks]

Answer 3

$$2x^3 + 4x^2 + 7x + 1 = 0$$

Divide through by 2

$$x^3 + 2x^2 + \frac{7}{2}x + \frac{1}{2} = 0$$

(i) $(\alpha + \beta + \gamma)^2 = \alpha^2 + \beta^2 + \gamma^2 + 2(\alpha\beta + \beta\gamma + \gamma\alpha)$

(ii) $\alpha + \beta + \gamma = -2$

(iii) $\alpha\beta + \beta\gamma + \gamma\alpha = \dfrac{7}{2}$

(iv) $\alpha\beta\gamma = -\dfrac{1}{2}$

(v) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$

$$= (-2)^2 - 2 \times \frac{7}{2} = -3$$

[1, 1, 1, 1, 2 marks]

Answer 4

$$x^3 - 8x^2 + 28x - 32 = 0$$

$$\alpha + \beta + \gamma = 8 \qquad \alpha\beta + \beta\gamma + \gamma\alpha = 28 \qquad \alpha\beta\gamma = 32$$

(i)
$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma}{\alpha\beta\gamma} + \frac{\alpha\gamma}{\alpha\beta\gamma} + \frac{\alpha\beta}{\alpha\beta\gamma}$$
$$= \frac{\alpha\beta + \beta\gamma + \gamma\alpha}{\alpha\beta\gamma}$$
$$= \frac{28}{32} = \frac{7}{8} \quad \text{or} \quad 0.875$$

(ii)
$$(\alpha + 2)(\beta + 2)(\gamma + 2) = (\alpha + 2)(\beta\gamma + 2\beta + 2\gamma + 4)$$
$$= \alpha\beta\gamma + 2\alpha\beta + 2\alpha\gamma + 4\alpha + 2\beta\gamma + 4\beta + 4\gamma + 8$$
$$= \alpha\beta\gamma + 2(\alpha\beta + \beta\gamma + \gamma\alpha) + 4(\alpha + \beta + \gamma) + 8$$
$$= 32 + 2 \times 28 + 4 \times 8 + 8$$
$$= 128$$

(iii)
$$(\alpha + \beta + \gamma)^2 = \alpha^2 + \beta^2 + \gamma^2 + 2(\alpha\beta + \beta\gamma + \gamma\alpha) \qquad \text{(see question 3)}$$
$$\therefore \alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$$
$$= 8^2 - 2 \times 28$$
$$= 8$$

[3, 3, 2 marks]

Answer 5

(i)

$$\left(6 + 4\sqrt{3}\ \mathrm{i} \right)^3 = \left(2\left(3 + 2\sqrt{3}\ \mathrm{i} \right) \right)^3$$
$$= 8\left(3 + 2\sqrt{3}\ \mathrm{i} \right)^3$$
$$= 8\left(3 + 2\sqrt{3}\ \mathrm{i} \right)\left(3 + 2\sqrt{3}\ \mathrm{i} \right)^2$$
$$= 8\left(3 + 2\sqrt{3}\ \mathrm{i} \right)\left(9 + 12\sqrt{3}\ \mathrm{i} + 12\,\mathrm{i}^2 \right)$$
$$= 8\left(3 + 2\sqrt{3}\ \mathrm{i} \right)\left(-3 + 12\sqrt{3}\ \mathrm{i} \right)$$
$$= 24\left(3 + 2\sqrt{3}\ \mathrm{i} \right)\left(-1 + 4\sqrt{3}\ \mathrm{i} \right)$$
$$= 24\left(-3 + 12\sqrt{3}\ \mathrm{i} - 2\sqrt{3}\ \mathrm{i} + 24\,\mathrm{i}^2 \right)$$
$$= 24\left(-27 + 10\sqrt{3}\ \mathrm{i} \right)$$
$$= -648 + 240\sqrt{3}\ \mathrm{i}$$

[3 marks]

(ii) $x^3 - 252x + 1296 = 0 \Rightarrow p = -252, \quad q = 1296$

$$4p^3 + 27q^2 = 4\,(-252)^3 + 27 \times 1296^2$$
$$= -64012032 + 45349632$$
$$= -18662400 \qquad \text{Which is } \neq 0 \text{ so OK to proceed}$$

$$C = \sqrt[3]{-\frac{1296}{2} + \sqrt{\frac{1296^2}{4} + \frac{(-252)^3}{27}}}$$
$$= \sqrt[3]{-648 + \sqrt{419904 - 592704}}$$
$$= \sqrt[3]{-648 + \sqrt{-172800}}$$
$$= \sqrt[3]{-648 + \sqrt{57600 \times 3 \times (-1)}}$$
$$= \sqrt[3]{-648 + 240\sqrt{3}\ \mathrm{i}}$$
$$= 6 + 4\sqrt{3}\ \mathrm{i} \qquad \text{from part (i)}$$

$$
\begin{aligned}
x &= 6 + 4\sqrt{3}\,i - \frac{(-252)}{3\,(6 + 4\sqrt{3}\,i)} \\
&= 6 + 4\sqrt{3}\,i + \frac{84\,(6 - 4\sqrt{3}\,i)}{(6 + 4\sqrt{3}\,i)(6 - 4\sqrt{3}\,i)} \\
&= 6 + 4\sqrt{3}\,i + \frac{84\,(6 - 4\sqrt{3}\,i)}{36 + 48} \\
&= 6 + 4\sqrt{3}\,i + 6 - 4\sqrt{3}\,i \\
&= 12
\end{aligned}
$$

[3 marks]

(iii) $\therefore\ (x - 12)$ is a factor of $x^3 - 252x + 1296$

Using the technique of polynomial division from the year 1 course,

$$
\begin{array}{r|l}
 & x^2 + 12x - 108 \\
\hline
x - 12 & x^3 + 0x^2 - 252x + 1296 \\
 & \underline{x^3 - 12x^2} \\
 & \quad 12x^2 - 252x \\
 & \quad \underline{12x^2 - 144x} \\
 & \qquad -108x + 1296 \\
 & \qquad \underline{-108x + 1296} \\
 & \qquad\qquad\qquad 0 \leftarrow \text{ as expected}
\end{array}
$$

For the quadratic,

$$
\begin{aligned}
x &= \frac{-12 \pm \sqrt{(-12)^2 - 4 \times 1 \times (-108)}}{2 \times 1} \\
&= \frac{-12 \pm \sqrt{576}}{2} \\
&= \frac{-12 \pm 24}{2} \\
&= -18 \text{ or } 6
\end{aligned}
$$

The three roots of the cubic equation are,

$12, \quad 6 \quad (-18)$

[2 marks]

Answer 6

(i) $(\alpha + \beta + \gamma)^3 =$

$$\alpha^3 + 3\alpha^2\beta + 3\alpha^2\gamma + 3\beta^2\alpha + \beta^3 + 3\beta^2\gamma + 3\gamma^2\alpha + 3\gamma^2\beta + \gamma^3 + 6\alpha\beta\gamma$$

[1 mark]

(ii)

$$3\alpha^3 + 3\alpha^2\beta + 3\alpha^2\gamma + 3\beta^2\alpha + 3\beta^3 + 3\beta^2\gamma + 3\gamma^2\alpha + 3\gamma^2\beta + 3\gamma^3 - 2(\alpha^3 + \beta^3 + \gamma^3) + 6\alpha\beta\gamma$$

$$= 3\alpha^2(\alpha + \beta + \gamma) + 3\beta^2(\alpha + \beta + \gamma) + 3\gamma^2(\alpha + \beta + \gamma) - 2(\alpha^3 + \beta^3 + \gamma^3) + 6\alpha\beta\gamma$$

$$= 3(\alpha^2 + \beta^2 + \gamma^2)(\alpha + \beta + \gamma) - 2(\alpha^3 + \beta^3 + \gamma^3) + 6\alpha\beta\gamma$$

So,

$$2(\alpha^3 + \beta^3 + \gamma^3) = 3(\alpha^2 + \beta^2 + \gamma^2)(\alpha + \beta + \gamma) - (\alpha + \beta + \gamma)^3 + 6\alpha\beta\gamma$$

From question 3,

$$(\alpha + \beta + \gamma)^2 = \alpha^2 + \beta^2 + \gamma^2 + 2(\alpha\beta + \beta\gamma + \gamma\alpha)$$

That is, $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$

Thus,

$$2(\alpha^3 + \beta^3 + \gamma^3) =$$

$$3\left((\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)\right)(\alpha + \beta + \gamma) - (\alpha + \beta + \gamma)^3 + 6\alpha\beta\gamma$$

And finally,

$$\alpha^3 + \beta^3 + \gamma^3 =$$

$$\frac{3}{2}\left((\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)\right)(\alpha + \beta + \gamma) - \frac{1}{2}(\alpha + \beta + \gamma)^3 + 3\alpha\beta\gamma$$

Other expressions that can result include,

$$\alpha^3 + \beta^3 + \gamma^3 = (\alpha + \beta + \gamma)^3 - 3(\alpha + \beta + \gamma)(\alpha\beta + \beta\gamma + \gamma\alpha) + 3\alpha\beta\gamma$$

$$\alpha^3 + \beta^3 + \gamma^3 = (\alpha + \beta + \gamma)\left((\alpha + \beta + \gamma)^2 - 3(\alpha\beta + \beta\gamma + \gamma\alpha)\right) + 3\alpha\beta\gamma$$

[2 marks]

Answer 7

$$x^3 - 2x^2 + 4x - 5 = 0$$

$$p + q + r = 2 \qquad pq + qr + rp = 4 \qquad pqr = 5$$

(i)

$$\frac{2}{p} + \frac{2}{q} + \frac{2}{r} = \frac{2pr}{pqr} + \frac{2pr}{pqr} + \frac{2pq}{pqr}$$

$$= \frac{2(pq + qr + rq)}{pqr}$$

$$= \frac{2 \times 4}{5}$$

$$= \frac{8}{5}$$

[3 marks]

(ii)

$$(p - 4)(q - 4)(r - 4)$$

$$= (p - 4)(qr - 4q - 4r + 16)$$

$$= pqr - 4pq - 4pr + 16p - 4qr + 16q + 16r - 64$$

$$= pqr - 4(pq + qr + rp) + 16(p + q + r) - 64$$

$$= 5 - 4 \times 4 + 16 \times 2 - 64$$

$$= -43$$

[3 marks]

(iii) From question 6

$$p^3 + q^3 + r^3 =$$

$$\frac{3}{2}\left((p + q + r)^2 - 2(pq + qr + rp) \right)(p + q + r) - \frac{1}{2}(p + q + r)^3 + 3pqr$$

$$= \frac{3}{2}(2^2 - 2 \times 4) \times 2 - \frac{1}{2} \times 2^3 + 3 \times 5$$

$$= -12 - 4 + 15$$

$$= -16 + 120$$

$$= -1$$

[2 marks]

4.1 Root Relations II

As with quadratics, with cubics it may be that the equation of a second cubic is sought that is related to the first through a transformation of the roots. Two methods of tackling such questions will be presented. In the first it is the roots that are worked on, in the second the equation is transformed directly.

4.2 Example

The roots of the cubic equation $3x^3 + x^2 + 2x + 6 = 0$ are α, β and γ.

Without solving the equation, find an all integer coefficient polynomial with roots

$$3\alpha, \; 3\beta \; \text{ and } \; 3\gamma$$

Teaching Video : http://www.NumberWonder.co.uk/v9093/4a.mp4 (Method 1)
http://www.NumberWonder.co.uk/v9093/4b.mp4 (Method 2)

<=Method 1 Method 2 =>

After watching the teaching videos write out your own version of the solution.

