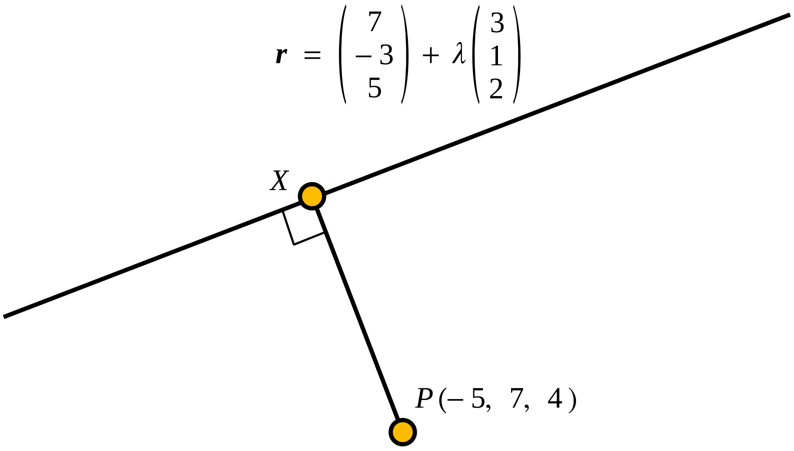
4.3 Answers

4.3.1 Solution to 4.1 Example

$$\boldsymbol{r} = \begin{pmatrix} 7 \\ -3 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$$



There must be a special value of λ that moves a point along the line from the start point of $(7, -3, 5)$ to the point X. Call this special value λ'.

Thus, $X\ (7 + 3\lambda', -3 + \lambda', 5 + 2\lambda')$

$$\overrightarrow{PX} = \boldsymbol{x} - \boldsymbol{p}$$

$$= \begin{pmatrix} 7 + 3\lambda' \\ -3 + \lambda' \\ 5 + 2\lambda' \end{pmatrix} - \begin{pmatrix} -5 \\ 7 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} 12 + 3\lambda' \\ -10 + \lambda' \\ 1 + 2\lambda' \end{pmatrix}$$

Making use of the scalar product,

$$\begin{pmatrix} 12 + 3\lambda' \\ -10 + \lambda' \\ 1 + 2\lambda' \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$$

$$36 + 9\lambda' - 10 + \lambda' + 2 + 4\lambda' = 0$$

$$28 + 14\lambda' = 0$$

$$\lambda' = -2$$

$$X\ (7 + 3\lambda', -3 + \lambda', 5 + 2\lambda') = X(1, -5, 1)$$

The distance between X and P is,

$$\sqrt{(1 - (-5))^2 + ((-5) - 7)^2 + (1 - 4)^2} = \sqrt{6^2 + (-12)^2 + (-3)^2}$$

$$= \sqrt{36 + 144 + 9}$$

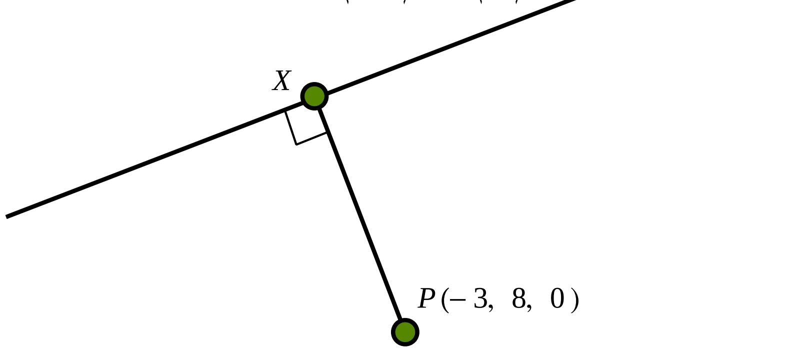
$$= \sqrt{189} \qquad (\text{ or } 3\sqrt{21} \text{ or } 13.75)$$

[4 marks]

4.3.2 Solution to 4.2 Exercise

Answer 1

$$\boldsymbol{r} = \begin{pmatrix} 10 \\ 3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix}$$



There must be a special value of λ that moves a point along the line from the start point of (10, 3, 1) to the point X. Call this special value λ'.

