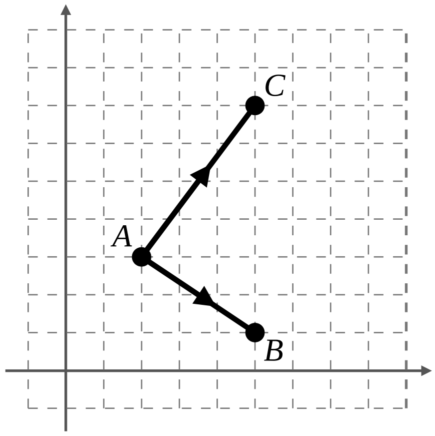
5.3 Answers

5.3.1 Solution to the introductory example



To get the correct angle, $\angle CAB$, it is necessary† to get both direction vectors ***away*** from the point A

Thus the calculation proceeds as follows...

$$\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a} = \begin{pmatrix} 5 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

$$\overrightarrow{AC} = \boldsymbol{c} - \boldsymbol{a} = \begin{pmatrix} 4 \\ 7 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} 3 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

$$3 \times 2 + (-2) \times 4 = \sqrt{3^2 + (-2)^2}\ \sqrt{2^2 + 4^2}\ \cos\theta$$

$$-2 = \sqrt{13}\ \sqrt{20}\ \cos\theta\ -\ 0.124$$

$$\cos\theta = -0.124 \qquad \therefore \quad \theta = 97.1°$$

[3 marks]

† Having both direction vectors towards the point A also works but having one towards A and one away from A will get the angle's compliment.
i.e. $180 - \angle CAB$.
Often examinations questions are about directions between lines in which case the angle wanted is not as tightly specified, and such questions usually ask for the acute angle between the lines. i.e. The angle less than 90°.

5.3.2 Solutions to 5.2 Exercise

Answer 1

(i) $\overrightarrow{BA} = \begin{pmatrix} 4 \\ -1 \\ 8 \end{pmatrix}$

[1 mark]

(ii) $\left|\overrightarrow{BA}\right| = 9$

[1 mark]

(iii) $\left|\overrightarrow{BC}\right| = 7$

[1 mark]

(iv)

$$\overrightarrow{BA} \bullet \overrightarrow{BC} = \begin{pmatrix} 4 \\ -1 \\ 8 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ -2 \\ -6 \end{pmatrix}$$

$$12 + 2 - 48 = 9 \times 7 \times cos\ (\angle ABC)$$

$$cos\ (\angle ABC) = \frac{-34}{63}$$

$$\angle ABC = 122.7°$$

[2 marks]

(v)

$$\text{Area } \Delta = 0.5 \times 7 \times 9 \times sin\ (122.7°)$$

$$= 26.5$$

[2 marks]

Answer 2

$$\overrightarrow{AB} = \begin{pmatrix} -8 \\ 5 \\ 2 \end{pmatrix} - \begin{pmatrix} -4 \\ 7 \\ 3 \end{pmatrix} = \begin{pmatrix} -4 \\ -2 \\ -1 \end{pmatrix}$$

$$\overrightarrow{AC} = \begin{pmatrix} 9 \\ 1 \\ -6 \end{pmatrix} - \begin{pmatrix} -4 \\ 7 \\ 3 \end{pmatrix} = \begin{pmatrix} 13 \\ -6 \\ -9 \end{pmatrix}$$

$$\overrightarrow{AB} \bullet \overrightarrow{AC} = \begin{pmatrix} -4 \\ -2 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} 13 \\ -6 \\ -9 \end{pmatrix}$$

$$-52 + 12 + 9 = \sqrt{21}\ \sqrt{286}\ cos\,(\angle BAC)$$

$$cos\ (\angle BAC) = \frac{-31}{77.498}$$

$$\angle ABC = 113.6°$$

[4 marks]

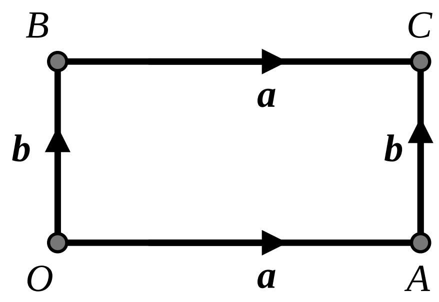
Answer 3

(a) $$\boldsymbol{c} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ -4 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ -3 \end{pmatrix}$$

$$c = 3\boldsymbol{i} + 3\boldsymbol{j} - 3\boldsymbol{k}$$

[1 mark]

(b) As $\boldsymbol{c} = \boldsymbol{a} + \boldsymbol{b}$ we could simply show that any one of the four vertex angles is 90° and it follows that all four angles are 90° and thus that *OABC* is a rectangle.



Alternatively, show that $\boldsymbol{a} \bullet \boldsymbol{b} = 0$
Thus,

$$a \bullet b = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 1 \\ -4 \end{pmatrix}$$

$$= 2 + 2 - 4$$

$$= 0$$

$\therefore$ As the scalar product is zero each angle of quadrilateral *OACB* is 90°

and so *OACB* is a rectangle. ☐

$$Area = \sqrt{2^2 + 2^2 + 1^2} \times \sqrt{1^2 + 1^2 + (-4)^2}$$

$$= 3 \times \sqrt{18}$$

$$= 9\sqrt{2}$$ $*$ Exact area asked for $*$

[6 marks]

(c) $$\overrightarrow{OD} = \frac{1}{2}\overrightarrow{OC}$$

$$= \frac{1}{2}(3\boldsymbol{i} + 3\boldsymbol{j} - 3\boldsymbol{k})$$

$$= 1.5\boldsymbol{i} + 1.5\boldsymbol{j} - 1.5\boldsymbol{k}$$

[1 mark]

(d)

$$\overrightarrow{DA} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 1.5 \\ 1.5 \\ -1.5 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0.5 \\ 2.5 \end{pmatrix}$$

$$\overrightarrow{DC} = \begin{pmatrix} 3 \\ 3 \\ -3 \end{pmatrix} - \begin{pmatrix} 1.5 \\ 1.5 \\ -1.5 \end{pmatrix} = \begin{pmatrix} 1.5 \\ 1.5 \\ -1.5 \end{pmatrix}$$

$$\overrightarrow{DA} \bullet \overrightarrow{DC} = \begin{pmatrix} 0.5 \\ 0.5 \\ 2.5 \end{pmatrix} \bullet \begin{pmatrix} 1.5 \\ 1.5 \\ -1.5 \end{pmatrix}$$

$$0.75 + 0.75 - 3.75 = \sqrt{6.75}\ \sqrt{6.75}\ cos\,(\angle ADC)$$

$$cos\ (\angle ADC) = \frac{-2.25}{6.75}$$

$$cos\ (\angle ADC) = \frac{-1}{3}$$

$$\angle ADC = 109.47122°$$

[6 marks]