☞

[6 marks]

Both methods are useful. For some questions either method will work, for others only one of the methods will. If one method stalls, try the other.

4.3 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable.
Make the method used clear.

Marks available : 40

Question 1

The roots of the cubic equation $x^3 + mx^2 + nx + 5 = 0$ are α, β and γ.

(i) Given that $\alpha + \beta + \gamma = 7$ and $\alpha\beta + \beta\gamma + \gamma\alpha = 6$ state the value of m and the value of n.

[2 marks]

(ii) Without solving the equation, find an all integer coefficient polynomial with roots 2α, 2β and 2γ.

[3 marks]

Question 2

The roots of the cubic equation $x^3 + 10x^2 + 3x - 7 = 0$ are α, β and γ.
Without solving the equation, find an all integer coefficient polynomial with roots

$$\alpha - 1, \ \beta - 1 \ \text{ and } \ \gamma - 1$$

[5 marks]

Question 3

Further A-Level Examination Question from May 2018, Core 1, Q2 (Edexcel)

The cubic equation $z^3 - 3z^2 + z + 5 = 0$ has roots α, β and γ

Without solving the equation, find the cubic equation whose roots are $(2\alpha + 1)$, $(2\beta + 1)$ and $(2\gamma + 1)$ giving your answer in the form $w^3 + pw^2 + qw + r$ where p, q and r are integers to be found.

[5 marks]

Question 4

The cubic equation $x^3 + 5x^2 - x + 3 = 0$ has roots α, β and γ

Without solving the equation, find the cubic equation whose roots are α^2, β^2 and γ^2 giving your answer in the form $w^3 + pw^2 + qw + r$ where p, q and r are integers.

[6 marks]

Question 5

The cubic equation $x^3 + 4x^2 - 5x - 7 = 0$ has roots α, β and γ.

Without solving the equation, find the cubic equation whose roots are,

$$\frac{1}{\alpha + 1}, \qquad \frac{1}{\beta + 1} \quad \text{and} \quad \frac{1}{\gamma + 1}$$

giving your answer in the form $w^3 + pw^2 + qw + r$ where p, q and r are integers.

[6 marks]

Question 6

A quintic polynomial has equation,

$$x^5 + x + 2 = 0$$

and its roots are $\alpha, \beta, \gamma, \delta, \sigma$

Without solving the equation, find the quintic with roots,

$$\alpha + 2,\ \beta + 2,\ \gamma + 2,\ \delta + 2 \text{ and } \sigma + 2$$

[5 marks]

Question 7

A cubic polynomial with roots α, β and γ has the equation,

$$x^3 + 2x^2 - 3x - 5 = 0$$

Without solving the equation, find an equation that has roots

$$\alpha + \beta,\ \beta + \gamma \text{ and } \gamma + \alpha$$

[8 marks]

4.4 Answers

4.4.1 Solution 4.2 Example (Teaching Video)

METHOD 1

$$3x^3 + x^2 + 2x + 6 = 0$$

Divide through by 3

$$x^3 + \frac{1}{3}x^2 + \frac{2}{3}x + 2 = 0$$

$$\alpha + \beta + \gamma = -\frac{1}{3} \qquad \alpha\beta + \beta\gamma + \gamma\alpha = \frac{2}{3} \qquad \alpha\beta\gamma = -2$$

Sum of new roots : $3\alpha + 3\beta + 3\gamma = 3(\alpha + \beta + \gamma)$

$$= 3 \times -\frac{1}{3}$$

$$= -1$$

Sum of product pairs of new roots :

$$3\alpha \times 3\beta + 3\beta \times 3\gamma + 3\gamma \times 3\alpha = 9(\alpha\beta + \beta\gamma + \gamma\alpha)$$

$$= 9 \times \frac{2}{3}$$

$$= 6$$

Sum of product triple of new roots :

$$3\alpha \times 3\beta \times 3\gamma = 27\alpha\beta\gamma$$

$$= 27 \times -2$$

$$= -54$$

The polynomial with roots 3α, 3β and 3γ is thus, $w^3 + w^2 + 6w + 54 = 0$

METHOD 2

$$w = 3x \Rightarrow x = \frac{w}{3}$$

$$3\left(\frac{w}{3}\right)^3 + \left(\frac{w}{3}\right)^2 + 2\left(\frac{w}{3}\right) + 6 = 0$$

$$\frac{w^3}{9} + \frac{w^2}{9} + \frac{2w}{3} + 6 = 0$$

Multiply through by 9

$w^3 + w^2 + 6w + 54 = 0$ as before with Method 1

[6 marks]

4.4.2 Solutions to 4.3 Exercise

Answer 1

(i) $x^3 - 7x^2 + 6x + 5 = 0 \Rightarrow m = -7, \; n = 6$

[2 marks]

(ii) **METHOD 1**

$$x^3 - 7x^2 + 6x + 5 = 0$$

$$\alpha + \beta + \gamma = 7 \qquad \alpha\beta + \beta\gamma + \gamma\alpha = 6 \qquad \alpha\beta\gamma = -5$$

Sum of new roots : $2\alpha + 2\beta + 2\gamma = 2(\alpha + \beta + \gamma)$

$$= 2 \times 7$$

$$= 14$$

Sum of product pairs of new roots :

$$2\alpha \times 2\beta + 2\beta \times 2\gamma + 2\gamma \times 2\alpha = 4(\alpha\beta + \beta\gamma + \gamma\alpha)$$

$$= 4 \times 6$$

$$= 24$$

Sum of product triple of new roots :

$$2\alpha \times 2\beta \times 2\gamma = 8\alpha\beta\gamma$$

$$= 8 \times -5$$

$$= -40$$

The polynomial with roots 2α, 2β and 2γ is thus, $w^3 - 14w^2 + 24w + 40 = 0$

METHOD 2

$$w = 2x \Rightarrow x = \frac{w}{2}$$

$$\left(\frac{w}{2}\right)^3 - 7\left(\frac{w}{2}\right)^2 + 6\left(\frac{w}{2}\right) + 5 = 0$$

$$\frac{w^3}{8} - \frac{7w^2}{4} + 3w + 5 = 0$$

Multiply through by 8

$w^3 - 14w^2 + 24w + 40 = 0$ as before with Method 1

[3 marks]

Answer 2

METHOD 1

$$x^3 + 10x^2 + 3x - 7 = 0$$

$$\alpha + \beta + \gamma = -10 \qquad \alpha\beta + \beta\gamma + \gamma\alpha = 3 \qquad \alpha\beta\gamma = 7$$

Sum of new roots : $\alpha - 1 + \beta - 1 + \gamma - 1 = \alpha + \beta + \gamma - 3$

$$= -10 - 3$$

$$= -13$$

Sum of product pairs of new roots :

$$(\alpha - 1)(\beta - 1) + (\beta - 1)(\gamma - 1) + (\gamma - 1)(\alpha - 1)$$

$$= \alpha\beta - \alpha - \beta + 1 + \beta\gamma - \beta - \gamma + 1 + \gamma\alpha - \alpha - \gamma + 1$$

$$= (\alpha\beta + \beta\gamma + \gamma\alpha) - 2(\alpha + \beta + \gamma) + 3$$

$$= 3 - 2 \times (-10) + 3$$

$$= 3 + 20 + 3$$

$$= 26$$

Sum of product triple of new roots :

$$(\alpha - 1)(\beta - 1)(\gamma - 1)$$

$$= (\alpha - 1)(\beta\gamma - \beta - \gamma + 1)$$

$$= \alpha\beta\gamma - \alpha\beta - \alpha\gamma + \alpha - \beta\gamma + \beta + \gamma - 1$$

$$= \alpha\beta\gamma + (\alpha + \beta + \gamma) - (\alpha\beta + \beta\gamma + \gamma\alpha) - 1$$

$$= 7 - 10 - 3 - 1$$

$$= -7$$

The polynomial sought is thus, $w^3 + 13w^2 + 26w + 7 = 0$

METHOD 2

$$w = x - 1 \Rightarrow x = w + 1$$

$$(w + 1)^3 + 10(w + 1)^2 + 3(w + 1) - 7 = 0$$

$$w^3 + 3w^2 + 3w + 1 + 10w^2 + 20w + 10 + 3w + 3 - 7 = 0$$

$$w^3 + 13w^2 + 26w + 7 = 0$$

As before with Method 1

[5 marks]

Answer 3

METHOD 1

$$z^3 - 3z^2 + z + 5 = 0$$

$$\alpha + \beta + \gamma = 3 \qquad \alpha\beta + \beta\gamma + \gamma\alpha = 1 \qquad \alpha\beta\gamma = -5$$

Sum of new roots : $2\alpha + 1 + 2\beta + 1 + 2\gamma + 1$

$$= 2(\alpha + \beta + \gamma) + 3$$

$$= 2 \times 3 + 3$$

$$= 9$$

Sum of product pairs of new roots :

$$(2\alpha + 1)(2\beta + 1) + (2\beta + 1)(2\gamma + 1) + (2\gamma + 1)(2\alpha + 1)$$

$$= 4\alpha\beta + 2\alpha + 2\beta + 1 + 4\beta\gamma + 2\beta + 2\gamma + 1 + 4\gamma\alpha + 2\alpha + 2\gamma + 1$$

$$= 4(\alpha\beta + \beta\gamma + \gamma\alpha) + 4(\alpha + \beta + \gamma) + 3$$

$$= 4 \times 1 + 4 \times 3 + 3$$

$$= 19$$

Sum of product triple of new roots :

$$(2\alpha + 1)(2\beta + 1)(2\gamma + 1)$$

$$= (2\alpha + 1)(4\beta\gamma + 2\beta + 2\gamma + 1)$$

$$= 8\alpha\beta\gamma + 4\alpha\beta + 4\alpha\gamma + 2\alpha + 4\beta\gamma + 2\beta + 2\gamma + 1$$

$$= 8\alpha\beta\gamma + 2(\alpha + \beta + \gamma) + 4(\alpha\beta + \beta\gamma + \gamma\alpha) + 1$$

$$= 8 \times (-5) + 2 \times 3 + 4 \times 1 + 1$$

$$= -29$$

The polynomial sought is thus, $w^3 - 9w^2 + 19w + 29 = 0$

METHOD 2

$$w = 2x + 1 \Rightarrow x = \frac{w - 1}{2}$$

$$\left(\frac{w - 1}{2}\right)^3 - 3\left(\frac{w - 1}{2}\right)^2 + \left(\frac{w - 1}{2}\right) + 5 = 0$$

Multiply through by 8

$$(w - 1)^3 - 6(w - 1)^2 + 4(w - 1) + 40 = 0$$

$$w^3 - 3w^2 + 3w - 1 - 6w^2 + 12w - 6 + 4w - 4 + 40 = 0$$

$$w^3 - 9w^2 + 19w + 29 = 0$$

As before with Method 1

[5 marks]

Answer 4

A question in which Method 2 does not work !