Thus, $X\ (\ 10 + 4\lambda',\ 3 - 2\lambda',\ 1 + \lambda'\ )$

$$\overrightarrow{PX} = \boldsymbol{x} - \boldsymbol{p}$$

$$= \begin{pmatrix} 10 + 4\lambda' \\ 3 - 2\lambda' \\ 1 + \lambda' \end{pmatrix} - \begin{pmatrix} -3 \\ 8 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 13 + 4\lambda' \\ -5 - 2\lambda' \\ 1 + \lambda' \end{pmatrix}$$

Making use of the scalar product,

$$\begin{pmatrix} 13 + 4\lambda' \\ -5 - 2\lambda' \\ 1 + \lambda' \end{pmatrix} \bullet \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix}$$

$$52 + 16\lambda' + 10 + 4\lambda' + 1 + \lambda' = 0$$

$$63 + 21\lambda' = 0$$

$$\lambda' = -3$$

$$X\ (\ 10 + 4\lambda',\ 3 - 2\lambda',\ 1 + \lambda'\ ) = X(-2,\ 9,\ -2)$$

The distance between X and P is,

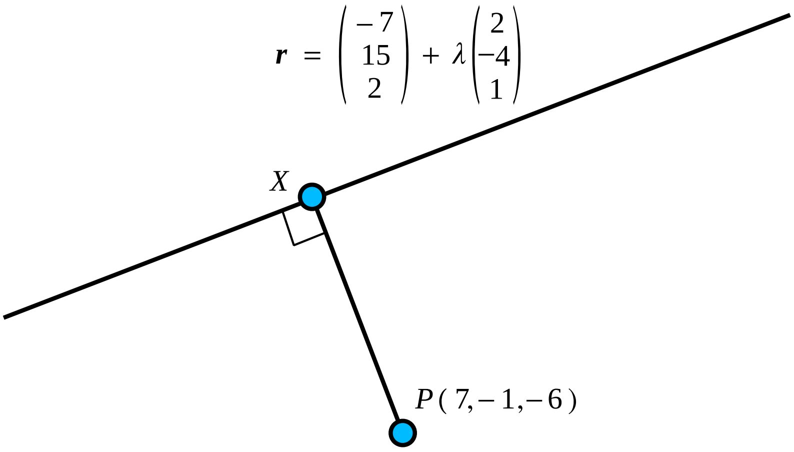
$$\sqrt{((-2) - (-3))^2 + (9 - 8)^2 + ((-2) - 0)^2} = \sqrt{1^2 + 1^2 + (-2)^2}$$

$$= \sqrt{6} \qquad (\text{ or } 2.45)$$

[4 marks]

Answer 2

$$\boldsymbol{r} = \begin{pmatrix} -7 \\ 15 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix}$$



There must be a special value of λ that moves a point along the line from the start point of $(-7,\ 15,\ 2)$ to the point X. Call this special value λ'.

Thus, $X\ (-7 + 2\lambda',\ 15 - 4\lambda',\ 2 + \lambda')$

$$\begin{aligned}\overrightarrow{PX} &= \boldsymbol{x} - \boldsymbol{p} \\ &= \begin{pmatrix} -7 + 2\lambda' \\ 15 - 4\lambda' \\ 2 + \lambda' \end{pmatrix} - \begin{pmatrix} 7 \\ -1 \\ -6 \end{pmatrix} \\ &= \begin{pmatrix} -14 + 2\lambda' \\ 16 - 4\lambda' \\ 8 + \lambda' \end{pmatrix}\end{aligned}$$

Making use of the scalar product,

$$\begin{pmatrix} -14 + 2\lambda' \\ 16 - 4\lambda' \\ 8 + \lambda' \end{pmatrix} \bullet \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix}$$

$$\begin{aligned} -28 + 4\lambda' - 64 + 16\lambda' + 8 + \lambda' &= 0 \\ -84 + 21\lambda' &= 0 \\ \lambda' &= 4 \end{aligned}$$

$$X\ (-7 + 2\lambda',\ 15 - 4\lambda',\ 2 + \lambda') = X\,(1,\ -1,\ 6)$$

The distance between X and P is,

$$\begin{aligned}\sqrt{(1 - 7)^2 + ((-1) - (-1))^2 + (6 - (-6))^2} &= \sqrt{(-6)^2 + 0^2 + (12)^2} \\ &= \sqrt{36 + 0 + 144} \\ &= \sqrt{180} \qquad (\text{ or } 6\sqrt{5} \text{ or } 13.42)\end{aligned}$$

[5 marks]

Answer 3

(a) $\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a} = \begin{pmatrix} -2 \\ 2 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ 5 \\ -3 \end{pmatrix}$

[2 marks]

(b) $\boldsymbol{r} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 5 \\ -3 \end{pmatrix}$ for example

[2 marks]

