2.4 Answers

2.4.1 Solutions to 2.2 Example

For the lines to intersect,

$$\begin{pmatrix}1\\2\\3\end{pmatrix} + \lambda\begin{pmatrix}3\\2\\-1\end{pmatrix} = \begin{pmatrix}9\\2\\5\end{pmatrix} + \mu\begin{pmatrix}-1\\-2\\1\end{pmatrix}$$

$$\text{Upper row : } 1 + 3\lambda = 9 - \mu \Rightarrow 3\lambda + \mu = 8 \qquad (1)$$

$$\text{Middle row : } 2 + 2\lambda = 2 - 2\mu \Rightarrow \lambda + \mu = 0 \Rightarrow \mu = -\lambda \qquad (2)$$

$$\text{Solve simultaneously : } 3\lambda + (-\lambda) = 8$$

$$2\lambda = 8$$

$$\lambda = 4 \quad \therefore \mu = -4$$

Crucial !

Check the lower row : $\quad$ LHS = $3 - \lambda$ $\quad$ RHS = $5 + \mu$

$$= 3 - 4 \qquad\qquad = 5 - 4$$

$$= -1 \qquad\qquad = 1$$

As LHS ≠ RHS the lines do not intersect, they are skew

[4 marks]

2.4.2 Solution to 2.3 Exercise

Answer 1

(i)
$$\begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} -6 \\ -3 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 5 \\ 1 \\ -2 \end{pmatrix}$$

Upper row : $2 - 2\lambda = -6 + 5\mu \Rightarrow 2\lambda + 5\mu = 8$ (1)

Middle row : $3 + 4\lambda = -3 + \mu \Rightarrow 4\lambda - \mu = -6$ (2)

Solve simultaneously : $4\lambda + 10\mu = 16$ (1) $\times$ 2

$4\lambda - \mu = -6$ (2)

$11\mu = 22$

$\mu = 2 \quad \therefore \lambda = -1$

Crucial !

Check the lower row : LHS $= -2 + \lambda$ RHS $= 1 - 2\mu$

$= -2 - 1$ $= 1 - 2 \times 2$

$= -3$ $= -3$

As LHS = RHS the lines intersect ☐

[4 marks]

(ii) To obtain the point of intersection,

$$\begin{pmatrix} -6 \\ -3 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 5 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -6 \\ -3 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} 5 \\ 1 \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix}$$ Intersection at $(4, -1, -3)$

[1 mark]

(iii)
$$\begin{pmatrix} -2 \\ 4 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 5 \\ 1 \\ -2 \end{pmatrix}$$

$$(-2) \times 5 + 4 \times 1 + 1 \times (-2) = \sqrt{(-2)^2 + 4^2 + 1^2}\ \sqrt{5^2 + 1^2 + (-2)^2}\ \cos\theta$$

$$-10 + 4 - 2 = \sqrt{21}\ \sqrt{30}\ \cos\theta$$

$$\cos\theta = -\frac{8}{\sqrt{21} \times \sqrt{30}}$$

$\therefore\ \theta = 108.6°$ which is an acute angle of $71.4°$

[3 marks]

Answer 2

(a) For the lines to intersect,

$$\begin{pmatrix}1\\0\\-1\end{pmatrix} + \lambda\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}1\\3\\6\end{pmatrix} + \mu\begin{pmatrix}2\\1\\-1\end{pmatrix}$$

Lower row : $-1 = 6 - \mu \Rightarrow \mu = 7$

Middle row : $\lambda = 3 + \mu \Rightarrow \lambda = 10$

Crucial !

Check the upper row : LHS $= 1 + \lambda$ RHS $= 1 + 2\mu$

$= 1 + 10$ $= 1 + 144$

$= 11$ $= 15$

As LHS $\neq$ RHS the lines do not intersect, they are skew

[4 marks]

(b) To find point A, $\begin{pmatrix}1\\0\\-1\end{pmatrix} + \lambda\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\-1\end{pmatrix} + \begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}2\\1\\-1\end{pmatrix}$ $A(2,\ 1,\ -1)$

To find point B, $\begin{pmatrix}1\\3\\6\end{pmatrix} + \mu\begin{pmatrix}2\\1\\-1\end{pmatrix} = \begin{pmatrix}1\\3\\6\end{pmatrix} + 2\begin{pmatrix}2\\1\\-1\end{pmatrix} = \begin{pmatrix}5\\5\\4\end{pmatrix}$ $B(5,\ 5,\ 4)$

$$\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a} = \begin{pmatrix}5\\5\\4\end{pmatrix} - \begin{pmatrix}2\\1\\-1\end{pmatrix} = \begin{pmatrix}3\\4\\5\end{pmatrix}$$

$$\begin{pmatrix}3\\4\\5\end{pmatrix} \bullet \begin{pmatrix}1\\1\\0\end{pmatrix}$$

$$3 \times 1 + 4 \times 1 + 5 \times 0 = \sqrt{3^2 + 4^2 + 5^2}\ \sqrt{1^2 + 1^2 + 0^2}\ \cos\theta$$

$$7 = \sqrt{50}\ \sqrt{2}\ \cos\theta$$

$$\cos\theta = \frac{7}{10}$$

[6 marks]

Answer 3

(a)

$$\boldsymbol{r}_{AB} = \begin{pmatrix} 8 \\ 13 \\ -2 \end{pmatrix} + \lambda\left(\begin{pmatrix} 10 \\ 14 \\ -4 \end{pmatrix} - \begin{pmatrix} 8 \\ 13 \\ -2 \end{pmatrix} \right)$$

$$= \begin{pmatrix} 8 \\ 13 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} \qquad \text{for example}$$

[3 marks]

(b)

$$\overrightarrow{CB} = \boldsymbol{b} - \boldsymbol{c} = \begin{pmatrix} 10 \\ 14 \\ -4 \end{pmatrix} - \begin{pmatrix} 9 \\ 9 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ -10 \end{pmatrix}$$

$$\left|\overrightarrow{CB}\right| = \sqrt{1^2 + 5^2 + (-10)^2} = \sqrt{126} = 3\sqrt{14} \text{ or } 11.2$$

[2 marks]

(c)

$$\begin{pmatrix} 1 \\ 5 \\ -10 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$$

$$1 \times 2 + 5 \times 1 + (-10) \times (-2) = \sqrt{1^2 + 5^2 + (-10)^2}\ \sqrt{2^2 + 1^2 + (-2)^2}\ \cos\theta$$

$$27 = \sqrt{126}\ \sqrt{9}\ \cos\theta$$

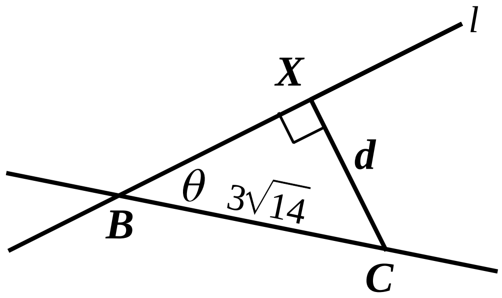
$$\cos\theta = -\frac{27}{\sqrt{126} \times \sqrt{9}}$$

$$= \frac{3\sqrt{14}}{14}$$

$$\therefore\ \theta = 36.7°$$

[3 marks]

(d)



The shortest distance, $d = 3\sqrt{14}\ \sin 36.7° = 6.7$

[3 marks]

(e) Area $= \dfrac{1}{2} \times 6.7 \times 3\sqrt{14} \times \sin(90 - 36.7)$

$= 30.1$

[3 marks]