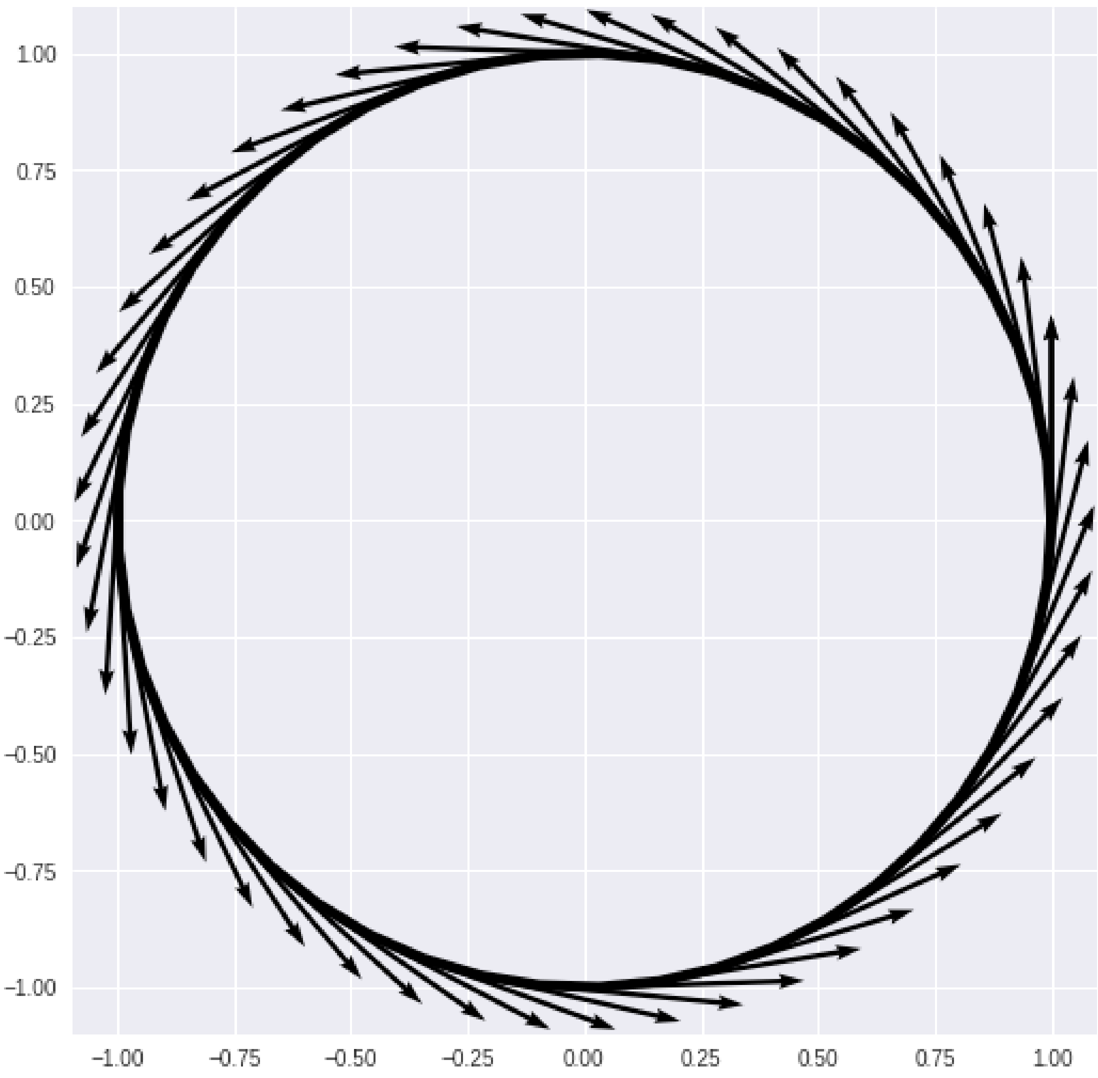
Further Pure A-Level Mathematics
Compulsory Course Component
Core 1

VECTORS

III



Vectors III

Lesson 1

Further A-Level Pure Mathematics
Vectors III : Core 1

1.1 Introduction

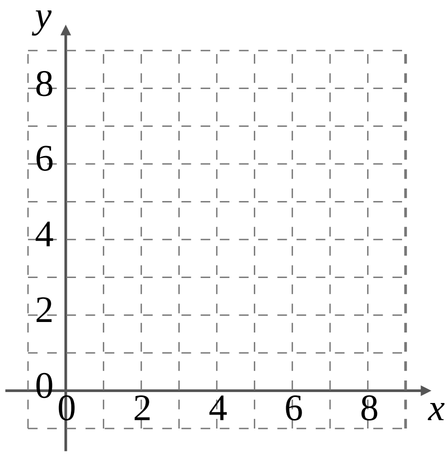
Vectors provide us with a powerful means of working with straight lines in two and three dimensions, extendable to higher dimensions if required. Unlike the approach based on trigonometry, vector methods do not become more intricate as the number of dimensions is increased.
To illuminate the vector approach, a two dimensional problem will be tackled first using familiar GCSE methods, then the Further A-Level vectors approach.

1.2 The GCSE Method

Consider the two straight lines;

$$y = 2x + 1 \qquad \text{and} \qquad y = \frac{1}{2}x + 4$$

(i) On the grid below, sketch the two lines.
(ii) Write down the point of intersection.
(iii) Show how to find this point of intersection using simple algebra.
(iv) Find θ, the acute angle in degrees, between the two lines using trigonometry.



[6 marks]

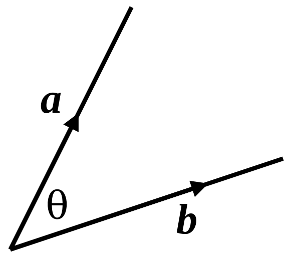
1.3 The Vector Method

[6 marks]

1.4 Angles and Vectors

The Scalar Product (Also called **The Dot Product**)

By definition : $\boldsymbol{a} \bullet \boldsymbol{b} = |\boldsymbol{a}|\ |\boldsymbol{b}|\ \cos\theta$



Where $|\boldsymbol{a}|$ and $|\boldsymbol{b}|$ are the magnitudes of vectors ***a*** and ***b***

θ is the angle between vectors ***a*** and ***b***

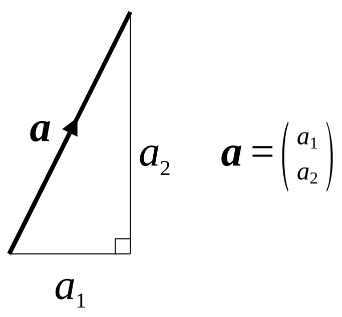
Whenever angles are involved in a problem involving vectors, use the scalar product with the **DIRECTION PARTS** of the vector equations of the lines.

The scalar product will be studied in Lesson 3, but for now work with;

In two dimensions;

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \bullet \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$a_1 b_1 + a_2 b_2 = \sqrt{a_1^2 + a_2^2}\ \sqrt{b_1^2 + b_2^2}\ \cos\theta$$



In three dimensions;

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \bullet \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$a_1 b_1 + a_2 b_2 + a_3 b_3 = \sqrt{a_1^2 + a_2^2 + a_3^2}\ \sqrt{b_1^2 + b_2^2 + b_3^2}\ \cos\theta$$

1.5 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 60

Question 1

Consider the two straight lines;

$$\boldsymbol{r_1} = \begin{pmatrix} -4 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

and $$\boldsymbol{r_2} = \begin{pmatrix} 3 \\ -8 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 7 \end{pmatrix}$$

(i) Write down the direction part of $\boldsymbol{r_1}$

[1 mark]

(ii) Use the scalar product to show that;

$$\cos\theta = \frac{1}{\sqrt{2}}$$

[3 marks]

(iii) Hence determine, in degrees, the angle between the two lines.

[1 mark]

Question 2

Consider the two straight lines;

$$\boldsymbol{r_1} = \begin{pmatrix} -7 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

and $$\boldsymbol{r_2} = \begin{pmatrix} -5 \\ 8 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

(i) Write down the direction part of $\boldsymbol{r_2}$

[1 mark]

(ii) Use the dot product to show that;

$$cos\ \theta = \frac{1}{\sqrt{10}}$$

[3 marks]

(iii) Hence determine the angle between the two lines.

[1 mark]

Question 3

Consider the two straight lines;

$$\boldsymbol{r_1} = \begin{pmatrix} -4 \\ -11 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\boldsymbol{r_2} = \begin{pmatrix} -8 \\ 11 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

Use vector methods to determine the point of intersection of the two lines.