(c) $\overrightarrow{AC} = \boldsymbol{c} - \boldsymbol{a} = \begin{pmatrix} 2 \\ p \\ -4 \end{pmatrix} - \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ p + 3 \\ -6 \end{pmatrix}$

As AC is perpendicular to l, $\begin{pmatrix} 1 \\ p + 3 \\ -6 \end{pmatrix} \bullet \begin{pmatrix} -3 \\ 5 \\ -3 \end{pmatrix} = 0$

$$-3 + 5p + 15 + 18 = 0$$

$$5p = -30$$

$$p = -6$$

[4 marks]

(d) $\left|\overrightarrow{AC}\right| = \left|\begin{pmatrix} 1 \\ -6 + 3 \\ -6 \end{pmatrix}\right| = \sqrt{1^2 + (-3)^2 + (-6)^2} = \sqrt{46}$ or 6.78

[2 marks]

Answer 4

(a)

$$\begin{pmatrix} 9 \\ 13 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

Upper row : $9 + \lambda = 2 + 2\mu \Rightarrow \lambda - 2\mu = -7$ (1)

Middle row : $13 + 4\lambda = -1 + \mu \Rightarrow 4\lambda - \mu = -14$ (2)

Solve simultaneously : $\lambda - 2\mu = -7$ (1)

$8\lambda - 2\mu = -28$ (2) $\times$ 2

$$-7\lambda = 21$$

$$\lambda = -3 \quad \therefore \quad \mu = 4\lambda + 14$$

$$\mu = 2$$

Crucial !

Check the lower row : LHS $= -3 - 2\lambda$ RHS $= 1 + \mu$

$= -3 + 6$ $= 1 + 2$

$= 3$ $= 3$

As LHS = RHS the lines intersect.

To get the point of intersection,

$$\begin{pmatrix} 9 \\ 13 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} = \begin{pmatrix} 9 \\ 13 \\ -3 \end{pmatrix} - 3 \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} 6 \\ 1 \\ 3 \end{pmatrix} \quad \text{i.e. } (6,\ 1,\ 3)$$

[5 marks]

(b) Using the scalar product $\begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

$$2 + 4 - 2 = \sqrt{1^2 + 4^2 + (-2)^2}\ \sqrt{2^2 + 1^2 + 1^2}\ \cos\theta$$

$$\cos\theta = \frac{4}{\sqrt{21}\ \sqrt{6}}$$

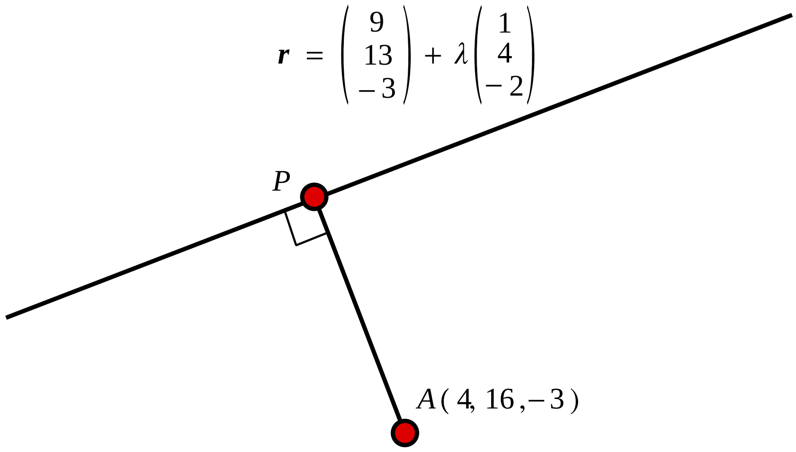
$$= \frac{4}{3\sqrt{14}}$$

$$\theta = 69.1^\circ$$

[3 marks]

(c)

$$\boldsymbol{r} = \begin{pmatrix} 9 \\ 13 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}$$



There must be a special value of λ that moves a point along the line from the start point of (9, 13, – 3) to the point P. Call this special value λ'.

