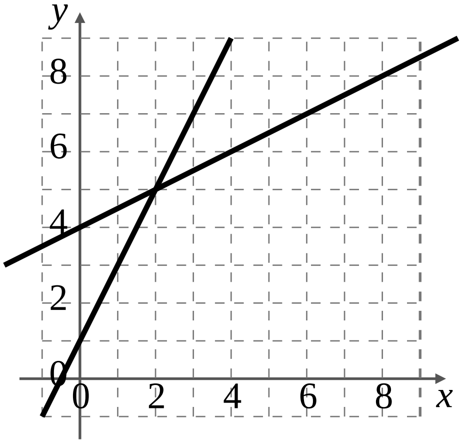
1.6 Answers

1.6.1 Solution to 1.2 The GCSE Method

(i)



[1 mark]

(ii) Point of intersection is (2, 5)

[1 mark]

(iii)

$$2x + 1 = \frac{1}{2}x + 4$$

multiply through by 2

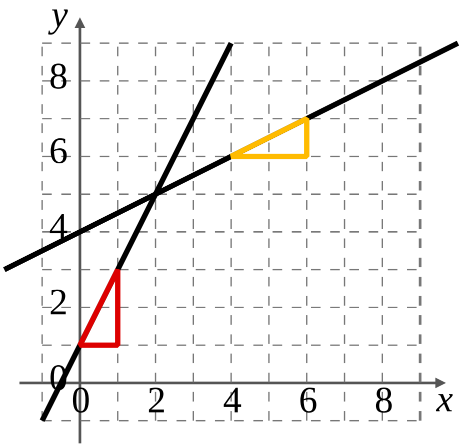
$$4x + 2 = x + 8$$

$$3x = 6$$

$$x = 2 \Rightarrow y = 5 \qquad \therefore \text{Intersection} \,(2,\ 5)$$

[2 marks]

(iv) For θ the angle of intersection,



$$\theta = arctan\left(\frac{2}{1}\right) - arctan\left(\frac{1}{2}\right)$$

$$= 63.4 - 26.6$$

$$= 36.9°$$

[2 marks]

1.6.2 Solution to 1.3 The Vector Method

$$\boldsymbol{r_1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \text{and} \quad \boldsymbol{r_2} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

From the upper row : $\lambda = 2\mu$

From the lower row : $1 + 2\lambda = 4 + \mu$

$$1 + 4\mu = 4 + \mu$$

$$3\mu = 3$$

$$\mu = 1 \Rightarrow \lambda = 2$$

Point of intersection : $\boldsymbol{r_1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$= \begin{pmatrix} 0 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 5 \end{pmatrix} \quad \text{i.e. } (2, 5)$$

Using Scalar Product : $\begin{pmatrix} 1 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$$1 \times 2 + 2 \times 1 = \sqrt{1^2 + 2^2} \times \sqrt{2^2 + 1^2} \times \cos\theta$$

$$4 = \sqrt{5}\,\sqrt{5}\,\cos\theta$$

$$\cos\theta = \frac{4}{5}$$

$$= 36.9°$$

[6 marks]

1.6.3 Solutions to 1.5 Exercise

Answer 1

(i) The direction part of $\boldsymbol{r_1} = \begin{pmatrix} -4 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 2 \end{pmatrix}$ is the $\begin{pmatrix} 5 \\ 2 \end{pmatrix}$

[1 mark]

(ii)

$$\begin{pmatrix} 5 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 7 \end{pmatrix}$$

$$\begin{aligned}
5 \times 3 + 2 \times 7 &= \sqrt{5^2 + 2^2} \times \sqrt{3^2 + 7^2} \times \cos\theta \\
15 + 14 &= \sqrt{29} \times \sqrt{58} \times \cos\theta \\
29 &= \sqrt{29} \times \sqrt{2} \times \sqrt{29} \times \cos\theta \\
29 &= 29\sqrt{2}\cos\theta \\
\cos\theta &= \frac{1}{\sqrt{2}} \qquad \square
\end{aligned}$$

[3 marks]

(iii) $\therefore \quad \theta = 45°$

[1 mark]

Answer 2

(i) The direction part of $\boldsymbol{r_1} = \begin{pmatrix} -5 \\ 8 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 3 \end{pmatrix}$ is the $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$

[1 mark]

(ii)

$$\begin{pmatrix} 4 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

$$\begin{aligned}
4 \times (-1) + 3 \times 3 &= \sqrt{4^2 + 3^2} \times \sqrt{(-1)^2 + 3^2} \times \cos\theta \\
-4 + 9 &= \sqrt{25} \times \sqrt{10} \times \cos\theta \\
5 &= 5\sqrt{10}\cos\theta \\
\cos\theta &= \frac{1}{\sqrt{10}} \qquad \square
\end{aligned}$$

[3 marks]

(iii) $\therefore \quad \theta = 71.6°$

[1 mark]

Answer 3

$$\begin{pmatrix} -4 \\ -11 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -8 \\ 11 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

Upper row : $-4 + \lambda = -8 + 2\mu \Rightarrow \lambda - 2\mu = -4$ (1)