METHOD 1

$$x^3 + 5x^2 - x + 3 = 0$$

$$\alpha + \beta + \gamma = -5 \qquad \alpha\beta + \beta\gamma + \gamma\alpha = -1 \qquad \alpha\beta\gamma = -3$$

Sum of new roots :

$$(\alpha + \beta + \gamma)^2 = \alpha^2 + \beta^2 + \gamma^2 + 2(\alpha\beta + \beta\gamma + \gamma\alpha)$$

$$\therefore\ \alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$$

$$= (-5)^2 - 2 \times (-1)$$

$$= 27$$

Sum of product pairs of new roots :

$$(\alpha\beta + \beta\gamma + \gamma\alpha)^2 = \alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 + 2\alpha\beta^2\gamma + 2\alpha^2\beta\gamma + 2\alpha\beta\gamma^2$$

$$= \alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 + 2\alpha\beta\gamma(\alpha + \beta + \gamma)$$

$$\therefore\ \alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 = (\alpha\beta + \beta\gamma + \gamma\alpha)^2 - 2\alpha\beta\gamma(\alpha + \beta + \gamma)$$

$$= (-1)^2 - 2 \times (-3) \times (-5)$$

$$= -29$$

Sum of product triple of new roots :

$$\alpha^2\beta^2\gamma^2 = (\alpha\beta\gamma)^2$$

$$= (-3)^2$$

$$= 9$$

The polynomial sought is thus, $w^3 - 27w^2 - 29w - 9 = 0$

[6 marks]

Answer 5

METHOD 2

$$w = \frac{1}{x + 1} \Rightarrow x + 1 = \frac{1}{w} \Rightarrow x = \frac{1 - w}{w}$$

$$\left(\frac{1 - w}{w}\right)^3 + 4\left(\frac{1 - w}{w}\right)^2 - 5\left(\frac{1 - w}{w}\right) - 7 = 0$$

Multiply through by w^3

$$(1 - w)^3 + 4w(1 - w)^2 - 5w^2(1 - w) - 7w^3 = 0$$

$$1 - 3w + 3w^2 + w^3 + 4w - 8w^2 + 4w^3 - 5w^2 + 5w^3 - 7w^3 = 0$$

$$w^3 - 10w^2 + w + 1 = 0$$

[5 marks]

Answer 6

$$w = x + 2 \Rightarrow x = w - 2$$

$$x^5 + x + 2 = 0$$

$$(w - 2)^5 + \left(w - 2\right) + 2 = 0$$

$$\begin{aligned} &1 \times w^5 \times (-2)^0 \\ &+ 5 \times w^4 \times (-2)^1 \\ &+ 10 \times w^3 \times (-2)^2 \\ &+ 10 \times w^2 \times (-2)^3 \\ &+ 5 \times w^1 \times (-2)^4 \\ &+ 1 \times w^0 \times (-2)^5 + w - 2 + 2 = 0 \end{aligned}$$

$$w^5 - 10\,w^4 + 40\,w^3 - 80\,w^2 + 81w - 32 = 0$$

[5 marks]

Answer 7

METHOD 1

$$x^3 + 2x^2 - 3x - 5 = 0$$

$$\alpha + \beta + \gamma = -2 \qquad \alpha\beta + \beta\gamma + \gamma\alpha = -3 \qquad \alpha\beta\gamma = 5$$

Sum of new roots :

$$\begin{aligned} \alpha + \beta + \beta + \gamma + \gamma + \alpha &= 2(\alpha + \beta + \gamma) \\ &= 2 \times (-2) \\ &= -4 \end{aligned}$$

Sum of product pairs of new roots :

$$(\alpha + \beta)(\beta + \gamma) + (\beta + \gamma)\ (\gamma + \alpha) + (\gamma + \alpha)(\alpha + \beta)$$

$$= \alpha\beta + \gamma a + \beta^2 + \beta\gamma + \beta\gamma + \alpha\beta + \gamma^2 + \gamma\alpha + \gamma\alpha + \beta\gamma + \alpha^2 + \alpha\beta$$

$$= 3(\alpha\beta + \beta\gamma + \gamma\alpha) + \alpha^2 + \beta^2 + \gamma^2$$

$$= 3(\alpha\beta + \beta\gamma + \gamma\alpha) + (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$$

Because...

$$(\alpha + \beta + \gamma)^2 = \alpha^2 + \beta^2 + \gamma^2 + 2(\alpha\beta + \beta\gamma + \gamma\alpha)$$

$$\therefore\ \alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$$

So we have that...

$$(\alpha + \beta)(\beta + \gamma) + (\beta + \gamma)\ (\gamma + \alpha) + (\gamma + \alpha)(\alpha + \beta)$$

$$= (\alpha\beta + \beta\gamma + \gamma\alpha) + (\alpha + \beta + \gamma)^2$$

$$= (-3) + (-2)^2$$

$$= 1$$

Sum of product triple of new roots :

$$\begin{aligned} (\alpha + \beta)(\beta + \gamma)(\gamma + \alpha) &= (-2 - \gamma)(-2 - \alpha)(-2 - \beta) \\ &= (-1)(2 + \gamma)(2 + \alpha)(2 + \beta) \\ &= (-1)(2 + \gamma)(4 + 2\beta + 2\alpha + \alpha\beta) \\ &= (-1)(8 + 4\beta + 4\alpha + 2\alpha\beta + 4\gamma + 2\beta\gamma + 2\alpha\gamma + \alpha\beta\gamma) \\ &= (-1)(8 + 4(\alpha + \beta + \gamma) + 2(\alpha\beta + \beta\gamma + \gamma\alpha) + \alpha\beta\gamma) \\ &= (-1)(8 + 4(-2) + 2(-3) + 5)\ ) \\ &= 1 \end{aligned}$$

The polynomial sought is thus, $w^3 + 4w^2 + w - 1 = 0$

METHOD 2

$$x^3 + 2x^2 - 3x - 5 = 0$$

$$\alpha + \beta + \gamma = -2 \qquad \alpha\beta + \beta\gamma + \gamma\alpha = -3 \qquad \alpha\beta\gamma = 5$$

The new roots are : $\alpha + \beta,\ \ \beta + \gamma$ and $\gamma + \alpha$

which can be written as : $(-2 - \gamma), (-2 - \alpha)$ and $(-2 - \beta)$

$$w = -2 - x \Rightarrow x = -2 - w \Rightarrow x = (-1)(w + 2)$$

$$(-1)^3(w + 2)^3 + 2(-1)^2(w + 2)^2 - 3(-1)(w + 2) - 5 = 0$$

$$(-1)\left(w^3 + 6w^2 + 12w + 8\right) + 2w^2 + 8w + 8 + 3w + 6 - 5 = 0$$

multiply through by (-1)

$$w^3 + 6w^2 + 12w + 8 - 2w^2 - 8w - 8 - 3w - 6 + 5 = 0$$

$$w^3 + 4w^2 + w - 1 = 0$$

[8 marks]

5.1 Root Algebra Galore

In questions about the roots of quadratic and cubic equations, the same pieces of algebra repeatedly occur and most mathematicians end up with their own personal catalogue of algebraic relationships that they have encountered and found useful. Here is presented a beginners catalogue, based on the work of this topic. In examinations, you may use these algebraic relationships without proof.

Reciprocals

Quadratic : $$\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta}$$

Cubic : $$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\alpha\beta + \beta\gamma + \gamma\alpha}{\alpha\beta\gamma}$$

Quartic : $$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta} = \frac{\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta}{\alpha\beta\gamma\delta}$$

Sums of Squares

Quadratic : $$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$$

Cubic : $$\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$$

Quartic : $$\alpha^2 + \beta^2 + \gamma^2 + \delta^2$$
$$= (\alpha + \beta + \gamma + \delta)^2 - 2(\alpha(\beta + \gamma + \delta) + \beta(\gamma + \delta) + \gamma(\delta))$$

Sums of Cubes

Quadratic : $$\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$$

Cubic : $$\alpha^3 + \beta^3 + \gamma^3$$
$$= (\alpha + \beta + \gamma)^3 - 3(\alpha + \beta + \gamma)(\alpha\beta + \beta\gamma + \gamma\alpha) + 3\alpha\beta\gamma$$

Sum of Quartics

Quadratic : $$\alpha^4 + \beta^4 = (\alpha^2 + \beta^2)^2 - 2(\alpha\beta)^2$$

5.2 The Roots of a Quartic

The Roots of a Quartic

If α, β, γ and δ are roots of

$$a x^4 + b x^3 + c x^2 + dx + e = 0 \qquad a, b, c, d, e \in \mathbb{C}$$

then, $\alpha + \beta + \gamma + \delta = -\dfrac{b}{a} \qquad \alpha(\beta + \gamma + \delta) + \beta(\gamma + \delta) + \gamma(\delta) = \dfrac{c}{a}$

$$\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta = -\frac{d}{a} \qquad \alpha\beta\gamma\delta = \frac{e}{a}$$

5.3 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable. Make the method used clear.

Marks available : 40

Question 1

α, β, γ and δ are the roots of the quartic, $x^4 + 3x^3 + 2x^2 - x + 4 = 0$

Without solving the equation, find the values of,

(i) $\alpha + \beta + \gamma + \delta$ **(ii)** $\alpha(\beta + \gamma + \delta) + \beta(\gamma + \delta) + \gamma(\delta)$

[1, 1 mark]

(iii) $\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta$ **(iv)** $\alpha\beta\gamma\delta$

[1, 1 mark]

(v) $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma} + \dfrac{1}{\delta}$

[2 marks]

(vi) $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$

[2 marks]

Question 2

The roots of a quartic equation $ax^4 + bx^3 + cx^2 + dx + e = 0$ are,

$$\alpha = -3, \ \beta = -1, \ \gamma = 2 \text{ and } \delta = 5$$

Find integer values for a, b, c, d and e

[4 marks]

Question 3

The quartic equation $2x^4 - 34x^3 + 202x^2 + dx + e = 0$ has roots,

$$\alpha, \quad \alpha + 1, \quad 2\alpha + 1 \quad \text{and} \quad 3\alpha + 1$$

(i) Find the value of α

[2 marks]

(ii) Find the values of d and e

[4 marks]

Question 4

(a) Prove the "Sum of Quartics" formula for a quadratic equation with roots α and β, that,

$$\alpha^4 + \beta^4 = \left(\alpha^2 + \beta^2 - \sqrt{2}\,\alpha\beta\right)\left(\alpha^2 + \beta^2 + \sqrt{2}\,\alpha\beta\right)$$

[2 marks]

(b) For the quadratic equation $7x^2 + 13x - 8 = 0$ with roots α and β, find, without solving the equation, the value of,

(i) $\alpha^2 + \beta^2$

[2 marks]

(ii) $\alpha^4 + \beta^4$

[4 marks]

Question 5

Further A-Level Examination Question from January 2015, Q5 (Edexcel)

The quadratic equation $4x^2 + 3x + 1 = 0$ has roots α and β.

(a) Write down the value of $(\alpha + \beta)$ and the value of $\alpha\beta$

[2 marks]

(b) Find the value of $\left(\alpha^2 + \beta^2\right)$

[2 marks]

(c) Find a quadratic equation with roots,

$$\left(4\alpha - \beta\right) \quad \text{and} \quad \left(4\beta - \alpha\right)$$

giving your answer in the form $px^2 + qx + r = 0$ where p, q and r are integers to be determined.

[4 marks]

Question 6

The cubic equation $x^3 - 14x^2 + 56x - 64 = 0$ has roots α, $k\alpha$ and $k^2\alpha$ for some real constant k.

Find the two possible values of α and the corresponding two values of k.