[4 marks]

Question 4

Consider the two straight lines;

$$\boldsymbol{r_1} = \begin{pmatrix} -15 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$\boldsymbol{r_2} = \begin{pmatrix} 15 \\ 17 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

Use vector methods to determine the point of intersection of the two lines.

[4 marks]

Question 5

Consider the two straight lines;

$$\boldsymbol{r_1} = \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 4 \\ 12 \end{pmatrix}$$

and $$\boldsymbol{r_2} = \begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} -3 \\ -4 \\ 5 \end{pmatrix}$$

(i) Write down the direction part of $\boldsymbol{r_1}$

[1 mark]

(ii) Use the scalar product to show that;

$$\cos\theta = \frac{7\sqrt{2}}{26}$$

[3 marks]

(iii) Hence determine, in degrees, the angle between the two lines.

[1 mark]

Question 6

A, B and C are the points;

$$A(2,\ 3,\ 7) \qquad B(8,\ 5,\ 9) \qquad C(3,\ 11,\ 4)$$

(i) Show that $\overrightarrow{AB} = \begin{pmatrix} 6 \\ 2 \\ 2 \end{pmatrix}$

[1 mark]

(ii) Determine $\overrightarrow{AC}$

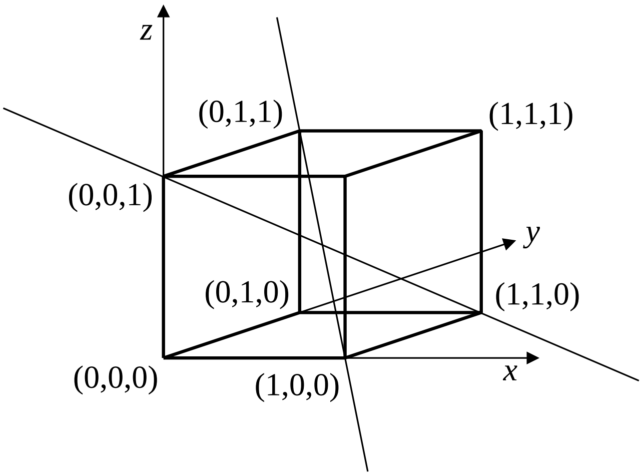
[1 mark]

(iii) Find $\angle$BAC by considering the scalar product, $\overrightarrow{AB} \bullet \overrightarrow{AC}$

[3 marks]

Question 7

A cube has the eight vertices shown in the following diagram,



(i) One long diagonal passes through the points *A* (1,0,0) and *B* (0,1,1)
Explain why this diagonal line has the vector equation;

$$\boldsymbol{r}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

[1 mark]

(ii) Another long diagonal passes through the points *C* (1,1,0) and *D* (0,0,1)
Determine the vector equation of this diagonal line.

[2 mark]

(iii) Use the scalar product to determine the acute angle between the two long diagonals. Give your answer in degrees, accurate to one decimal place.

[3 marks]

Question 8

C4 Examination Question from January 2010, Q4

The line l_1 has vector equation

$$\boldsymbol{r_1} = \begin{pmatrix} -6 \\ 4 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix}$$

and the line l_2 has vector equation

$$\boldsymbol{r_2} = \begin{pmatrix} -6 \\ 4 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix}$$

where λ and μ are parameters.

The lines l_1 and l_2 intersect at the point A and the acute angle between l_1 and l_2 is θ

(a) Write down the coordinates of A

[1 mark]

(b) Find the value of $\cos\theta$

[3 marks]

The point X lies on l_1 where $\lambda = 4$

(c) Find the coordinates of X

[1 mark]

(d) Find the vector $\overrightarrow{AX}$

[2 marks]

(e) Hence, or otherwise, show that

$$\left|\overrightarrow{AX}\right| = 4\sqrt{26}$$

[2 marks]

The point Y lies on l_2.

Given that the vector $\overrightarrow{YX}$ is perpendicular to l_1

(f) find the length of AY, giving your answer to 3 significant figures

[3 marks]

Question 9

The lines l_1 and l_2 have equations;

$$l_1 \quad : \quad \boldsymbol{r_1} = \boldsymbol{i} - \boldsymbol{j} - 5\boldsymbol{k} + \lambda(2\boldsymbol{i} + 3\boldsymbol{j} + 4\boldsymbol{k})$$

$$l_2 \quad : \quad \boldsymbol{r_2} = 16\boldsymbol{i} + 5\boldsymbol{j} - 2\boldsymbol{k} + \mu(7\boldsymbol{i} - 6\boldsymbol{j} - 10\boldsymbol{k})$$

Find the acute angle, to the nearest 0.1°, between l_1 and l_2

[4 marks]

Question 10

A straight line has equation;

$$\boldsymbol{r} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

(i) Which part of the equation is the direction part ?

[1 mark]

(ii) Show that the point (13, 23) is on the line.

[2 marks]

(iii) Where does this line cross the x-axis ?

[2 marks]

Question 11

A straight line has equation;

$$\boldsymbol{r} = \begin{pmatrix} 3 \\ 7 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}$$

(i) Which part of the equation is the direction part ?

[1 mark]

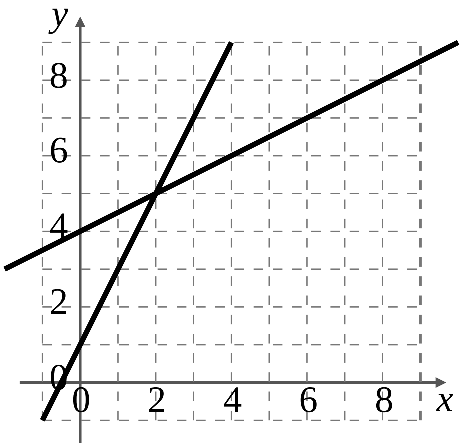
(ii) Show that the point (19, 47, 11) is on the line.

(iii) Show that this line does not cross the x-axis.

1.6 Answers

1.6.1 Solution to 1.2 The GCSE Method

(i)



[1 mark]

(ii) Point of intersection is (2, 5)

[1 mark]

(iii)

$$2x + 1 = \frac{1}{2}x + 4$$

multiply through by 2

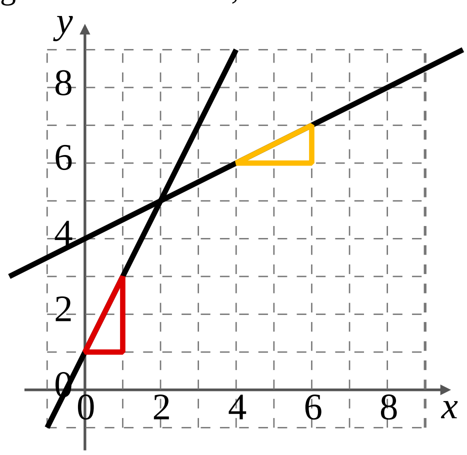
$$4x + 2 = x + 8$$

$$3x = 6$$

$$x = 2 \Rightarrow y = 5 \qquad \therefore \text{Intersection} (2,\ 5)$$

[2 marks]

(iv) For θ the angle of intersection,



$$\theta = arctan\left(\frac{2}{1}\right) - arctan\left(\frac{1}{2}\right)$$

$$= 63.4 - 26.6$$

$$= 36.9°$$

[2 marks]

1.6.2 Solution to 1.3 The Vector Method

$$\boldsymbol{r_1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \text{and} \quad \boldsymbol{r_2} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

From the upper row : $\lambda = 2\mu$

From the lower row : $1 + 2\lambda = 4 + \mu$

$$1 + 4\mu = 4 + \mu$$

$$3\mu = 3$$

$$\mu = 1 \Rightarrow \lambda = 2$$

Point of intersection : $\boldsymbol{r_1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$= \begin{pmatrix} 0 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 5 \end{pmatrix} \quad \text{i.e. } (2, 5)$$

Using Scalar Product : $\begin{pmatrix} 1 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$$1 \times 2 + 2 \times 1 = \sqrt{1^2 + 2^2} \times \sqrt{2^2 + 1^2} \times \cos\theta$$

$$4 = \sqrt{5}\sqrt{5} \cos\theta$$

$$\cos\theta = \frac{4}{5}$$

$$= 36.9°$$

[6 marks]

1.6.3 Solutions to 1.5 Exercise

Answer 1

(i) The direction part of $\boldsymbol{r_1} = \begin{pmatrix} -4 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 2 \end{pmatrix}$ is the $\begin{pmatrix} 5 \\ 2 \end{pmatrix}$