Lower row : $-11 + 2\lambda = 11 - \mu \Rightarrow 2\lambda + \mu = 22$ (2)

Solve (1) and (2) simultaneously to get $\mu = 6$ and $\lambda = 8$

To get the point of intersection using $\lambda = 8$,

$$\begin{aligned} \boldsymbol{r_1} &= \begin{pmatrix} -4 \\ -11 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} -4 \\ -11 \end{pmatrix} + 8 \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} 4 \\ 5 \end{pmatrix} \qquad \therefore \text{ Intersection at } (4, 5) \end{aligned}$$

[4 marks]

Answer 4

$$\begin{pmatrix} -15 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 15 \\ 17 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

Upper row : $-15 + 3\lambda = 15 + 3\mu \Rightarrow \lambda - \mu = 10$ (1)

Lower row : $-4 + \lambda = 17 + 2\mu \Rightarrow \lambda - 2\mu = 21$ (2)

Solve (1) and (2) simultaneously to get $\mu = -11$ and $\lambda = -1$

To get the point of intersection using $\lambda = -1$,

$$\begin{aligned} \boldsymbol{r_1} &= \begin{pmatrix} -15 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} -15 \\ -4 \end{pmatrix} - 1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} -18 \\ -5 \end{pmatrix} \qquad \therefore \text{ Intersection at } (-18, -5) \end{aligned}$$

[4 marks]

Answer 5

(i) The direction part of $\boldsymbol{r_1} = \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 4 \\ 12 \end{pmatrix}$ is the $\begin{pmatrix} 3 \\ 4 \\ 12 \end{pmatrix}$

[1 mark]

(ii) $\begin{pmatrix} 3 \\ 4 \\ 12 \end{pmatrix} \bullet \begin{pmatrix} -3 \\ -4 \\ 5 \end{pmatrix}$

$$3 \times (-3) + 4 \times (-4) + 12 \times 5 = \sqrt{3^2 + 4^2 + 12^2}\ \sqrt{(-3)^2 + (-4)^2 + 5^2}\ \cos\theta$$

$$35 = 13 \times 5\sqrt{2}\ \cos\theta$$

$$\cos\theta = \frac{7}{13\sqrt{2}}$$

$$= \frac{7\sqrt{2}}{26} \qquad \square$$

[3 marks]

(iii) $\therefore\ \theta = 67°$

[1 mark]

Answer 6

(i) $\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$

$$= \begin{pmatrix} 8 \\ 5 \\ 9 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 2 \end{pmatrix}$$

[1 mark]

(ii) $\overrightarrow{AC} = \boldsymbol{c} - \boldsymbol{a}$

$$= \begin{pmatrix} 3 \\ 11 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix} = \begin{pmatrix} 1 \\ 8 \\ -3 \end{pmatrix}$$

[1 mark]

(iii) $\begin{pmatrix} 6 \\ 2 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 8 \\ -3 \end{pmatrix}$

$$6 \times 1 + 2 \times 8 + 2 \times (-3) = \sqrt{6^2 + 2^2 + 2^2}\ \sqrt{1^2 + 8^2 + (-3)^2}\ \cos\theta$$

$$16 = 2\sqrt{11}\ \sqrt{74}\ \cos\theta$$

$$\cos\theta = \frac{8}{\sqrt{11}\ \sqrt{74}}$$

$$\theta = 73.7°$$

[3 marks]

Answer 7

(i) Let A be the point (1, 0, 0), and be B be (0, 1, 1)

$$\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$$

$$= \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

$$\boldsymbol{r_1} = \overrightarrow{OA} + \lambda \overrightarrow{AB}$$

$$= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

[1 mark]

(ii) Let C be the point (1, 1, 0), and be D be (0, 0, 1)

$$\overrightarrow{CD} = \boldsymbol{d} - \boldsymbol{c}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

$$\boldsymbol{r_2} = \overrightarrow{OC} + \lambda \overrightarrow{CD}$$

$$= \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

[2 mark]

(iii)

$$\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

$$(-1) \times (-1) + 1 \times (-1) + 1 \times 1 = \sqrt{(-1)^2 + 1^2 + 1^2}\ \sqrt{(-1)^2 + (-1)^2 + 1^2}\ \cos\theta$$

$$1 - 1 + 1 = \sqrt{3}\,\sqrt{3}\ \cos\theta$$

$$\cos\theta = \frac{1}{3}$$

$$\theta = 70.5°$$

[3 marks]

Answer 8

(a) $\quad \boldsymbol{r_1} = \begin{pmatrix} -6 \\ 4 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix}$ and $\boldsymbol{r_2} = \begin{pmatrix} -6 \\ 4 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix}$ intersect at $\begin{pmatrix} -6 \\ 4 \\ -1 \end{pmatrix}$

[1 mark]

(b)

$$\begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix}$$

$$4 \times 3 + (-1) \times (-4) + 3 \times 1 = \sqrt{4^2 + (-1)^2 + 3^2}\ \sqrt{3^2 + (-4)^2 + 1^2}\ \cos\theta$$

$$12 + 4 + 3 = \sqrt{26}\ \sqrt{26}\ \cos\theta$$

$$\cos\theta = \frac{19}{26}$$

[3 marks]