[6 marks]

5.4 Solutions to 5.3 Exercise

Answer 1

(i) $\alpha + \beta + \gamma + \delta = -3$

(ii) $\alpha(\beta + \gamma + \delta) + \beta(\gamma + \delta) + \gamma(\delta) = 2$

(iii) $\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta = 1$

(iv) $\alpha\beta\gamma\delta = 4$

(v) $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma} + \dfrac{1}{\delta} = \dfrac{\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta}{\alpha\beta\gamma\delta} = \dfrac{1}{4}$

(vi) $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$

$$= (\alpha + \beta + \gamma + \delta)^2 - 2(\alpha(\beta + \gamma + \delta) + \beta(\gamma + \delta) + \gamma(\delta))$$
$$= (-3)^2 - 2(2)$$
$$= 5$$

[1, 1, 1, 1, 2, 2 marks]

Answer 2

$$a x^4 + b x^3 + c x^2 + dx + e = 0$$

Divide through by a

$$x^4 + \frac{b}{a}x^3 + \frac{c}{a}x^2 + \frac{d}{a}x + \frac{e}{a} = 0$$

$$\alpha + \beta + \gamma + \delta = (-3) + (-1) + 2 + 5$$
$$= 3$$

$$\alpha(\beta + \gamma + \delta) + \beta(\gamma + \delta) + \gamma(\delta)$$
$$= (-3)(-1 + 2 + 5) + (-1)(2 + 5) + (2)(5)$$
$$= -18 - 7 + 10$$
$$= -15$$

$$\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta$$
$$= (-3)(-1)(2) + (-1)(2)(5) + (2)(5)(-3) + (5)(-3)(-1)$$
$$= 6 - 10 - 30 + 15$$
$$= -19$$

$$\alpha\beta\gamma\delta = (-3)(-1)(2)(5)$$
$$= 30$$

$$x^4 - 3x^3 - 15x^2 + 19x + 30 = 0$$

[4 marks]

Answer 3

(i) $$2x^4 - 34x^3 + 202x^2 + dx + e = 0$$

Divide through by 2

$$x^4 - 17x^3 + 101x^2 + \frac{d}{2}x + \frac{e}{2} = 0$$

$$\text{Sum of roots} = 17$$

$$\therefore \alpha + \alpha + 1 + 2\alpha + 1 + 3\alpha + 1 = 17$$

$$7\alpha + 3 = 17$$

$$7\alpha = 14$$

$$\alpha = 2$$

So the roots are 2, 3, 5 and 7

(ii) $$\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta = -\frac{d}{2}$$

$$\therefore d = -2(2 \times 3 \times 5 + 3 \times 5 \times 7 + 5 \times 7 \times 2 + 7 \times 2 \times 3)$$

$$= -2(30 + 105 + 70 + 42)$$

$$= -2 \times 247$$

$$= -494$$

$$\alpha\beta\gamma\delta = \frac{e}{2}$$

$$e = 2 \times 2 \times 3 \times 5 \times 7$$

$$= 420$$

Answer 4

(a)

$$\begin{aligned} \text{LHS} &= \alpha^4 + \beta^4 \\ &= \left(\alpha^2\right)^2 + \left(\beta^2\right)^2 \\ &= \left(\alpha^2 + \beta^2\right)^2 - 2\,\alpha^2\beta^2 \\ &= \left(\alpha^2 + \beta^2\right)^2 - 2\left(\alpha\beta\ \right)^2 \\ &= \text{RHS} \qquad\qquad \square \end{aligned}$$

[2 marks]

(b) **(i)**

$$\begin{aligned} \alpha^2 + \beta^2 &= (\alpha + \beta)^2 - 2\alpha\beta \\ &= \left(-\frac{13}{7}\right)^2 - 2 \times \left(-\frac{8}{7}\right) \\ &= \frac{169}{49} + \frac{16}{7} \times \frac{7}{7} \\ &= \frac{281}{49} \end{aligned}$$

[2 marks]

(ii)

$$\begin{aligned} \alpha^4 + \beta^4 &= \left(\alpha^2 + \beta^2\right)^2 - 2(\alpha\beta)^2 \\ &= \left(\frac{281}{49}\right)^2 - 2\left(-\frac{8}{7}\right)^2 \\ &= \frac{78961}{2401} - \frac{128}{49} \times \frac{49}{49} \\ &= \frac{72689}{2401} \end{aligned}$$

[4 marks]

Answer 5

(a)

$$4x^2 + 3x + 1 = 0$$

Divide through by 4

$$x^2 + \frac{3}{4}x + \frac{1}{4} = 0$$

$$\therefore \; \left(\alpha + \beta\right) = -\frac{3}{4} \quad \text{and} \quad \alpha\beta = \frac{1}{4}$$

[2 marks]

(b)

$$\left(\alpha^2 + \beta^2\right) = (\alpha + \beta)^2 - 2\alpha\beta$$

$$= \left(-\frac{3}{4}\right) - 2 \times \frac{1}{4}$$

$$= \frac{1}{16} \quad \text{or} \quad 0.0625$$

[2 marks]

METHOD 1

(c) Sum of new roots,

$$4\alpha - \beta + 4\beta - \alpha = 3(\alpha + \beta)$$

$$= 3 \times \left(-\frac{3}{4}\right)$$

$$= -\frac{9}{4}$$

Product of new roots,

$$\left(4\alpha - \beta\right)\left(4\beta - \alpha\right) = \left(16\alpha\beta - 4\alpha^2 - 4\beta^2 + \alpha\beta\right)$$

$$= 17\alpha\beta - 4\left(\alpha^2 + \beta^2\right)$$

$$= 17 \times \left(\frac{1}{4}\right) - 4 \times \left(\frac{1}{16}\right)$$

$$= 4$$

The new quadratic with roots $\left(4\alpha - \beta\right)$ and $\left(4\beta - \alpha\right)$ is,

$$w^2 + \frac{9}{4}w + 4 = 0$$

$$4w^2 + 9w + 16 = 0$$

[4 marks]

METHOD 2

(c) From $\left(\alpha + \beta\right) = -\dfrac{3}{4} \quad \Rightarrow \quad \alpha = -\dfrac{3}{4} - \beta$ and $\beta = -\dfrac{3}{4} - \alpha$

So the new roots can be expressed like this;

$$4\alpha - \beta = 4\alpha + \frac{3}{4} + \alpha \quad \text{and} \quad 4\beta - \alpha = 4\beta + \frac{3}{4} + \beta$$

$$= 5\alpha + \frac{3}{4} \qquad\qquad = 5\beta + \frac{3}{4}$$

The transformation sought is thus,

$$w = 5x + \frac{3}{4}$$

$$w = \frac{20x + 3}{4} \Leftrightarrow x = \frac{4w - 3}{20}$$

$$4x^2 + 3x + 1 = 0$$

$$4\left(\frac{4w - 3}{20}\right)^2 + 3\left(\frac{4w - 3}{20}\right) + 1 = 0$$

Multiply through by 100

$$(4w - 3)^2 + 15(4w - 3) + 100 = 0$$

$$16w^2 - 24w + 9 + 60w - 45 + 100 = 0$$

$$16w^2 + 36w + 64 = 0$$

Divide through by 4

$$4w^2 + 9w + 16 = 0$$

[4 marks]

Answer 6

$$\text{Product of the roots}: \ \alpha \times k\alpha \times k^2\alpha = 64$$

$$(k\alpha)^3 = 64$$

$$k\alpha = 4$$

$$k = \frac{4}{\alpha}$$

$$\text{Sum of the roots}: \ \alpha + k\alpha + k^2\alpha = 14$$

$$\alpha + \frac{4}{\alpha} \times \alpha + \left(\frac{4}{\alpha}\right)^2 \times \alpha = 14$$

$$\alpha + 4 + \frac{16}{\alpha} = 14$$

Multiply through by α

$$\alpha^2 - 10\alpha + 16 = 0$$

$$(\alpha - 2)(\alpha - 8) = 0$$

Either $\alpha = 2$ or $\alpha = 8$

When $\alpha = 2$, $k = 2$ and when $\alpha = 8$, $k = \frac{1}{2}$

[6 marks]

6.1 Test Preparation

Gerolamo Cardano's first step in solving the general cubic equation was to make a substitution that turned it into a depressed cubic equation.

Depressed Cubic Formation Rule

Faced with a general cubic,

$$a\,x^3 + b\,x^2 + cx + d = 0 \qquad\qquad a,\ b,\ c,\ d \in \mathbb{C}$$

initiate a change of variable by replacing x with $t - \dfrac{b}{3\,a}$

This will always result in what is termed a depressed cubic, one of the form,

$$t^3 + pt + q = 0 \qquad\qquad p,\ q \in \mathbb{C}$$

The depressed cubic equation has some useful properties one of which shall now be studied, as it has a beautiful proof.

6.2 Sum of Cubes for a Depressed Cubic

Sum of Cubes for a Depressed Cubic

For a depressed cubic.

$$t^3 + pt + q = 0 \qquad\qquad p,\ q \in \mathbb{C}$$

with roots α, β and γ

$$\alpha^3 + \beta^3 + \gamma^3 = 3\alpha\beta\gamma$$

Proof

As there is no term in t^2, $\alpha + \beta + \gamma = 0$

$$\therefore\ \alpha + \beta = -\gamma$$

Cube both sides

$$(\,\alpha + \beta\,)^3 = -\gamma^3$$

$$\alpha^3 + 3\,\alpha^2\,\beta + 3\alpha\,\beta^2 + \beta^3 = -\gamma^3$$

$$\alpha^3 + \beta^3 + \gamma^3 = -3\alpha\beta(\,\alpha + \beta\,)$$

$$\alpha^3 + \beta^3 + \gamma^3 = -3\alpha\beta(-\gamma\,)$$

$$\alpha^3 + \beta^3 + \gamma^3 = 3\alpha\beta\gamma \qquad \square$$

6.3 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable.
Make the method used clear.
Marks available : 50

Question 1

The roots of the quadratic equation $a x^2 + bx + c = 0$ are $\dfrac{-3 \pm 4\mathrm{i}}{2}$

Find integer values for a, b and c

[3 marks]

Question 2

The roots of the equation $a x^4 + 7x^3 + 5x^2 + 3x - 4 = 0$ are α, β, γ and δ

(a) Given that $\alpha\beta\gamma\delta = -1$, write down the value of a

[1 mark]

(b) $\sum \alpha\beta$ is (sloppy) shorthand for “sum of the product pairs of roots”.
For a quartic $\sum \alpha\beta = \alpha(\beta + \gamma + \delta) + \beta(\gamma + \delta) + \gamma(\delta)$
$= \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta$

Write down the values of $\sum \alpha$, $\sum \alpha\beta$ and $\sum \alpha\beta\gamma$

[3 marks]

(c) Hence find the value of $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$

[3 marks]

Question 3

Further A-Level Examination Question from May 2018, IAL, F1, Q7 (Edexcel)

It is given that α and β are roots of the equation $5x^2 - 4x + 3 = 0$

Without solving the equation,

(a) find the exact value of $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$

[5 marks]

(b) find a quadratic equation which has roots $\frac{3}{\alpha^2}$ and $\frac{3}{\beta^2}$ giving your answer in the form $ax^2 + bx + c = 0$, where a, b and c are integers.

[4 marks]

Question 4

(i) For the general cubic, $a\,x^3 + b\,x^2 + cx + d = 0$, write down from memory the formula for $\alpha^3 + \beta^3 + \gamma^3$

[2 marks]

(ii) Show that your part (i) formula yields the same expression as proven in section "6.2 Sum of Cubes for a Depressed Cubic" when $b = 0$ in the general cubic equation.

[2 marks]

(iii) Consider the cubic equation,

$$x^3 + 3\,x^2 + 50 = 0$$

Using a suitable substitution, transform this into a depressed cubic.

[2 marks]

(iv) By thinking of $\alpha^3\beta^3 + \beta^3\gamma^3 + \gamma^3\alpha^3$ as $P^3 + Q^3 + R^3$ and using the same formula as in part (i) derive a formula for $\alpha^3\beta^3 + \beta^3\gamma^3 + \gamma^3\alpha^3$ with component parts that are either $\alpha + \beta + \gamma$ or $\alpha\beta + \beta\gamma + \gamma\alpha$ or $\alpha\beta\gamma$.