[1 mark]

(ii)

$$\begin{pmatrix} 5 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 7 \end{pmatrix}$$

$$\begin{aligned} 5 \times 3 + 2 \times 7 &= \sqrt{5^2 + 2^2} \times \sqrt{3^2 + 7^2} \times \cos\theta \\ 15 + 14 &= \sqrt{29} \times \sqrt{58} \times \cos\theta \\ 29 &= \sqrt{29} \times \sqrt{2} \times \sqrt{29} \times \cos\theta \\ 29 &= 29\sqrt{2}\,\cos\theta \\ \cos\theta &= \frac{1}{\sqrt{2}} \qquad \square \end{aligned}$$

[3 marks]

(iii) $\therefore \quad \theta = 45°$

[1 mark]

Answer 2

(i) The direction part of $\boldsymbol{r_1} = \begin{pmatrix} -5 \\ 8 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 3 \end{pmatrix}$ is the $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$

[1 mark]

(ii)

$$\begin{pmatrix} 4 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

$$\begin{aligned} 4 \times (-1) + 3 \times 3 &= \sqrt{4^2 + 3^2} \times \sqrt{(-1)^2 + 3^2} \times \cos\theta \\ -4 + 9 &= \sqrt{25} \times \sqrt{10} \times \cos\theta \\ 5 &= 5\sqrt{10}\,\cos\theta \\ \cos\theta &= \frac{1}{\sqrt{10}} \qquad \square \end{aligned}$$

[3 marks]

(iii) $\therefore \quad \theta = 71.6°$

[1 mark]

Answer 3

$$\begin{pmatrix} -4 \\ -11 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -8 \\ 11 \end{pmatrix} + \mu\begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

Upper row : $-4 + \lambda = -8 + 2\mu \Rightarrow \lambda - 2\mu = -4$ (1)

Lower row : $-11 + 2\lambda = 11 - \mu \Rightarrow 2\lambda + \mu = 22$ (2)

Solve (1) and (2) simultaneously to get $\mu = 6$ and $\lambda = 8$

To get the point of intersection using $\lambda = 8$,

$$\begin{aligned} \boldsymbol{r_1} &= \begin{pmatrix} -4 \\ -11 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} -4 \\ -11 \end{pmatrix} + 8\begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} 4 \\ 5 \end{pmatrix} \qquad \therefore \text{ Intersection at } (4,\ 5) \end{aligned}$$

[4 marks]

Answer 4

$$\begin{pmatrix} -15 \\ -4 \end{pmatrix} + \lambda\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 15 \\ 17 \end{pmatrix} + \mu\begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

Upper row : $-15 + 3\lambda = 15 + 3\mu \Rightarrow \lambda - \mu = 10$ (1)

Lower row : $-4 + \lambda = 17 + 2\mu \Rightarrow \lambda - 2\mu = 21$ (2)

Solve (1) and (2) simultaneously to get $\mu = -11$ and $\lambda = -1$

To get the point of intersection using $\lambda = -1$,

$$\begin{aligned} \boldsymbol{r_1} &= \begin{pmatrix} -15 \\ -4 \end{pmatrix} + \lambda\begin{pmatrix} 3 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} -15 \\ -4 \end{pmatrix} - 1\begin{pmatrix} 3 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} -18 \\ -5 \end{pmatrix} \qquad \therefore \text{ Intersection at } (-18,\ -5) \end{aligned}$$

[4 marks]

Answer 5

(i) The direction part of $\boldsymbol{r_1} = \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 4 \\ 12 \end{pmatrix}$ is the $\begin{pmatrix} 3 \\ 4 \\ 12 \end{pmatrix}$

[1 mark]

(ii) $\begin{pmatrix} 3 \\ 4 \\ 12 \end{pmatrix} \bullet \begin{pmatrix} -3 \\ -4 \\ 5 \end{pmatrix}$

$$3 \times (-3) + 4 \times (-4) + 12 \times 5 = \sqrt{3^2 + 4^2 + 12^2}\ \sqrt{(-3)^2 + (-4)^2 + 5^2}\ \cos\theta$$

$$35 = 13 \times 5\sqrt{2}\cos\theta$$

$$\cos\theta = \frac{7}{13\sqrt{2}}$$

$$= \frac{7\sqrt{2}}{26} \qquad \square$$

[3 marks]

(iii) $\therefore \quad \theta = 67°$

[1 mark]

Answer 6

(i) $\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$

$$= \begin{pmatrix} 8 \\ 5 \\ 9 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 2 \end{pmatrix}$$

[1 mark]

(ii) $\overrightarrow{AC} = \boldsymbol{c} - \boldsymbol{a}$

$$= \begin{pmatrix} 3 \\ 11 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix} = \begin{pmatrix} 1 \\ 8 \\ -3 \end{pmatrix}$$

[1 mark]

(iii) $\begin{pmatrix} 6 \\ 2 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 8 \\ -3 \end{pmatrix}$

$$6 \times 1 + 2 \times 8 + 2 \times (-3) = \sqrt{6^2 + 2^2 + 2^2}\ \sqrt{1^2 + 8^2 + (-3)^2}\ \cos\theta$$

$$16 = 2\sqrt{11}\ \sqrt{74}\ \cos\theta$$

$$\cos\theta = \frac{8}{\sqrt{11}\ \sqrt{74}}$$

$$\theta = 73.7°$$

[3 marks]

Answer 7

(i) Let A be the point (1, 0, 0), and be B be (0, 1, 1)

$$\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$$
$$= \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$
$$\boldsymbol{r_1} = \overrightarrow{OA} + \lambda \overrightarrow{AB}$$
$$= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

[1 mark]

(ii) Let C be the point (1, 1, 0), and be D be (0, 0, 1)

$$\overrightarrow{CD} = \boldsymbol{d} - \boldsymbol{c}$$
$$= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$
$$\boldsymbol{r_2} = \overrightarrow{OC} + \lambda \overrightarrow{CD}$$
$$= \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

[2 mark]

(iii)
$$\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

$$(-1) \times (-1) + 1 \times (-1) + 1 \times 1 = \sqrt{(-1)^2 + 1^2 + 1^2}\ \sqrt{(-1)^2 + (-1)^2 + 1^2}\ cos\,\theta$$
$$1 - 1 + 1 = \sqrt{3}\,\sqrt{3}\ cos\,\theta$$
$$cos\,\theta = \frac{1}{3}$$
$$\theta = 70.5°$$

[3 marks]

Answer 8

(a) $\boldsymbol{r_1} = \begin{pmatrix} -6 \\ 4 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix}$ and $\boldsymbol{r_2} = \begin{pmatrix} -6 \\ 4 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix}$ intersect at $\begin{pmatrix} -6 \\ 4 \\ -1 \end{pmatrix}$

[1 mark]

(b) $$\begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix}$$

$$4 \times 3 + (-1) \times (-4) + 3 \times 1 = \sqrt{4^2 + (-1)^2 + 3^2}\,\sqrt{3^2 + (-4)^2 + 1^2}\,\cos\theta$$

$$12 + 4 + 3 = \sqrt{26}\,\sqrt{26}\,\cos\theta$$

$$\cos\theta = \frac{19}{26}$$

[3 marks]

(c) $\boldsymbol{r_1} = \begin{pmatrix} -6 \\ 4 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} -6 \\ 4 \\ -1 \end{pmatrix} + 4 \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 10 \\ 0 \\ 11 \end{pmatrix} \quad \therefore\ X\ (10,\ 0,\ 11)$

[1 mark]

(d) $\overrightarrow{AX} = \overrightarrow{AO} + \overrightarrow{OX} = \overrightarrow{OX} - \overrightarrow{OA}$

$$= \begin{pmatrix} 10 \\ 0 \\ 11 \end{pmatrix} - \begin{pmatrix} -6 \\ 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 16 \\ -4 \\ 12 \end{pmatrix}$$

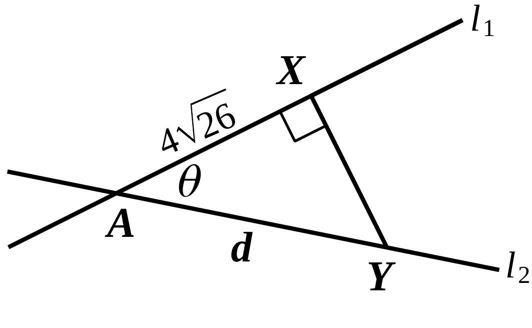
[2 marks]