(c) $\quad \boldsymbol{r_1} = \begin{pmatrix} -6 \\ 4 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} -6 \\ 4 \\ -1 \end{pmatrix} + 4 \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 10 \\ 0 \\ 11 \end{pmatrix} \quad \therefore\ X\ (\ 10,\ 0,\ 11\ )$

[1 mark]

(d) $\quad \overrightarrow{AX} = \overrightarrow{AO} + \overrightarrow{OX} = \overrightarrow{OX} - \overrightarrow{OA}$

$$= \begin{pmatrix} 10 \\ 0 \\ 11 \end{pmatrix} - \begin{pmatrix} -6 \\ 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 16 \\ -4 \\ 12 \end{pmatrix}$$

[2 marks]

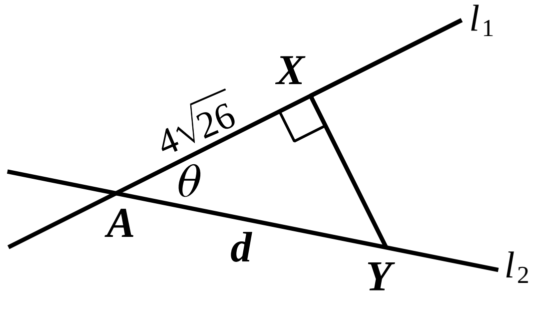
(e)

$$\left|\overrightarrow{AX}\right| = \sqrt{16^2 + (-4)^2 + 12^2}$$

$$= \sqrt{256 + 16 + 144} = \sqrt{416} = 4\sqrt{26}$$

[2 marks]

(f)



$$|AY| = \frac{4\sqrt{26}}{\left(\frac{19}{26}\right)} = 27.9$$

[3 marks]

Answer 9

$$\boldsymbol{r_1} = \begin{pmatrix} 1 \\ -1 \\ -5 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \quad \text{and} \quad \boldsymbol{r_2} = \begin{pmatrix} 16 \\ 5 \\ -2 \end{pmatrix} + \mu\begin{pmatrix} 7 \\ -6 \\ -10 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} 7 \\ -6 \\ -10 \end{pmatrix}$$

$$2 \times 7 + 3 \times (-6) + 4 \times (-10) = \sqrt{2^2 + 3^2 + 4^2}\ \sqrt{7^2 + (-6)^2 + (-10)^2}\ \cos\theta$$

$$14 - 18 - 40 = \sqrt{29}\sqrt{185}\cos\theta$$

$$\cos\theta = -\frac{44}{\sqrt{29} \times \sqrt{185}}$$

$$\therefore\ \theta = 126.9° \text{ which is an acute angle of } 53.1°$$

[4 marks]

Answer 10

(i) The direction part of $\boldsymbol{r} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ 5 \end{pmatrix}$ is the $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$

[1 mark]

(ii) If (13, 23) is on the line then there will be a value of λ for which,

$$\begin{pmatrix} 3 \\ -2 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 13 \\ 23 \end{pmatrix}$$

Upper row : $3 + 2\lambda = 13 \Rightarrow 2\lambda = 10 \Rightarrow \lambda = 5$

Lower row : $-2 + 5\lambda = 23 \Rightarrow 5\lambda = 25 \Rightarrow \lambda = 5$

$\therefore$ (13, 23) is on the line and occurs when $\lambda = 5$

[2 marks]

(iii) To cross the x-axis at p, $\begin{pmatrix} 3 \\ -2 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} p \\ 0 \end{pmatrix}$

Lower row : $-2 + 5\lambda = 0 \Rightarrow \lambda = \frac{2}{5}$

Upper row : $3 + 2 \times \frac{2}{5} = p \Rightarrow p = 3.8$

The line crosses the x-axis at (3.8, 0)

[2 marks]

Answer 11

(i) The direction part of $\boldsymbol{r_1} = \begin{pmatrix} 3 \\ 7 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}$ is the $\begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}$

[1 mark]

(ii) If (19, 47, 11) is on the line then there will be a value of λ for which,

$$\begin{pmatrix} 3 \\ 7 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 19 \\ 47 \\ 11 \end{pmatrix}$$

Upper row : $3 + 2\lambda = 19 \Rightarrow 2\lambda = 16 \Rightarrow \lambda = 8$

Middle row : $7 + 5\lambda = 47 \Rightarrow 5\lambda = 40 \Rightarrow \lambda = 8$

Lower row : $3 + \lambda = 11 \Rightarrow \lambda = 8$

$\therefore$ (19, 47, 11) is on the line and occurs when $\lambda = 8$

[2 marks]

(iii) To cross the x-axis at p, $\begin{pmatrix} 3 \\ 7 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} p \\ 0 \\ 0 \end{pmatrix}$

Middle row : $7 + 5\lambda = 0 \Rightarrow \lambda = -\frac{7}{5}$

Lower row : $3 + \lambda = 0 \Rightarrow \lambda = -3$

There is no value of λ for which both the y component and the z component are zero

$\therefore$ The line does not cross the x-axis ☐

[2 marks]