[4 marks]

(v) Let α, β and γ be the roots of your part (iii) depressed cubic. Without solving the equation find a cubic with roots, α^3, β^3 and γ^3 in the form $w^3 + p\,w^2 + qw + r$ where p, q and r are integers.

[6 marks]

Question 5

$$f(x) = mx^3 + 12x^2 + 4x + 16 \qquad \text{where } m \text{ is a real constant}$$

Given that,

- f has at least one root on the imaginary axis

Solve completely,

$$f(x) = 0$$

[7 marks]

Question 6

Prove that the roots of $x^3 + px + qx + r = 0$ form an Arithmetic Progression if and only if $2p^3 + 27r = 9pq$

[8 marks]

Answer 1

$$a x^2 + bx + c = 0$$

Divide through by a

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

$$\text{Sum of the roots} = -\frac{b}{a}$$

$$-\frac{b}{a} = \frac{-3 + 4\text{i}}{2} + \frac{-3 - 4\text{i}}{2}$$

$$= -3$$

$$\text{Product of the roots} = \frac{c}{a}$$

$$\frac{c}{a} = \left(\frac{-3 + 4\text{i}}{2}\right)\left(\frac{-3 - 4\text{i}}{2}\right)$$

$$= \frac{9 + 12\text{i} - 12\text{i} - 16\text{i}^2}{4}$$

$$= \frac{25}{4}$$

$$\text{Quadratic : } x^2 + 3x + \frac{25}{4} = 0$$

Multiply through by 4

$$4x^2 + 12x + 25 = 0$$

[3 marks]

Answer 2

(a)

$$a x^4 + 7x^3 + 5x^2 + 3x - 4 = 0$$

Divide through by a

$$x^4 + \frac{7}{a}x^3 + \frac{5}{a}x^2 + \frac{3}{a}x - \frac{4}{a} = 0$$

$$-\frac{4}{a} = \alpha\beta\gamma\delta$$

$$-\frac{4}{a} = -1$$

$$\therefore a = 4$$

[1 mark]

(b)

$$x^4 + \frac{7}{4}x^3 + \frac{5}{4}x^2 + \frac{3}{4}x - 1 = 0$$

$$\sum\alpha = \alpha + \beta + \gamma + \delta = -\frac{7}{4}$$

$$\sum\alpha\beta = \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \frac{5}{4}$$

$$\sum\alpha\beta\gamma = \alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta = -\frac{3}{4}$$

[3 marks]

(c)

$$\begin{aligned} &\alpha^2 + \beta^2 + \gamma^2 + \delta^2 \\ &= (\alpha + \beta + \gamma + \delta)^2 - 2\big(\alpha(\beta + \gamma + \delta) + \beta(\gamma + \delta) + \gamma(\delta)\big) \\ &= \left(\frac{7}{4}\right)^2 - 2\left(\frac{5}{4}\right) \\ &= \frac{49}{16} - \frac{10}{4} \\ &= \frac{9}{16} \end{aligned}$$

[3 marks]

Answer 3

(a)

$$5x^2 - 4x + 3 = 0$$

Divide through by 5

$$x^2 - \frac{4}{5}x + \frac{3}{5} = 0$$

$$\alpha + \beta = \frac{4}{5} \text{ and } \alpha\beta = \frac{3}{5}$$

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$$

$$= \left(\frac{4}{5}\right)^2 - 2 \times \frac{3}{5}$$

$$= \frac{16 - 30}{25}$$

$$= -\frac{14}{25}$$

$$\frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{\beta^2 + \alpha^2}{\alpha^2\beta^2}$$

$$= \frac{(-14) \times 5^2}{25 \times 3^2}$$

$$= -\frac{14}{9}$$

[5 marks]

(b) Sum of new roots : $\frac{3}{\alpha^2} + \frac{3}{\beta^2} = 3\left(\frac{1}{\alpha^2} + \frac{1}{\beta^2}\right)$

$$= 3\left(-\frac{14}{9}\right)$$

$$= -\frac{14}{3}$$

Product of new roots : $\left(\frac{3}{\alpha^2}\right)\left(\frac{3}{\beta^2}\right) = \frac{9}{(\alpha\beta)^2}$

$$= \frac{9 \times 5^2}{3^2}$$

$$= 25$$

The required polynomial is thus,

$$x^2 + \frac{14}{3}x + 25 = 0$$

$$3x^2 + 14x + 75 = 0$$

[4 marks]

Answer 4

(i)

$$\alpha^3 + \beta^3 + \gamma^3 = (\alpha + \beta + \gamma)^3 - 3(\alpha + \beta + \gamma)(\alpha\beta + \beta\gamma + \gamma\alpha) + 3\alpha\beta\gamma$$

[2 marks]

(ii) For a depressed cubic, there is no term in x^2, so $b = 0$ and therefore $\alpha + \beta + \gamma = 0$, in which case,

$$\begin{aligned}\alpha^3 + \beta^3 + \gamma^3 &= (0)^3 - 3(0)(\alpha\beta + \beta\gamma + \gamma\alpha) + 3\alpha\beta\gamma \\ &= 3\alpha\beta\gamma\end{aligned}$$

which is the same as section “6.2 Sum of Cubes for a depressed cubic” ☐

[2 marks]

(iii) The suitable transformation is,

$$x = t - \frac{b}{3a} \Rightarrow x = t - 1$$

$$\begin{aligned}x^3 + 3x^2 + 50 &= 0 \\ (t-1)^3 + 3(t-1)^2 + 50 &= 0 \\ t^3 - 3t^2 + 3t - 1 + 3t^2 - 6t + 3 + 50 &= 0 \\ t^3 - 3t + 52 &= 0\end{aligned}$$

[2 marks]

(iv)

$$\begin{aligned}&P^3 + Q^3 + R^3 \\ &= (P + Q + R)^3 - 3(P + Q + R)(PQ + QR + RP) + 3PQR \\ &= (\alpha\beta + \beta\gamma + \gamma\alpha)^3 - 3(\alpha\beta + \beta\gamma + \gamma\alpha)(\alpha\beta\beta\gamma + \beta\gamma\gamma\alpha + \gamma\alpha\alpha\beta) + 3\alpha\beta\beta\gamma\gamma\alpha \\ &= (\alpha\beta + \beta\gamma + \gamma\alpha)^3 - 3\alpha\beta\gamma(\alpha\beta + \beta\gamma + \gamma\alpha)(\beta + \gamma + \alpha) + 3(\alpha\beta\gamma)^2 \\ &= (\alpha\beta + \beta\gamma + \gamma\alpha)^3 - 3\alpha\beta\gamma(\alpha\beta + \beta\gamma + \gamma\alpha)(\alpha + \beta + \gamma) + 3(\alpha\beta\gamma)^2\end{aligned}$$

[4 marks]

(v) For the depressed cubic with roots α, β and γ,

$$t^3 - 3t + 52 = 0$$

$$\alpha + \beta + \gamma = 0, \quad \alpha\beta + \beta\gamma + \gamma\alpha = -3 \text{ and } \alpha\beta\gamma = -52$$

Sum of new roots : $\alpha^3 + \beta^3 + \gamma^3 = 3\alpha\beta\gamma$ as $\alpha + \beta + \gamma = 0$

$$= 3 \times (-52)$$

$$= -156$$

Sum of product pairs of new roots, from part (iv) :

$$\alpha^3\beta^3 + \beta^3\gamma^3 + \gamma^3\alpha^3$$

$$= (\alpha\beta + \beta\gamma + \gamma\alpha)^3 - 3\alpha\beta\gamma(\alpha\beta + \beta\gamma + \gamma\alpha)(\alpha + \beta + \gamma) + 3(\alpha\beta\gamma)^2$$

but for a depressed cubic $\alpha + \beta + \gamma = 0$ so this can be simplified to,

$$\Rightarrow \alpha^3\beta^3 + \beta^3\gamma^3 + \gamma^3\alpha^3 = (\alpha\beta + \beta\gamma + \gamma\alpha)^3 + 3(\alpha\beta\gamma)^2$$

$$= (-3)^3 + 3 \times (-52)^2$$

$$= -27 + 8112$$

$$= 8085$$

Sum of product triple of new roots : $\alpha^3\beta^3\gamma^3 = (\alpha\beta\gamma)^3$

$$= (-52)^3$$

$$= -140608$$

The cubic equation with roots α^3, β^3 and γ^3 is,

$$w^3 + 156w^2 + 8085w + 140608 = 0$$

[6 marks]

Answer 5

$$f(x) = mx^3 + 12x^2 + 4x + 16$$

Let f have roots α, β and γ in which case,

$$\alpha + \beta + \gamma = -\frac{12}{m} \qquad \alpha\beta + \beta\gamma + \gamma\alpha = \frac{4}{m} \text{ and } \alpha\beta\gamma = -\frac{16}{m}$$

From

- f has at least one root on the imaginary axis

Let α be that root, $\alpha = k\mathrm{i}$, and so $f(k\mathrm{i}) = 0$

$$\begin{aligned} f(k\mathrm{i}) &= 0 \\ m(k\mathrm{i})^3 + 12(k\mathrm{i})^2 + 4(k\mathrm{i}) + 16 &= 0 \\ -mk^3\mathrm{i} - 12k^2 + 4k\mathrm{i} + 16 &= 0 \end{aligned}$$

Equating real parts : $-12k^2 + 16 = 0$

$$\Rightarrow k^2 = \frac{16}{12} = \frac{4}{3}$$

Equating imaginary parts : $-mk^3 + 4k = 0$

$$\begin{aligned} mk^3 &= 4k \\ m &= \frac{4}{k^2} \\ &= \frac{4 \times 3}{4} \\ &= 3 \end{aligned}$$

$$3x^3 + 12x^2 + 4x + 16 = 0$$

As $k^2 = \dfrac{4}{3}$ and a root was $\alpha = k\mathrm{i}$, $\alpha = \pm\dfrac{2}{\sqrt{3}}\mathrm{i} = \pm\dfrac{2\sqrt{3}}{3}\mathrm{i}$

$f(x)$ had all real coefficients and so has a conjugate pair or roots

$$\alpha = \frac{2\sqrt{3}}{3}\mathrm{i} \qquad \beta = -\frac{2\sqrt{3}}{3}\mathrm{i}$$

$$\begin{aligned} \alpha\beta\gamma &= -\frac{16}{m} \\ \frac{2\sqrt{3}}{3}\mathrm{i} \times \left(-\frac{2\sqrt{3}}{3}\mathrm{i}\right) \times \gamma_g &= -\frac{16}{3} \\ -\frac{4 \times 3 \times (-1)}{9} \times \gamma_g &= -\frac{16}{3} \\ \gamma_g &= -4 \end{aligned}$$

[7 marks]

Answer 6

Clever Method

Depressed Cubic Formation Rule
Faced with a general cubic,

$$a\,x^3 + b\,x^2 + cx + d = 0$$

initiate a change of variable by replacing x with $t - \dfrac{b}{3\,a}$
This will always result in what is termed a depressed cubic, one of the form,

$$t^3 + pt + q = 0$$

Let $f\,(x) = x^3 + p\,x^2 + qx + r = 0$
Let $g\,(x)$ be the related depressed cubic,

$$g\,(x) = f\left(x - \frac{p}{3}\right) = x^3 - \frac{p^2 - 3q}{3}\,x + \frac{2\,p^3 - 9pq + 27r}{r}$$

Note that the roots of $f\,(x)$ form an Arithmetic Progression if and only if the roots of $g\,(x)$ are of the form $(-\,d)$, 0, $(+\,d)$ for some complex number d which is equivalent to $g\,(x) = (\,x + d\,)\;x\;(\,x - d\,) = x^3 - d^2\,x$. That is, the roots of $f\,(x)$ form an Arithmetic Progression if and only if the constant term of $g\,(x)$ is 0, which is the same as saying that $2\,p^3 - 9pq + 27r = 0$.