(e) $$\left|\overrightarrow{AX}\right| = \sqrt{16^2 + (-4)^2 + 12^2}$$

$$= \sqrt{256 + 16 + 144} = \sqrt{416} = 4\sqrt{26}$$

[2 marks]

(f)



$$|AY| = \frac{4\sqrt{26}}{\left(\frac{19}{26}\right)} = 27.9$$

[3 marks]

Answer 9

$$\boldsymbol{r_1} = \begin{pmatrix} 1 \\ -1 \\ -5 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \quad \text{and} \quad \boldsymbol{r_2} = \begin{pmatrix} 16 \\ 5 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 7 \\ -6 \\ -10 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} 7 \\ -6 \\ -10 \end{pmatrix}$$

$$2 \times 7 + 3 \times (-6) + 4 \times (-10) = \sqrt{2^2 + 3^2 + 4^2}\ \sqrt{7^2 + (-6)^2 + (-10)^2}\ \cos\theta$$

$$14 - 18 - 40 = \sqrt{29}\ \sqrt{185}\ \cos\theta$$

$$\cos\theta = -\frac{44}{\sqrt{29} \times \sqrt{185}}$$

$$\therefore\ \theta = 126.9° \text{ which is an acute angle of } 53.1°$$

[4 marks]

Answer 10

(i) The direction part of $\boldsymbol{r} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 5 \end{pmatrix}$ is the $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$

[1 mark]

(ii) If (13, 23) is on the line then there will be a value of λ for which,

$$\begin{pmatrix} 3 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 13 \\ 23 \end{pmatrix}$$

Upper row : $3 + 2\lambda = 13 \Rightarrow 2\lambda = 10 \Rightarrow \lambda = 5$

Lower row : $-2 + 5\lambda = 23 \Rightarrow 5\lambda = 25 \Rightarrow \lambda = 5$

$\therefore$ (13, 23) is on the line and occurs when $\lambda = 5$

[2 marks]

(iii) To cross the x-axis at p, $\begin{pmatrix} 3 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} p \\ 0 \end{pmatrix}$

Lower row : $-2 + 5\lambda = 0 \Rightarrow \lambda = \frac{2}{5}$

Upper row : $3 + 2 \times \frac{2}{5} = p \Rightarrow p = 3.8$

The line crosses the x-axis at (3.8, 0)

[2 marks]

Answer 11

(i) The direction part of $\boldsymbol{r_1} = \begin{pmatrix} 3 \\ 7 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}$ is the $\begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}$

[1 mark]

(ii) If (19, 47, 11) is on the line then there will be a value of λ for which,

$$\begin{pmatrix} 3 \\ 7 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 19 \\ 47 \\ 11 \end{pmatrix}$$

Upper row : $3 + 2\lambda = 19 \Rightarrow 2\lambda = 16 \Rightarrow \lambda = 8$

Middle row : $7 + 5\lambda = 47 \Rightarrow 5\lambda = 40 \Rightarrow \lambda = 8$

Lower row : $3 + \lambda = 11 \Rightarrow \lambda = 8$

$\therefore$ (19, 47, 11) is on the line and occurs when $\lambda = 8$

[2 marks]

(iii) To cross the x-axis at p, $\begin{pmatrix} 3 \\ 7 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} p \\ 0 \\ 0 \end{pmatrix}$

Middle row : $7 + 5\lambda = 0 \Rightarrow \lambda = -\dfrac{7}{5}$

Lower row : $3 + \lambda = 0 \Rightarrow \lambda = -3$

There is no value of λ for which both the y component and the z component are zero

$\therefore$ The line does not cross the x-axis $\square$

[2 marks]

2.1 Intersecting Lines In Three Dimensions

In two dimensions if two distinct lines are not parallel they must have a point of intersection. In three dimensions the same is not true; it is possible for two lines that are not parallel lines to not intersect. Such lines are said to be **SKEW**.

2.2 Example

Determine if the following lines intersect or if they are skew.

$$\boldsymbol{r}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{r}_2 = \begin{pmatrix} 9 \\ 2 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$$

[4 marks]

2.3 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 32

Question 1

(i) Show that the following lines intersect;

$$\boldsymbol{r}_1 = \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 4 \\ 1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{r}_2 = \begin{pmatrix} -6 \\ -3 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 5 \\ 1 \\ -2 \end{pmatrix}$$

[4 marks]

(ii) Find the coordinates of the point of intersection.

[1 mark]

Recall that in three dimensions the scalar product is;

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \bullet \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$a_1b_1 + a_2b_2 + a_3b_3 = \sqrt{a_1{}^2 + a_2{}^2 + a_3{}^2}\ \sqrt{b_1{}^2 + b_2{}^2 + b_3{}^2}\ \cos\theta$$

(iii) Use this to find, to the nearest 0.1°, the acute angle between the lines. (Remember to use the direction part of the lines !)

[3 marks]

Question 2

C4 Examination Question from June 2007, Q5

The line l_1 has equation;

$$\boldsymbol{r} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

The line l_2 has equation;

$$\boldsymbol{r} = \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

(a) Show that l_1 and l_2 do not meet.

[4 marks]

The point A is on l_1 where $\lambda = 1$, and the point B is on l_2 where $\mu = 2$

(b) Find the cosine of the acute angle between AB and l_1

[6 marks]

Question 3

C4 Examination Question from June 2009, Q7

Relative to a fixed origin O, the point A has position vector $8\,\boldsymbol{i} + 13\,\boldsymbol{j} - 2\,\boldsymbol{k}$,
the point B has position vector $10\,\boldsymbol{i} + 14\,\boldsymbol{j} - 4\,\boldsymbol{k}$
and the point C has position vector $9\,\boldsymbol{i} + 9\,\boldsymbol{j} + 6\,\boldsymbol{k}$

The line l passes through the points A and B

(a) Find a vector equation for the line l

[3 marks]

(b) Find $\left|\overrightarrow{CB}\right|$

[2 marks]

(c) Find the size of the acute angle between the line segment CB and the line l, giving your answer in degrees to 1 decimal place.

[3 marks]

(d) Find the shortest distance from the point C to the line l

[3 marks]

The point X lies on l
Given that the vector $\overrightarrow{CX}$ is perpendicular to l

(e) find the area of triangle CXB, giving your answer to 3 significant figures.

[3 marks]

2.4 Answers

2.4.1 Solutions to 2.2 Example

For the lines to intersect,

$$\begin{pmatrix}1\\2\\3\end{pmatrix} + \lambda\begin{pmatrix}3\\2\\-1\end{pmatrix} = \begin{pmatrix}9\\2\\5\end{pmatrix} + \mu\begin{pmatrix}-1\\-2\\1\end{pmatrix}$$

$$\text{Upper row}: 1 + 3\lambda = 9 - \mu \Rightarrow 3\lambda + \mu = 8 \qquad (1)$$

$$\text{Middle row}: 2 + 2\lambda = 2 - 2\mu \Rightarrow \lambda + \mu = 0 \Rightarrow \mu = -\lambda \qquad (2)$$

$$\text{Solve simultaneously}: 3\lambda + (-\lambda) = 8$$

$$2\lambda = 8$$

$$\lambda = 4 \quad \therefore \mu = -4$$

Crucial !

Check the lower row : LHS = $3 - \lambda$ RHS = $5 + \mu$

$$= 3 - 4 \qquad = 5 - 4$$

$$= -1 \qquad = 1$$

As LHS ≠ RHS the lines do not intersect, they are skew

[4 marks]

2.4.2 Solution to 2.3 Exercise

Answer 1

(i)

$$\begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} -6 \\ -3 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 5 \\ 1 \\ -2 \end{pmatrix}$$

Upper row : $2 - 2\lambda = -6 + 5\mu \Rightarrow 2\lambda + 5\mu = 8$ (1)

Middle row : $3 + 4\lambda = -3 + \mu \Rightarrow 4\lambda - \mu = -6$ (2)

Solve simultaneously : $4\lambda + 10\mu = 16$ (1) $\times$ 2

$$4\lambda - \mu = -6 \qquad (2)$$

$$11\mu = 22$$

$$\mu = 2 \quad \therefore \lambda = -1$$

Crucial !