Thus, $P\ (9 + \lambda',\ 13 + 4\lambda', -3 - 2\lambda')$

$$\overrightarrow{AP} = \boldsymbol{p} - \boldsymbol{a}$$

$$= \begin{pmatrix} 9 + \lambda' \\ 13 + 4\lambda' \\ -3 - 2\lambda' \end{pmatrix} - \begin{pmatrix} 4 \\ 16 \\ -3 \end{pmatrix}$$

$$= \begin{pmatrix} 5 + \lambda' \\ -3 + 4\lambda' \\ -2\lambda' \end{pmatrix}$$

Making use of the scalar product,

$$\begin{pmatrix} 5 + \lambda' \\ -3 + 4\lambda' \\ -2\lambda' \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}$$

$$5 + \lambda' - 12 + 16\lambda' + 4\lambda' = 0$$

$$-7 + 21\lambda' = 0$$

$$\lambda' = \frac{1}{3}$$

$$P\ (9 + \lambda',\ 13 + 4\lambda', -3 - 2\lambda') = P\left(9\frac{1}{3},\ 14\frac{1}{3}, -3\frac{2}{3}\right)$$

[6 marks]

Answer 5

(a) $\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a} = \begin{pmatrix} 7 \\ 14 \\ 5 \end{pmatrix} - \begin{pmatrix} 4 \\ 8 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 6 \end{pmatrix}$ or $\overrightarrow{AB} = 3\boldsymbol{i} + 6\boldsymbol{j} + 6\boldsymbol{k}$

[2 marks]

(b) $\overrightarrow{AO} \bullet \overrightarrow{AB} = \begin{pmatrix} -4 \\ -8 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 6 \\ 6 \end{pmatrix}$

$$(-4) \times 3 + (-8) \times 6 + 1 \times 6 = \sqrt{16 + 64 + 1}\ \sqrt{9 + 36 + 36}\, cos\, \theta$$

$$-12 - 48 + 6 = \sqrt{81}\ \sqrt{81}\ cos\, \theta$$

$$cos\, \theta = -\frac{2}{3}$$ Direction Important, see lesson 5

[3 marks]

(c) The line given is $\boldsymbol{r} = \begin{pmatrix} 0 \\ 0 \\ -9 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

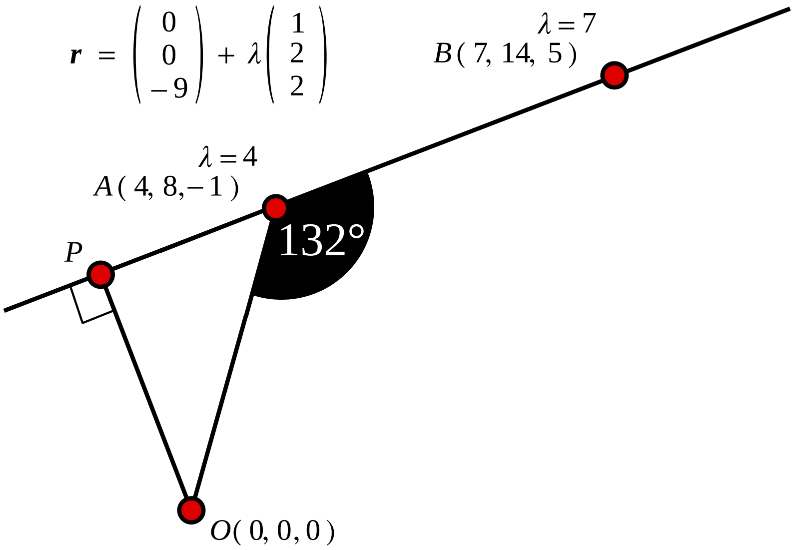
By letting $\lambda = 4$ point A is reached

By letting $\lambda = 7$ point B is reached

$\therefore$ The point P represented by the line $\boldsymbol{r}$ passes through A and B

[3 marks]

(d)



There must be a special value of λ that moves a point along the line from the start point of $(0, \; 0, -9)$ to the point P. Call this special value λ'.

Thus, $P \; (0 + \lambda', \; 0 + 2\lambda', -9 + 2\lambda')$

$$\begin{aligned} \overrightarrow{OP} &= \boldsymbol{p} - \boldsymbol{o} \\ &= \begin{pmatrix} \lambda' \\ 2\lambda' \\ -9 + 2\lambda' \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \lambda' \\ 2\lambda' \\ -9 + 2\lambda' \end{pmatrix} \end{aligned}$$

Making use of the scalar product,

$$\begin{pmatrix} \lambda' \\ 2\lambda' \\ -9 + 2\lambda' \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

$$\begin{aligned} \lambda' + 4\lambda' - 18 + 4\lambda' &= 0 \\ -18 + 9\lambda' &= 0 \\ \lambda' &= 2 \end{aligned}$$

[3 marks]

$$P \; (0 + \lambda', \; 0 + 2\lambda', -9 + 2\lambda') = P(2, \; 4, -5)$$

[2 marks]