$$\therefore\; 2\,p^3 + 27r = 9pq \qquad \square$$

[8 marks]

Standard Method

$$x^3 + px + qx + r = 0$$

Without loss of generality, let the roots be

$$\alpha = m - d, \qquad \beta = m \quad \text{and} \quad \gamma = m + d \qquad \text{for } m,\ d \in \mathbb{C}$$

$$\alpha + \beta + \gamma = p$$

$$m - d + m + m + d = p$$

$$3m = p \qquad (1)$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = q$$

$$(m - d)m + m(m + d) + (m + d)(m - d) = q$$

$$m^2 - md + m^2 + md + m^2 - d^2 = q$$

$$3m^2 - d^2 = q \qquad (2)$$

$$\alpha\beta\gamma = r$$

$$(m - d)\ m\ (m + d) = r$$

$$m(m^2 - d^2) = r$$

$$m\left(m^2 - \left(3m^2 - q\right)\right) = r \qquad \text{from (2)}$$

$$m\left(-2m^2 + q\right) = r$$

$$-2m^3 + mq = r$$

$$-2\left(\frac{p}{3}\right)^3 + \left(\frac{p}{3}\right)q = r$$

Multiply through by 27

$$-2p^3 + 9pq = 27r$$

$$\therefore 2p^3 + 27r = 9pq$$

As all of the steps used are reversible the theorem is proven $\square$

[8 marks]

7.1 Test

Any solution based entirely on graphical
or numerical methods is not acceptable.
Make the method used clear.
Marks available : 40

Question 1

Consider the equation $ax^2 + bx + c = 0$ whose roots are α and β.
What would $\alpha - \beta$ be in terms of a, b and c

[4 marks]

Question 2

The cubic equation $x^3 + mx + n = 0$ has roots $1, -4$ and γ

(i) State, with a reason, whether γ is real

[1 mark]

(ii) Find the values of m, n and γ

[4 marks]

Question 3

The roots of the equation $3x^3 - px + 11 = 0$ are α, β and γ

(i) Given that $\alpha\beta + \beta\gamma + \gamma\alpha = 4$, write down the value of p

[1 mark]

(ii) Write down the values of $\alpha + \beta + \gamma$ and $\alpha\beta\gamma$

[2 marks]

(iii) Hence find the value of $(3 - \alpha)(3 - \beta)(3 - \gamma)$

[3 marks]

Question 4

It is given that α and β are roots of $3x^2 + 2x + 1$.

Without solving the equation, find the exact value of $\left(\frac{1-\alpha}{1+\alpha}\right)^3 + \left(\frac{1-\beta}{1+\beta}\right)^3$

[8 marks]

Question 5

Further A-Level Examination Question from SAM, 2018, Core 1, Q6 (Edexcel)

$$f(x) = kx^2 + 3x - 11 \qquad\qquad g(x) = mx^3 - 2x^2 + 3x - 9$$

where k and m are real constants.

Given that,

- the sum of the roots of f is equal to the product of the roots of g
- g has at least one root on the imaginary axis

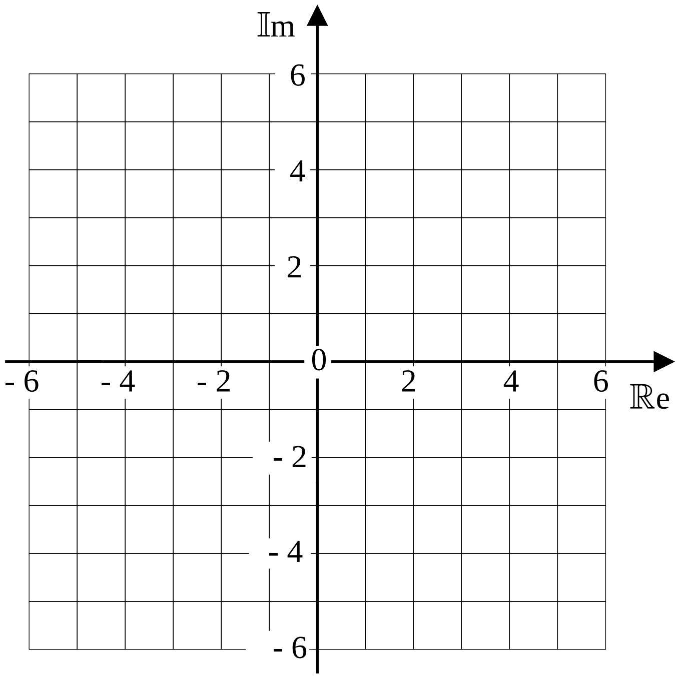
(a) Solve completely,

(i) $f(x) = 0$

(ii) $g(x) = 0$

[7 marks]

(b) Plot the roots of f and the roots of g on a single Argand diagram



[2 marks]

Question 6

If α, β, γ and δ are the roots of $x^4 + x^3 + px^2 + 4x - 2 = 0$, where p is a constant, then,

(i) find (in terms of p) an equation whose roots are $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ and $\frac{1}{\delta}$

[3 marks]

(ii) Given that $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = \frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} + \frac{1}{\delta^2}$ find the value of p.

[5 marks]

Answer 1

$$\begin{aligned}(\alpha - \beta)^2 &= \alpha^2 - 2\alpha\beta + \beta^2 \\ &= \alpha^2 + 2\alpha\beta + \beta^2 - 4\alpha\beta \\ &= (\alpha + \beta)^2 - 4\alpha\beta \\ &= \left(-\frac{b}{a}\right)^2 - 4 \times \left(\frac{c}{a}\right) \\ &= \frac{b^2 - 4ac}{a^2} \\ \therefore \alpha - \beta &= \frac{\pm\sqrt{b^2 - 4ac}}{a}\end{aligned}$$

[4 marks]

Answer 2

(i) As there is no term in x^2, $\alpha + \beta + \gamma = 0$

As α and β are real, γ must also be real to satisfy the above constraint.

[1 mark]

(ii) As there is no term in x^2, $\alpha + \beta + \gamma = 0$

$$\begin{aligned}\therefore \quad 1 + (-4) + \gamma &= 0 \\ \gamma &= 3\end{aligned}$$

Furthermore, $\alpha\beta + \beta\gamma + \gamma\alpha = m$

$$\begin{aligned}m &= 1 \times (-4) + (-4) \times 3 + 3 \times 1 \\ &= -13\end{aligned}$$

Finally, $\alpha\beta\gamma = -n$

$$\begin{aligned}n &= -(1 \times (-4) \times 3) \\ &= 12\end{aligned}$$

So, the equation was,

$$x^3 - 13x + 12 = 0$$

[4 marks]

Answer 3

(i)

$$3x^3 - px + 11 = 0$$

Divide through by 3

$$x^3 - \frac{p}{3}x + \frac{11}{3} = 0$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = -\frac{p}{3}$$

$$4 = -\frac{p}{3}$$

$$p = -12$$

[1 mark]

(ii) The cubic is thus, $x^3 + 4x + \dfrac{11}{3} = 0$

$$\alpha + \beta + \gamma = 0 \quad \alpha\beta + \beta\gamma + \gamma\alpha = 4 \quad \text{and} \quad \alpha\beta\gamma = -\frac{11}{3}$$

[2 marks]

(iii) $(3 - \alpha)(3 - \beta)(3 - \gamma)$

$$= (3 - \alpha)(9 - 3\gamma - 3\beta + \beta\gamma)$$

$$= 27 - 9\gamma - 9\beta + 3\beta\gamma - 9\alpha + 3\alpha\gamma + 3\alpha\beta - \alpha\beta\gamma$$

$$= 27 - 9(\alpha + \beta + \gamma) + 3(\alpha\beta + \beta\gamma + \gamma\alpha) - \alpha\beta\gamma$$

$$= 27 - 9(0) + 3(4) - \left(-\frac{11}{3}\right)$$

$$= \frac{128}{3}$$

[3 marks]

Answer 4
Clever Method

$$3x^2 + 2x + 1 = 0$$

$$x^2 + \frac{2}{3}x + \frac{1}{3} = 0$$

$$\alpha + \beta = -\frac{2}{3} \quad \text{and} \quad \alpha\beta = \frac{1}{3}$$

As α is a root, $3\alpha^2 + 2\alpha + 1 = 0$

$$\Rightarrow 3\alpha^2 + 3\alpha + 1 = a$$

$$1 - \alpha = -3\alpha(\alpha + 1)$$

As β is a root, $3\beta^2 + 2\beta + 1 = 0$

$$\Rightarrow 3\beta^2 + 3\beta + 1 = \beta$$

$$1 - \beta = -3\beta(\beta + 1)$$

So,

$$\left(\frac{1-\alpha}{1+\alpha}\right)^3 + \left(\frac{1-\beta}{1+\beta}\right)^3 = \left(\frac{-3\alpha(\alpha+1)}{(1+\alpha)}\right)^3 + \left(\frac{-3\beta(\beta+1)}{(1+\beta)}\right)^3$$

$$= -27\left(\alpha^3 + \beta^3\right)$$

$$= -27\left((\alpha+\beta)^3 - 3\alpha^2\beta - 3\alpha\beta^2\right)$$

$$= -27\left((\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)\right)$$

$$= -27(\alpha+\beta)^3 + 81\alpha\beta(\alpha+\beta)$$

$$= -27 \times \left(-\frac{2}{3}\right)^3 + 81 \times \left(\frac{1}{3}\right) \times \left(-\frac{2}{3}\right)$$

$$= 8 - 18$$

$$= -10$$

[8 marks]

Straight Forward Method

$$3x^2 + 2x + 1 = 0$$

$$x^2 + \frac{2}{3}x + \frac{1}{3} = 0$$

$$\alpha + \beta = -\frac{2}{3} \text{ and } \alpha\beta = \frac{1}{3}$$

Let $x = \left(\frac{1-\alpha}{1+\alpha}\right)$ and $y = \left(\frac{1-\beta}{1+\beta}\right)$ so $x^3 + y^3$ is sought

Then, $x + y = \left(\frac{1-\alpha}{1+\alpha}\right) + \left(\frac{1-\beta}{1+\beta}\right)$

$$= \frac{(1-\alpha)(1+\beta) + (1-\beta)(1+\alpha)}{(1+\alpha)\ (1+\beta)}$$

$$= \frac{2\ (1-\alpha\beta)}{1+(\alpha+\beta)+\alpha\beta}$$

$$= \frac{2\left(1-\left(\frac{1}{3}\right)\right)}{1+\left(-\frac{2}{3}\right)+\frac{1}{3}}$$

$$= 2$$

and, $xy = \left(\frac{1-\alpha}{1+\alpha}\right)\left(\frac{1-\beta}{1+\beta}\right)$

$$= \frac{1-(\alpha+\beta)+\alpha\beta}{1+(\alpha+\beta)+\alpha\beta}$$

$$= \frac{1-\left(-\frac{2}{3}\right)+\frac{1}{3}}{1+\left(-\frac{2}{3}\right)+\frac{1}{3}}$$

$$= 3$$

From the Binomial theorem, $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

deduce that, $x^3 + y^3 = (x+y)^3 - 3xy(x+y)$

$$\therefore \left(\frac{1-\alpha}{1+\alpha}\right)^3 + \left(\frac{1-\beta}{1+\beta}\right)^3 = 2^3 - 3 \times 2 \times 3$$

$$= 8 - 18$$

$$= -10$$

[8 marks]