Check the lower row : LHS $= -2 + \lambda$ RHS $= 1 - 2\mu$

$= -2 - 1$ $= 1 - 2 \times 2$

$= -3$ $= -3$

As LHS = RHS the lines intersect ☐

[4 marks]

(ii) To obtain the point of intersection,

$$\begin{pmatrix} -6 \\ -3 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 5 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -6 \\ -3 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} 5 \\ 1 \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix}$$ Intersection at $(4, -1, -3)$

[1 mark]

(iii)

$$\begin{pmatrix} -2 \\ 4 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 5 \\ 1 \\ -2 \end{pmatrix}$$

$$(-2) \times 5 + 4 \times 1 + 1 \times (-2) = \sqrt{(-2)^2 + 4^2 + 1^2} \sqrt{5^2 + 1^2 + (-2)^2} \ cos\,\theta$$

$$-10 + 4 - 2 = \sqrt{21}\sqrt{30}\ cos\,\theta$$

$$cos\,\theta = -\frac{8}{\sqrt{21} \times \sqrt{30}}$$

$\therefore\ \theta = 108.6°$ which is an acute angle of $71.4°$

[3 marks]

Answer 2

(a) For the lines to intersect,

$$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

Lower row : $-1 = 6 - \mu \Rightarrow \mu = 7$

Middle row : $\lambda = 3 + \mu \Rightarrow \lambda = 10$

Crucial !

Check the upper row : LHS $= 1 + \lambda$ RHS $= 1 + 2\mu$

$$= 1 + 10 \qquad = 1 + 144$$

$$= 11 \qquad = 15$$

As LHS $\neq$ RHS the lines do not intersect, they are skew

[4 marks]

(b) To find point A, $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ $A(2, 1, -1)$

To find point B, $\begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix} + 2 \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \\ 4 \end{pmatrix}$ $B(5, 5, 4)$

$$\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a} = \begin{pmatrix} 5 \\ 5 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$3 \times 1 + 4 \times 1 + 5 \times 0 = \sqrt{3^2 + 4^2 + 5^2}\,\sqrt{1^2 + 1^2 + 0^2}\,\cos\theta$$

$$7 = \sqrt{50}\,\sqrt{2}\,\cos\theta$$

$$\cos\theta = \frac{7}{10}$$

[6 marks]

Answer 3

(a)

$$\boldsymbol{r}_{AB} = \begin{pmatrix} 8 \\ 13 \\ -2 \end{pmatrix} + \lambda \left(\begin{pmatrix} 10 \\ 14 \\ -4 \end{pmatrix} - \begin{pmatrix} 8 \\ 13 \\ -2 \end{pmatrix} \right)$$

$$= \begin{pmatrix} 8 \\ 13 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} \quad \text{for example}$$

[3 marks]

(b) $\overrightarrow{CB} = \boldsymbol{b} - \boldsymbol{c} = \begin{pmatrix} 10 \\ 14 \\ -4 \end{pmatrix} - \begin{pmatrix} 9 \\ 9 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ -10 \end{pmatrix}$

$$\left| \overrightarrow{CB} \right| = \sqrt{1^2 + 5^2 + (-10)^2} = \sqrt{126} = 3\sqrt{14} \ \text{ or } \ 11.2$$

[2 marks]

(c)

$$\begin{pmatrix} 1 \\ 5 \\ -10 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$$

$$1 \times 2 + 5 \times 1 + (-10) \times (-2) = \sqrt{1^2 + 5^2 + (-10)^2} \sqrt{2^2 + 1^2 + (-2)^2} \ cos\,\theta$$

$$27 = \sqrt{126}\sqrt{9}\ cos\,\theta$$

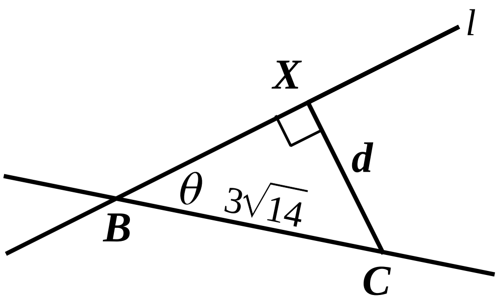
$$cos\,\theta = -\frac{27}{\sqrt{126} \times \sqrt{9}}$$

$$= \frac{3\sqrt{14}}{14}$$

$$\therefore \ \theta = 36.7°$$

[3 marks]

(d)



The shortest distance, $d = 3\sqrt{14}\ sin\,36.7° = 6.7$

[3 marks]

(e) Area $= \dfrac{1}{2} \times 6.7 \times 3\sqrt{14} \times sin\,(90 - 36.7)$

$= 30.1$

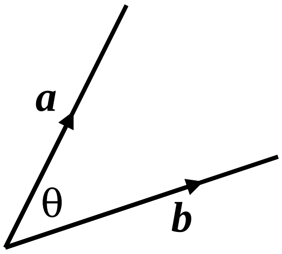
[3 marks]

Lesson 3

3.1 Mutually Perpendicular Vectors

The Scalar Product (Also called **The Dot Product**)

By definition : $\boldsymbol{a} \bullet \boldsymbol{b} = |\boldsymbol{a}|\ |\boldsymbol{b}|\ cos\ \theta$



Where $|\boldsymbol{a}|$ and $|\boldsymbol{b}|$ are the magnitudes of vectors ***a*** and ***b***

θ is the angle between vectors ***a*** and ***b***

Whenever angles are involved in a problem involving vectors, use the scalar product with the **DIRECTION PARTS** of the vector equations of the lines.

A consequence of the definition is that The Scalar Product is a very good detector of 90° angles, because if $\theta = 90°$, then $cos\ \theta = 0$.

Mutually Perpendicular Vectors

- If two vectors are mutually perpendicular, their scalar product will be zero.
- If the scalar product is zero, vectors ***a*** and ***b*** are mutually perpendicular.

3.2 Example

Prove that the following two lines are mutually perpendicular;

$$\boldsymbol{r}_1 = \begin{pmatrix} -6 \\ 5 \\ -7 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 7 \\ -14 \end{pmatrix} \quad \text{and} \quad \boldsymbol{r}_2 = \begin{pmatrix} 12 \\ 3 \\ -7 \end{pmatrix} + \mu \begin{pmatrix} 7 \\ 12 \\ 8 \end{pmatrix}$$

[3 marks]

3.3 Proof of The Scalar Product Formula In Two Dimensions

Prove that for any two vectors;

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

it follows that;

$$a_1 b_1 + a_2 b_2 = \sqrt{a_1^{\,2} + a_2^{\,2}}\ \sqrt{b_1^{\,2} + b_2^{\,2}}\ \cos\theta$$

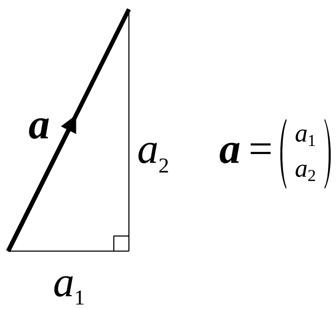
The Proof

By definition;

$$\boldsymbol{a} \bullet \boldsymbol{b} = |\boldsymbol{a}|\ |\boldsymbol{b}|\ \cos\theta$$

Let

$$\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \text{ in which case } |\boldsymbol{a}| = \sqrt{a_1^{\,2} + a_2^{\,2}}$$



Similarly

$$\boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \text{ in which case } |\boldsymbol{b}| = \sqrt{b_1^{\,2} + b_2^{\,2}}$$

Thus, from the definition

$$\begin{aligned} \boldsymbol{a} \bullet \boldsymbol{b} &= |\boldsymbol{a}|\ |\boldsymbol{b}|\ \cos\theta \\ &= \sqrt{a_1^{\,2} + a_2^{\,2}}\ \sqrt{b_1^{\,2} + b_2^{\,2}}\ \cos\theta \end{aligned}$$

However,

$$\begin{aligned} \boldsymbol{a} \bullet \boldsymbol{b} &= (\, a_1 \boldsymbol{i} + a_2 \boldsymbol{j}\,) \bullet (\, b_1 \boldsymbol{i} + b_2 \boldsymbol{j}) \\ &= a_1 b_1\, \boldsymbol{i} \bullet \boldsymbol{i} + a_1 b_2\, \boldsymbol{i} \bullet \boldsymbol{j} + a_2 b_1\, \boldsymbol{j} \bullet \boldsymbol{i} + a_2 b_2\, \boldsymbol{j} \bullet \boldsymbol{j} \\ &= a_1 b_1 + a_2 b_2 \end{aligned}$$

because $\quad \boldsymbol{i} \bullet \boldsymbol{i} = \boldsymbol{j} \bullet \boldsymbol{j} = 1,$ and $\quad \boldsymbol{i} \bullet \boldsymbol{j} = \boldsymbol{j} \bullet \boldsymbol{i} = 0$

Thus,

$$a_1 b_1 + a_2 b_2 = \sqrt{a_1^{\,2} + a_2^{\,2}}\ \sqrt{b_1^{\,2} + b_2^{\,2}}\ \cos\theta$$

$\square$

3.4 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 32

Question 1

Show that the following two lines are mutually perpendicular;

$$\boldsymbol{r}_1 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 3 \\ -4 \end{pmatrix} \qquad \textit{and} \qquad \boldsymbol{r}_2 = \begin{pmatrix} 4 \\ 7 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 7 \\ -9 \\ 2 \end{pmatrix}$$

[3 marks]

Question 2

Given that the following two lines are mutually perpendicular, find the value of a

$$\boldsymbol{r}_1 = \begin{pmatrix} 5 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} a \\ 4 \\ -3 \end{pmatrix} \qquad \textit{and} \qquad \boldsymbol{r}_2 = \begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 2a \\ 5 \end{pmatrix}$$

[3 marks]

Question 3

$$\boldsymbol{a} = p\,\boldsymbol{i} + 6\boldsymbol{j} - 4\boldsymbol{k}$$

$$\boldsymbol{b} = 5\,\boldsymbol{i} - 3\boldsymbol{j} - 7\boldsymbol{k}$$

Find the value of p if $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular

[3 marks]

Question 4

A, B and C are three points in three dimensional space with coordinates;

$$A(6,\ 3,\ 7) \qquad B(-3,\ 8,\ 5) \qquad C(-4,\ -13,\ 12)$$

(a) Determine;

(i) $\overrightarrow{AB}$

[1 mark]

(ii) $\overrightarrow{AC}$

[1 mark]

(b) Show that $\angle BAC$ is a right angle

[2 marks]

(c) Hence, or otherwise, find the area of $\triangle ABC$

[3 marks]

Question 5

Determine;

$$\begin{pmatrix} 1 \\ m \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -\frac{1}{m} \end{pmatrix}$$

[1 mark]

What well known GCSE result have you proved ?

[1 mark]

Question 6

Write down an equation of the vector line that passes through the points;

$$A(3,\ 2,\ -1) \qquad \text{and} \qquad B(5,\ 7,\ 3)$$

[2 marks]

Question 7

Write down a vector line equation for a line that passes through the point $4\,\boldsymbol{i} + \boldsymbol{j} - 2\,\boldsymbol{k}$ and which is parallel to the line with equation

$$\boldsymbol{r} = 3\,\boldsymbol{i} + 3\,\boldsymbol{j} - 5\,\boldsymbol{k} + \lambda(-2\,\boldsymbol{i} + 8\,\boldsymbol{j} + \boldsymbol{k})$$

[2 marks]

Question 8

Given that the vectors

$$\boldsymbol{u} = \begin{pmatrix} \lambda + 1 \\ 4 \\ 1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{v} = \begin{pmatrix} \lambda - 5 \\ \lambda \\ -11 \end{pmatrix}$$

are perpendicular, find the possible values of λ

[2 marks]

Question 9

This question is about finding a vector that is perpendicular to both of the lines

$$\boldsymbol{r}_1 = \begin{pmatrix} 10 \\ 2 \\ 10 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{r}_2 = \begin{pmatrix} 7 \\ 4 \\ 7 \end{pmatrix} + \mu \begin{pmatrix} 6 \\ -2 \\ 7 \end{pmatrix}$$

Without loss of generality, assume the direction vector sought is of the form;

$$\boldsymbol{d} = \begin{pmatrix} 1 \\ P \\ Q \end{pmatrix}$$

(a) Obtain an equation, in P and Q, from the fact that;

$$\begin{pmatrix} 1 \\ P \\ Q \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} = 0$$

[1 mark]

(b) Obtain a second equation, in P and Q, from the fact that;

$$\begin{pmatrix} 1 \\ P \\ Q \end{pmatrix} \bullet \begin{pmatrix} 6 \\ -2 \\ -7 \end{pmatrix} = 0$$

[1 mark]

(c) Solve the two equations simultaneously to determine the value of P and the value of Q and hence give the required direction vector which is perpendicular to both of the lines $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$.

[2 marks]

Question 10

Find a vector that is perpendicular to both of the lines

$$\boldsymbol{r}_1 = \begin{pmatrix} -4 \\ 7 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 10 \\ -3 \end{pmatrix} \quad \text{and} \quad \boldsymbol{r}_2 = \begin{pmatrix} 8 \\ -7 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 5 \\ -9 \\ 7 \end{pmatrix}$$

[4 marks]

3.5 Answers

3.5.1 Solution to 3.2 Example

$$\begin{pmatrix} 4 \\ 7 \\ -14 \end{pmatrix} \bullet \begin{pmatrix} 7 \\ 12 \\ 8 \end{pmatrix} = 4 \times 7 + 7 \times 12 + (-14) \times 8$$

$$= 28 + 84 - 112$$

$$= 0$$

$\therefore$ $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are mutually perpendicular ☐

[3 marks]

3.5.2 Solution to 3.4

Answer 1

$$\begin{pmatrix} 5 \\ 3 \\ -4 \end{pmatrix} \bullet \begin{pmatrix} 7 \\ -9 \\ 2 \end{pmatrix} = 5 \times 7 + 3 \times (-9) + (-4) \times 2$$

$$= 35 - 27 - 8$$

$$= 0$$

$\therefore$ $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are mutually perpendicular ☐

[3 marks]

Answer 2

As the lines are perpendicular, $\begin{pmatrix} a \\ 4 \\ -3 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 2a \\ 5 \end{pmatrix} = 0$

$$a \times 2 + 4 \times 2a + (-3) \times 5 = 0$$

$$2a + 8a - 15 = 0$$

$$10a = 15$$

$$a = \frac{3}{2}$$

[3 marks]

Answer 3

As the lines are perpendicular, $\begin{pmatrix} p \\ 6 \\ -4 \end{pmatrix} \bullet \begin{pmatrix} 5 \\ -3 \\ -7 \end{pmatrix} = 0$

$$p \times 5 + 6 \times (-3) + (-4) \times (-7) = 0$$

$$5p - 18 + 28 = 0$$

$$5p = -10$$

$$p = -2$$

[3 marks]

Answer 4

(a) **(i)** $\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a} = \begin{pmatrix} -3 \\ 8 \\ 5 \end{pmatrix} - \begin{pmatrix} 6 \\ 3 \\ 7 \end{pmatrix} = \begin{pmatrix} -9 \\ 5 \\ -2 \end{pmatrix}$

[1 mark]

(ii) $\overrightarrow{AC} = \boldsymbol{c} - \boldsymbol{a} = \begin{pmatrix} -4 \\ -13 \\ 12 \end{pmatrix} - \begin{pmatrix} 6 \\ 3 \\ 7 \end{pmatrix} = \begin{pmatrix} -10 \\ -16 \\ 5 \end{pmatrix}$

[1 mark]

(b) $\begin{pmatrix} -9 \\ 5 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} -10 \\ -16 \\ 5 \end{pmatrix} = (-9) \times (-10) + 5 \times (-16) + (-2) \times 5$

$= 90 - 80 - 10$

$= 0$ $\quad \therefore \angle BAC$ is a right angle ☐

[2 marks]

(c) Area $= \frac{1}{2} b\,h$

$= \frac{1}{2} \times |AB| \times |AC|$

$= \frac{1}{2} \times \sqrt{(-9)^2 + 5^2 + (-2)^2} \times \sqrt{(-10)^2 + (-16)^2 + 5^2}$

$= \frac{1}{2} \times \sqrt{110} \times \sqrt{381}$

$= 102.4$

[3 marks]

Answer 5

$$\begin{pmatrix} 1 \\ m \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -\frac{1}{m} \end{pmatrix} = 1 \times 1 + m \times \left(-\frac{1}{m}\right)$$

$$= 1 - 1$$

$$= 0$$

[1 mark]

$\therefore$ The line which is perpendicular to a line with gradient m has a gradient which is the sign changed reciprocal, that is $-\dfrac{1}{m}$

[1 mark]

Answer 6

$$r_{AB} = \overrightarrow{OA} + \lambda \overrightarrow{AB}$$

$$= \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} + \lambda \left(\begin{pmatrix} 5 \\ 7 \\ 3 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} \right)$$

$$= \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix} \quad \text{for example}$$

[2 marks]

Answer 7

$$\boldsymbol{s} = \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 8 \\ 1 \end{pmatrix}$$

[2 marks]