Solution was written and posted by MHH on Math.StackExchange.com, July 9th 2020

Answer 5

$$f(x) = kx^2 + 3x - 11 \qquad\qquad g(x) = mx^3 - 2x^2 + 3x - 9$$

Let f have roots α_f and β_f in which case $\alpha_f + \beta_f = -\dfrac{3}{k}$ and $\alpha_f\beta_f = -\dfrac{11}{k}$

Let g have roots α_g, β_g and γ_g in which case,

$$\alpha_g + \beta_g + \gamma_g = \frac{2}{m} \qquad \alpha_g\beta_g + \beta_g\gamma_g + \gamma_g\alpha_g = \frac{3}{m} \text{ and } \alpha_g\beta_g\gamma_g = \frac{9}{m}$$

We are told that,

- the sum of the roots of f is equal to the product of the roots of g

$$\alpha_f + \beta_f = \alpha_g\beta_g\gamma_g$$
$$-\frac{3}{k} = \frac{9}{m}$$
$$\frac{k}{3} = -\frac{m}{9}$$
$$k = -\frac{m}{3}$$

- g has at least one root on the imaginary axis

Let α_g be that root, $\alpha_g = K\mathrm{i}$, and so $g(K\mathrm{i}) = 0$

$$g(K\mathrm{i}) = 0$$
$$m(K\mathrm{i})^3 - 2(K\mathrm{i})^2 + 3(K\mathrm{i}) - 9 = 0$$
$$-mK^3\mathrm{i} + 2K^2 + 3K\mathrm{i} - 9 = 0$$
$$\text{Equating real parts : } 2K^2 - 9 = 0 \Rightarrow K^2 = \frac{9}{2}$$
$$\text{Equating imaginary parts : } -mK^3 + 3K = 0$$
$$mK^3 = 3K$$
$$m = \frac{3}{K^2}$$
$$= \frac{3 \times 2}{9}$$
$$= \frac{2}{3} \qquad \therefore\ k = -\frac{2}{9}$$

(a) **(i)**

$$f(x) = 0$$
$$kx^2 + 3x - 11 = 0$$
$$-\frac{2}{9}x^2 + 3x - 11 = 0$$
$$2x^2 - 27x + 99 = 0$$
$$x = \frac{27 \pm \sqrt{27^2 - 4 \times 2 \times 99}}{2 \times 2}$$
$$= \frac{27 \pm \sqrt{-63}}{4}$$
$$= \frac{27 \pm 3\sqrt{7}\,\mathrm{i}}{4}$$

(ii)

$$g(x) = 0$$
$$mx^3 - 2x^2 + 3x - 9 = 0$$
$$\frac{2}{3}x^3 - 2x^2 + 3x - 9 = 0$$
$$2x^3 - 6x^2 + 9x - 27 = 0$$

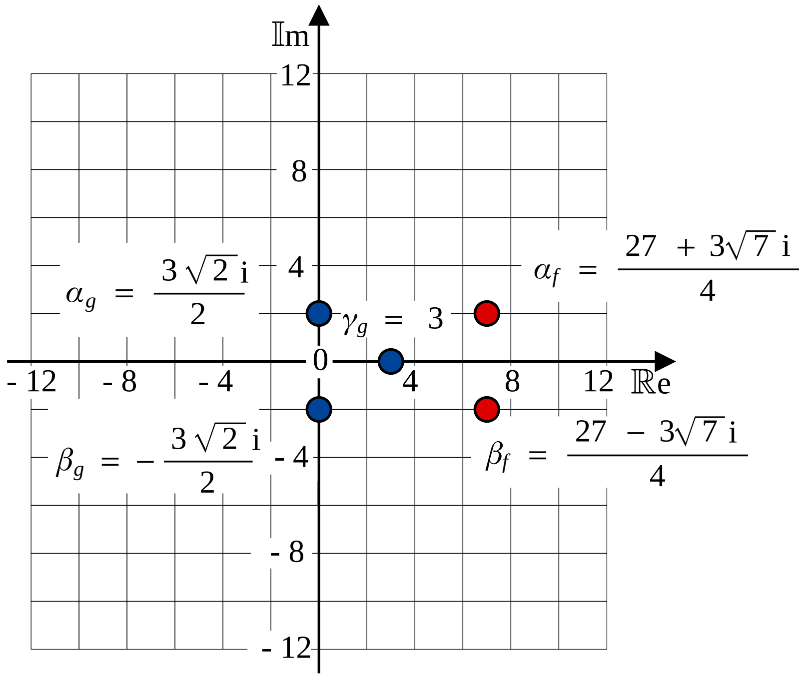
As $K^2 = \dfrac{9}{2}$ and a root was $\alpha_g = K\mathrm{i}$, $\alpha_g = \pm\dfrac{3}{\sqrt{2}}\mathrm{i} = \pm\dfrac{3\sqrt{2}}{2}\mathrm{i}$

$g(x)$ had all real coefficients and so has a conjugate pair or roots

$$\alpha_g = \frac{3\sqrt{2}}{2}\mathrm{i} \qquad \beta_g = -\frac{3\sqrt{2}}{2}\mathrm{i}$$
$$\alpha_g\beta_g\gamma_g = \frac{9}{m}$$
$$\frac{3\sqrt{2}}{2}\mathrm{i} \times \left(-\frac{3\sqrt{2}}{2}\mathrm{i} \right) \times \gamma_g = \frac{9 \times 3}{2}$$
$$-\frac{9 \times 2 \times (-1)}{4} \times \gamma_g = \frac{27}{2}$$
$$\gamma_g = 3$$

[7 marks]

(b)



[2 marks]

Answer 6

(i) Using the substitution method,

$$w = \frac{1}{x} \Rightarrow x = \frac{1}{w}$$

$$x^4 + x^3 + p\,x^2 + 4x - 2 = 0$$

$$\left(\frac{1}{w}\right)^4 + \left(\frac{1}{w}\right)^3 + p\left(\frac{1}{w}\right)^2 + 4\left(\frac{1}{w}\right) - 2 = 0$$

Multiply through by w^4

$$1 + w + p\,w^2 + 4\,w^3 - 2\,w^4 = 0$$

$$2\,w^4 - 4\,w^3 - p\,w^2 - w - 1 = 0$$

$$w^4 - 2\,w^3 - \frac{p}{2}\,w^2 - \frac{1}{2}\,w - \frac{1}{2} = 0$$

[3 marks]

(ii) For the original quartic,

$$\alpha + \beta + \gamma + \delta = -1 \qquad \alpha(\beta + \gamma + \delta) + \beta(\gamma + \delta) + \gamma(\delta) = p$$

$$\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta = -4 \qquad \alpha\beta\gamma\delta = 2$$

Standard result,

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2$$

$$= (\alpha + \beta + \gamma + \delta)^2 - 2(\alpha(\beta + \gamma + \delta) + \beta(\gamma + \delta) + \gamma(\delta))$$

$$\therefore \alpha^2 + \beta^2 + \gamma^2 + \delta^2 = (-1)^2 - 2p$$

$$= 1 - 2p$$

For the transformed quartic,

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta} = 2 \qquad \frac{1}{\alpha}\left(\frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta}\right) + \frac{1}{\beta}\left(\frac{1}{\gamma} + \frac{1}{\delta}\right) + \frac{1}{\gamma}\left(\frac{1}{\delta}\right) = -\frac{p}{2}$$

$$\frac{1}{\alpha\beta\gamma} + \frac{1}{\beta\gamma\delta} + \frac{1}{\gamma\delta\alpha} + \frac{1}{\delta\alpha\beta} = \frac{1}{2} \qquad \frac{1}{\alpha\beta\gamma\delta} = -\frac{1}{2}$$

So the standard result will become,

$$\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} + \frac{1}{\delta^2}$$

$$= \left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta}\right)^2 - 2\left(\frac{1}{\alpha}\left(\frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta}\right) + \frac{1}{\beta}\left(\frac{1}{\gamma} + \frac{1}{\delta}\right) + \frac{1}{\gamma}\left(\frac{1}{\delta}\right)\right)$$

$$\therefore \frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} + \frac{1}{\delta^2} = 2^2 - 2\left(-\frac{p}{2}\right)$$

$$= 4 + p$$

As $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = \frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} + \frac{1}{\delta^2}$

$$1 - 2p = 4 + p$$

$$3p = -3$$

$$p = -1$$

[5 marks]

8.1 Extension Material

In order to solve a general cubic equation, Cardano first reduced it to depressed cubic form,

$$t^3 + pt + q = 0$$

This provides a motivation to look at the algebra of the roots of this equation which has the special property of $\alpha + \beta + \gamma = 0$

The general result that,

$$\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$$

becomes,

$$\alpha^2 + \beta^2 + \gamma^2 = -2(\alpha\beta + \beta\gamma + \gamma\alpha)$$

and another general result that,

$$\alpha^3 + \beta^3 + \gamma^3$$
$$= (\alpha + \beta + \gamma)^3 - 3(\alpha + \beta + \gamma)(\alpha\beta + \beta\gamma + \gamma\alpha) + 3\alpha\beta\gamma$$

becomes

$$\alpha^3 + \beta^3 + \gamma^3 = 3\alpha\beta\gamma$$

In this extension lesson a further couple of formulae are derived and then a question given in which applying them may be practiced.

A Cubic's Sum of Squares of Product Pairs of Roots

If α, β and γ are roots of $a x^3 + b x^2 + cx + d = 0$ $\qquad a, b, c, d \in \mathbb{C}$

then, $\quad \alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 = (\alpha\beta + \beta\gamma + \gamma\alpha)^2 - 2\alpha\beta\gamma(\alpha + \beta + \gamma)$

For a depressed cubic,

$$\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 = (\alpha\beta + \beta\gamma + \gamma\alpha)^2$$

Proof

$$P^2 + Q^2 + R^2 = (P + Q + R)^2 - 2(PQ + QR + RP)$$

$$(\alpha\beta)^2 + (\beta\gamma)^2 + (\gamma\alpha)^2 = (\alpha\beta + \beta\gamma + \gamma\alpha)^2 - 2(\alpha\beta^2\gamma + \beta\gamma^2\alpha + \gamma\alpha^2\beta)$$

$$= (\alpha\beta + \beta\gamma + \gamma\alpha)^2 - 2\alpha\beta\gamma(\alpha + \beta + \gamma) \qquad \square$$

For a depressed cubic, $\alpha + \beta + \gamma = 0$

$$\therefore \quad \alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 = (\alpha\beta + \beta\gamma + \gamma\alpha)^2 \qquad \square$$

A Cubic's Sum of Cubes of Product Pairs of Roots

If α, β and γ are roots of $a x^3 + b x^2 + cx + d = 0$ $\qquad a,\ b,\ c,\ d \in \mathbb{C}$

then,

$$\alpha^3 \beta^3 + \beta^3 \gamma^3 + \gamma^3 \alpha^3$$
$$= (\alpha\beta + \beta\gamma + \gamma\alpha)^3 - 3\alpha\beta\gamma(\alpha\beta + \beta\gamma + \gamma\alpha)(\alpha + \beta + \gamma) + 3(\alpha\beta\gamma)^2$$

For a depressed cubic,

$$\alpha^3 \beta^3 + \beta^3 \gamma^3 + \gamma^3 \alpha^3 = (\alpha\beta + \beta\gamma + \gamma\alpha)^3 + 3(\alpha\beta\gamma)^3$$