Answer 8

$$\boldsymbol{u} \bullet \boldsymbol{v} = 0$$

$$\begin{pmatrix} \lambda + 1 \\ 4 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} \lambda - 5 \\ \lambda \\ -11 \end{pmatrix} = 0$$

$$(\lambda + 1)(\lambda - 5) + 4\lambda - 11 = 0$$

$$\lambda^2 - 4\lambda - 5 + 4\lambda - 11 = 0$$

$$\lambda^2 - 16 = 0$$

$$\lambda^2 = 16$$

$$\lambda = \pm 4$$

[2 marks]

Answer 9

(a) $\begin{pmatrix}1\\P\\Q\end{pmatrix} \bullet \begin{pmatrix}3\\1\\2\end{pmatrix} = 0 \Rightarrow 3 + P + 2Q = 0 \Rightarrow P + 2Q = -3 \qquad (1)$

[1 mark]

(b) $\begin{pmatrix}1\\P\\Q\end{pmatrix} \bullet \begin{pmatrix}6\\-2\\-7\end{pmatrix} = 0 \Rightarrow 6 - 2P - 7Q = 0 \Rightarrow 2P + 7Q = 6 \qquad (2)$

[1 mark]

(c)

$$2P + 7Q = 6 \qquad (2)$$

$$2P + 4Q = -6 \qquad (1) \times 2$$

$$3Q = 12$$

$$Q = 4 \quad \Rightarrow P = -11$$

The vector perpendicular to both r_1 and r_2 is $\begin{pmatrix}1\\-11\\4\end{pmatrix}$

[2 marks]

Answer 10

$\begin{pmatrix}1\\P\\Q\end{pmatrix} \bullet \begin{pmatrix}4\\10\\-3\end{pmatrix} = 0 \Rightarrow 4 + 10P - 3Q = 0 \Rightarrow 10P - 3Q = -4 \qquad (1)$

[1 mark]

$\begin{pmatrix}1\\P\\Q\end{pmatrix} \bullet \begin{pmatrix}5\\-9\\7\end{pmatrix} = 0 \Rightarrow 5 - 9P + 7Q = 0 \Rightarrow -9P + 7Q = -5 \qquad (2)$

[1 mark]

$$70P - 21Q = -28 \qquad (1) \times 7$$

$$-27P + 21Q = -15 \qquad (2) \times 3$$

$$43P = -43$$

$$P = -1 \quad \Rightarrow Q = -2$$

The vector perpendicular to both r_1 and r_2 is $\begin{pmatrix}1\\-1\\-2\end{pmatrix}$

[2 marks]

4.1 Shortest Distance from Point to Line

Example

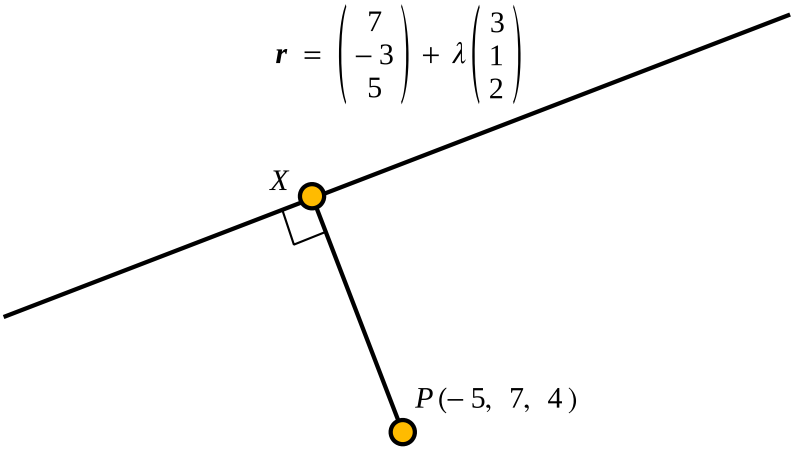
Question :

Find the shortest distance between the point $P(-5, 7, 4)$

and the line with equation $\boldsymbol{r} = \begin{pmatrix} 7 \\ -3 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$

Answer :

Call the location on the line that is closest to P the point X



[4 marks]

4.2 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 50

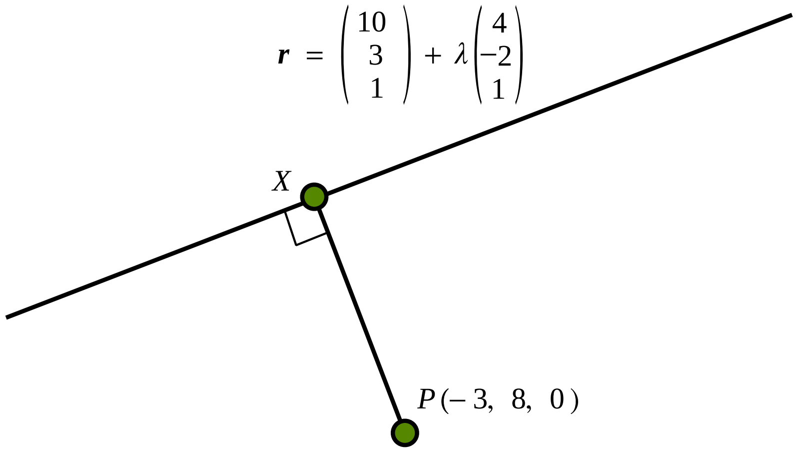
Question 1

Find the shortest distance between the point $P(-3, 8, 0)$ and the line with equation

$$\boldsymbol{r} = \begin{pmatrix} 10 \\ 3 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix}$$

Begin by calling the location on the line that is closest to P the point X

$$\boldsymbol{r} = \begin{pmatrix} 10 \\ 3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix}$$



[4 marks]

Question 2

Find the shortest distance between the point $P(7, -1, -6)$ and the line with equation

$$\boldsymbol{r} = \begin{pmatrix} -7 \\ 15 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix}$$

[5 marks]

Question 3

C4 Examination Question from January 2011, Q4

Relative to a fixed origin O, the point A has position vector $\boldsymbol{i} - 3\boldsymbol{j} + 2\boldsymbol{k}$ and the point B has position vector $-2\boldsymbol{i} + 2\boldsymbol{j} - \mathbf{k}$

The points A and B lie on a straight line l.

(a) Find $\overrightarrow{AB}$

[2 marks]

(b) Find a vector equation of l

[2 marks]

The point C has position vector $2\boldsymbol{i} + p\boldsymbol{j} - 4\boldsymbol{k}$ with respect to O, where p is a constant.

Given that AC is perpendicular to l, find

(c) the value of p

[4 marks]

(d) the distance AC

[2 marks]

Question 4

With respect to a fixed origin O, the lines l_1 and l_2 are given by the equations

$$l_1 \; : \; \mathbf{r} = (9\mathbf{i} + 13\mathbf{j} - 3\mathbf{k}) + \lambda(\mathbf{i} + 4\mathbf{j} - 2\mathbf{k})$$

$$l_2 \; : \; \mathbf{r} = (2\mathbf{i} - \mathbf{j} + \mathbf{k}) + \mu(2\mathbf{i} + \mathbf{j} + \mathbf{k})$$

where λ and μ are scalar parameters.

(a) Given that l_1 and l_2 meet, find the position vector of their point of intersection.

[5 marks]

(b) Find the acute angle between l_1 and l_2, giving your answer in degrees to 1 decimal place

[3 marks]

Given that the point A has position vector $4\,\boldsymbol{i} + 16\,\boldsymbol{j} - 3\,\boldsymbol{k}$ and that the point P lies on l_1 such that AP is perpendicular to l_1,

(c) Find the exact coordinates of P

[6 marks]

Question 5

P3 Examination Question from January 2002, Q6

Relative to a fixed origin O, the point A has position vector $4\,\boldsymbol{i} + 8\,\boldsymbol{j} - \boldsymbol{k}$, and the point B has position vector $7\,\boldsymbol{i} + 14\,\boldsymbol{j} + 5\,\boldsymbol{k}$

(a) Find the vector $\overrightarrow{AB}$

[1 mark]

(b) Calculate the cosine of $\angle OAB$

[3 marks]

(c) Show that, for all values of λ, the point P with position vector $\lambda\,\boldsymbol{i} + 2\lambda\,\boldsymbol{j} + (2\lambda - 9)\,\boldsymbol{k}$ lies on the line through A and B

[3 marks]

(d) Find the value of λ for which OP is perpendicular to AB

[3 marks]

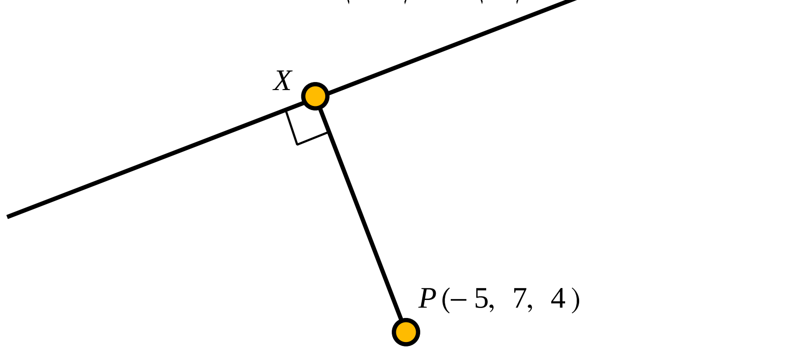
(e) Hence find the coordinates of the foot of the perpendicular from O to AB

[2 marks]

4.3 Answers

4.3.1 Solution to 4.1 Example

$$\boldsymbol{r} = \begin{pmatrix} 7 \\ -3 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$$



There must be a special value of λ that moves a point along the line from the start point of $(7, -3, 5)$ to the point X. Call this special value λ'.