Proof

$$P^3 + Q^3 + R^3 = (P + Q + R)^3 - 3(P + Q + R)(PQ + QR + RP) + 3PQR$$

$$(\alpha\beta)^3 + (\beta\gamma)^3 + (\gamma\alpha)^3 = (\alpha\beta + \beta\gamma + \gamma\alpha)^3$$
$$- 3(\alpha\beta + \beta\gamma + \gamma\alpha)(\alpha \beta^2 \gamma + \beta \gamma^2 \alpha + \gamma \alpha^2 \beta) + 3(\alpha\beta)(\beta\gamma)(\gamma\alpha)$$

$$\alpha^3 \beta^3 + \beta^3 \gamma^3 + \gamma^3 \alpha^3 = (\alpha\beta + \beta\gamma + \gamma\alpha)^3$$
$$- 3\alpha\beta\gamma(\alpha\beta + \beta\gamma + \gamma\alpha)(\alpha + \beta + \gamma) + 3(\alpha\beta\gamma)^2 \qquad \square$$

For a depressed cubic, $\alpha + \beta + \gamma = 0$

$$\therefore\ \alpha^3 \beta^3 + \beta^3 \gamma^3 + \gamma^3 \alpha^3 = (\alpha\beta + \beta\gamma + \gamma\alpha)^3 + 3(\alpha\beta\gamma)^2 \qquad \square$$

A Cubic's Sum of Quartics of Roots

If α, β and γ are roots of $a x^3 + b x^2 + cx + d = 0 \qquad a, \ b, \ c, \ d \in \mathbb{C}$

then, $\alpha^4 + \beta^4 + \gamma^4$

$$= \Big((\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha) \Big)^2$$
$$- 2 \Big((\alpha\beta + \beta\gamma + \gamma\alpha)^2 - 2\alpha\beta\gamma(\alpha + \beta + \gamma) \Big)$$

For a depressed cubic,

$$\alpha^4 + \beta^4 + \gamma^4 = 2(\alpha\beta + \beta\gamma + \gamma\alpha)^2$$

Proof

$$P^2 + Q^2 + R^2 = (P + Q + R)^2 - 2(PQ + QR + RP)$$

$$(\alpha^2)^2 + (\beta^2)^2 + (\gamma^2)^2 = (\alpha^2 + \beta^2 + \gamma^2)^2 - 2(\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2)$$

$$\alpha^4 + \beta^4 + \gamma^4$$
$$= \Big((\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha) \Big)^2 - 2\left(\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2\right)$$
$$= \Big((\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha) \Big)^2$$
$$- 2 \Big((\alpha\beta + \beta\gamma + \gamma\alpha)^2 - 2\alpha\beta\gamma(\alpha + \beta + \gamma) \Big) \qquad \square$$

For a depressed cubic, $\alpha + \beta + \gamma = 0$

$$\therefore \ \alpha^4 + \beta^4 + \gamma^4 = 2(\alpha\beta + \beta\gamma + \gamma\alpha)^2 \qquad \square$$

8.2 Exercise

Question 1

If α, β and γ are the roots of $x^3 + x + 1 = 0$, then, without solving the equation, find the equation whose roots are $(\alpha - \beta)^2$, $(\beta - \gamma)^2$ and $(\gamma - \alpha)^2$

[8 marks]

8.3 Answers to 8.2 Exercise

Answer 1
METHOD 1

$$x^3 + x + 1 = 0$$

$$\alpha + \beta + \gamma = 0 \quad \alpha\beta + \beta\gamma + \gamma\alpha = 1 \quad \alpha\beta\gamma = -1$$

Standard result :

$$\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$$
$$= 0^2 - 2 \times 1$$
$$= -2$$

Standard result :

$$\alpha^3 + \beta^3 + \gamma^3 = (\alpha + \beta + \gamma)^3 - 3(\alpha + \beta + \gamma)(\alpha\beta + \beta\gamma + \gamma\alpha) + 3\alpha\beta\gamma$$
$$= 0^3 - 3 \times 0 \times 1 + 3 \times (-1)$$
$$= -3$$

Standard result for a depressed cubic: (proven above)

$$\alpha^4 + \beta^4 + \gamma^4 = 2(\alpha\beta + \beta\gamma + \gamma\alpha)^2$$
$$= 2 \times 1^2$$
$$= 2$$

Standard result : (proven above)

$$\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 = (\alpha\beta + \beta\gamma + \gamma\alpha)^2 - 2\alpha\beta\gamma(\alpha + \beta + \gamma)$$
$$= 1^2 - 2 \times (-1) \times 0$$
$$= 1$$

Standard result : (proven above)

$$\alpha^3\beta^3 + \beta^3\gamma^3 + \gamma^3\alpha^3$$
$$= (\alpha\beta + \beta\gamma + \gamma\alpha)^3 - 3\alpha\beta\gamma(\alpha\beta + \beta\gamma + \gamma\alpha)(\alpha + \beta + \gamma) + 3(\alpha\beta\gamma)^2$$
$$= 1^3 - 3 \times (-1) \times 1 \times 0 + 3 \times (-1)^2$$
$$= 4$$

Also as α is a root, $\alpha^3 + \alpha + 1 = 0$ likewise for β and γ.

Finally, as $\alpha + \beta + \gamma = 0$

$$\alpha + \beta = -\gamma$$
$$(\alpha + \beta)^2 = \gamma^2$$
$$\alpha^2 + \beta^2 = \gamma^2 - 2\alpha\beta$$

Similarly, $\beta^2 + \gamma^2 = \alpha^2 - 2\beta\gamma$ and $\gamma^2 + \alpha^2 = \beta^2 - 2\gamma\alpha$

Sum of the new roots :

$$(\alpha - \beta)^2 + (\beta - \gamma)^2 + (\gamma - \alpha)^2$$

$$= \alpha^2 - 2\alpha\beta + \beta^2 + \beta^2 - 2\beta\gamma + \gamma^2 + \gamma^2 - 2\gamma\alpha + \alpha^2$$

$$= 2\left(\alpha^2 + \beta^2 + \gamma^2\right) - 2\left(\alpha\beta + \beta\gamma + \gamma\alpha\right)$$

$$= 2 \times (-2) - 2 \times 1$$

$$= -6$$

Sum of product pairs :

$$(\alpha - \beta)^2(\beta - \gamma)^2 + (\beta - \gamma)^2(\gamma - \alpha)^2 + (\gamma - \alpha)^2(\alpha - \beta)^2$$

$$= \left(\alpha^2 - 2\alpha\beta + \beta^2\right)\left(\beta^2 - 2\beta\gamma + \gamma^2\right)$$
$$+ \left(\beta^2 - 2\beta\gamma + \gamma^2\right)\left(\gamma^2 - 2\gamma\alpha + \alpha^2\right)$$
$$+ \left(\gamma^2 - 2\gamma\alpha + \alpha^2\right)\left(\alpha^2 - 2\alpha\beta + \beta^2\right)$$

$$= \alpha^2\beta^2 - 2\alpha^2\beta\gamma + \alpha^2\gamma^2 - 2\alpha\beta^3 + 4\alpha\beta^2\gamma - 2\alpha\beta\gamma^2 + \beta^4 - 2\beta^3\gamma + \beta^2\gamma^2$$
$$+ \beta^2\gamma^2 - 2\beta^2\gamma\alpha + \beta^2\alpha^2 - 2\beta\gamma^3 + 4\beta\gamma^2\alpha - 2\beta\gamma\alpha^2 + \gamma^4 - 2\gamma^3\alpha + \gamma^2\alpha^2$$
$$+ \gamma^2\alpha^2 - 2\gamma^2\alpha\beta + \gamma^2\beta^2 - 2\gamma\alpha^3 + 4\gamma\alpha^2\beta - 2\gamma\alpha\beta^2 + \alpha^4 - 2\alpha^3\beta + \alpha^2\beta^2$$

$$= \alpha^4 + \beta^4 + \gamma^4 + 3\left(\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2\right)$$
$$- 2\left(\alpha\beta^3 + \beta\gamma^3 + \gamma\alpha^3 + \beta^3\gamma + \gamma^3\alpha + \alpha^3\beta\right)$$

$$= \alpha^4 + \beta^4 + \gamma^4 + 3\left(\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2\right)$$
$$- 2\left(\alpha^3(\beta + \gamma) + \beta^3(\alpha + \gamma) + \gamma^3(\alpha + \beta)\right)$$

$$= \alpha^4 + \beta^4 + \gamma^4 + 3\left(\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2\right)$$
$$- 2\left(\alpha^3(-\alpha) + \beta^3(-\beta) + \gamma^3(-\gamma)\right)$$

$$= 3\left(\alpha^4 + \beta^4 + \gamma^4 + \alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2\right)$$

$$= 3 \times \left(2 + 1\right)$$

$$= 9$$

Sum of product triples :

$$
\begin{aligned}
(\alpha - \beta)^2(\beta - \gamma)^2(\gamma - \alpha)^2 \\
&= \left(\alpha^2 + \beta^2 - 2\alpha\beta\right)\left(\beta^2 + \gamma^2 - 2\beta\gamma\right)\left(\gamma^2 + \alpha^2 - 2\gamma\alpha\right) \\
&= \left(\gamma^2 - 4\alpha\beta\right)\left(\alpha^2 - 4\beta\gamma\right)\left(\beta^2 - 4\gamma\alpha\right) \\
&= \left(\gamma^2 - 4\alpha\beta\right)\left(\alpha^2\beta^2 - 4\gamma\alpha^3 - 4\beta^3\gamma + 16\alpha\beta\gamma^2\right) \\
&= \alpha^2\beta^2\gamma^2 - 4\gamma^3\alpha^3 - 4\beta^3\gamma^3 \\
&\qquad + 16\alpha\beta\gamma^4 - 4\alpha^3\beta^3 + 16\alpha^4\beta\gamma + 16\alpha\beta^4\gamma - 64\alpha^2\beta^2\gamma^2 \\
&= 16\alpha\beta\gamma\left(\alpha^3 + \beta^3 + \gamma^3\right) - 4\left(\alpha^3\beta^3 + \beta^3\gamma^3 + \gamma^3\alpha^3\right) - 63(\alpha\beta\gamma)^2 \\
&= 16 \times (-1) \times (-3) - 4 \times 4 - 63 \times (-1)^2 \\
&= 48 - 16 - 63 \\
&= -31
\end{aligned}
$$

Hence the equation is,

$$w^3 + 6w^2 + 9w + 31 = 0$$

[8 marks]

METHOD 2

Trying to find a transformation from x to w,

$$\begin{aligned}
\text{let } w &= (\alpha - \beta)^2 \\
&= \alpha^2 + \beta^2 - 2\alpha\beta \\
&= \gamma^2 - 4\alpha\beta \quad \text{but } \alpha\beta\gamma = -1 \ \therefore\ \alpha\beta = -\frac{1}{\gamma} \\
&= \gamma^2 + \frac{4}{\gamma} \\
\gamma w &= \gamma^3 + 4 \\
&\qquad \text{but } \gamma^3 + \gamma + 1 = 0 \Rightarrow \gamma^3 = -\gamma - 1 \\
\gamma w &= -\gamma - 1 + 4 \\
\gamma w + \gamma &= 3 \\
\gamma(w + 1) &= 3 \\
\gamma &= \frac{3}{w + 1} \qquad \text{That is, } x = \frac{3}{w+1}
\end{aligned}$$

$$x^3 + x + 1 = 0$$

$$\left(\frac{3}{w+1}\right)^3 + \left(\frac{3}{w+1}\right) + 1 = 0$$

Multiply through by $(w + 1)^3$

$$3^3 + 3(w + 1)^2 + (w + 1)^3 = 0$$

$$27 + 3w^2 + 6w + 3 + w^3 + 3w^2 + 3w + 1 = 0$$

$$w^3 + 6w^2 + 9w + 31 = 0$$

[8 marks]