Thus, $X\ (7 + 3\lambda', -3 + \lambda', 5 + 2\lambda')$

$$\overrightarrow{PX} = \boldsymbol{x} - \boldsymbol{p}$$

$$= \begin{pmatrix} 7 + 3\lambda' \\ -3 + \lambda' \\ 5 + 2\lambda' \end{pmatrix} - \begin{pmatrix} -5 \\ 7 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} 12 + 3\lambda' \\ -10 + \lambda' \\ 1 + 2\lambda' \end{pmatrix}$$

Making use of the scalar product,

$$\begin{pmatrix} 12 + 3\lambda' \\ -10 + \lambda' \\ 1 + 2\lambda' \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$$

$$36 + 9\lambda' - 10 + \lambda' + 2 + 4\lambda' = 0$$

$$28 + 14\lambda' = 0$$

$$\lambda' = -2$$

$$X\ (7 + 3\lambda', -3 + \lambda', 5 + 2\lambda') = X (1, -5, 1)$$

The distance between X and P is,

$$\sqrt{(1 - (-5))^2 + ((-5) - 7)^2 + (1 - 4)^2} = \sqrt{6^2 + (-12)^2 + (-3)^2}$$

$$= \sqrt{36 + 144 + 9}$$

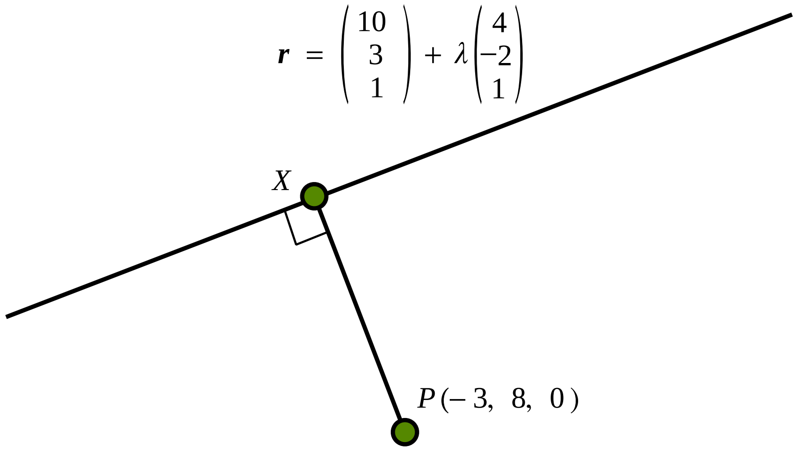
$$= \sqrt{189} \qquad (\text{ or } 3\sqrt{21} \text{ or } 13.75)$$

[4 marks]

4.3.2 Solution to 4.2 Exercise

Answer 1

$$\boldsymbol{r} = \begin{pmatrix} 10 \\ 3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix}$$



There must be a special value of λ that moves a point along the line from the start point of (10, 3, 1) to the point X. Call this special value λ'.

Thus, $X\ (\ 10 + 4\lambda',\ 3 - 2\lambda',\ 1 + \lambda'\ )$

$$\overrightarrow{PX} = \boldsymbol{x} - \boldsymbol{p}$$

$$= \begin{pmatrix} 10 + 4\lambda' \\ 3 - 2\lambda' \\ 1 + \lambda' \end{pmatrix} - \begin{pmatrix} -3 \\ 8 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 13 + 4\lambda' \\ -5 - 2\lambda' \\ 1 + \lambda' \end{pmatrix}$$

Making use of the scalar product,

$$\begin{pmatrix} 13 + 4\lambda' \\ -5 - 2\lambda' \\ 1 + \lambda' \end{pmatrix} \bullet \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix}$$

$$52 + 16\lambda' + 10 + 4\lambda' + 1 + \lambda' = 0$$

$$63 + 21\lambda' = 0$$

$$\lambda' = -3$$

$$X\ (\ 10 + 4\lambda',\ 3 - 2\lambda',\ 1 + \lambda'\ ) = X\,(-2,\ 9,\ -2)$$

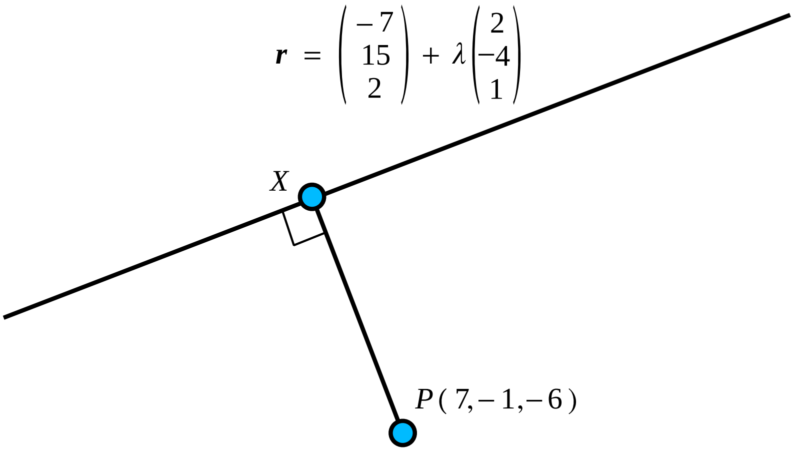
The distance between X and P is,

$$\sqrt{((-2) - (-3))^2 + (9 - 8)^2 + ((-2) - 0)^2} = \sqrt{1^2 + 1^2 + (-2)^2}$$

$$= \sqrt{6} \qquad (\text{ or } 2.45)$$

[4 marks]

Answer 2



There must be a special value of λ that moves a point along the line from the start point of $(-7,\ 15,\ 2)$ to the point X. Call this special value λ'.

Thus, $X\ (-7 + 2\lambda',\ 15 - 4\lambda',\ 2 + \lambda')$

$$\overrightarrow{PX} = \mathbf{x} - \mathbf{p}$$

$$= \begin{pmatrix} -7 + 2\lambda' \\ 15 - 4\lambda' \\ 2 + \lambda' \end{pmatrix} - \begin{pmatrix} 7 \\ -1 \\ -6 \end{pmatrix}$$

$$= \begin{pmatrix} -14 + 2\lambda' \\ 16 - 4\lambda' \\ 8 + \lambda' \end{pmatrix}$$

Making use of the scalar product,

$$\begin{pmatrix} -14 + 2\lambda' \\ 16 - 4\lambda' \\ 8 + \lambda' \end{pmatrix} \bullet \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix}$$

$$-28 + 4\lambda' - 64 + 16\lambda' + 8 + \lambda' = 0$$

$$-84 + 21\lambda' = 0$$

$$\lambda' = 4$$

$$X\ (-7 + 2\lambda',\ 15 - 4\lambda',\ 2 + \lambda') = X\,(1, -1,\ 6)$$

The distance between X and P is,

$$\sqrt{(1 - 7)^2 + ((-1) - (-1))^2 + (6 - (-6))^2} = \sqrt{(-6)^2 + 0^2 + (12)^2}$$

$$= \sqrt{36 + 0 + 144}$$

$$= \sqrt{180} \qquad (\text{ or } 6\sqrt{5} \text{ or } 13.42)$$

[5 marks]

Answer 3

(a) $\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a} = \begin{pmatrix} -2 \\ 2 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ 5 \\ -3 \end{pmatrix}$

[2 marks]

(b) $\boldsymbol{r} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 5 \\ -3 \end{pmatrix}$ for example

[2 marks]

(c) $\overrightarrow{AC} = \boldsymbol{c} - \boldsymbol{a} = \begin{pmatrix} 2 \\ p \\ -4 \end{pmatrix} - \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ p + 3 \\ -6 \end{pmatrix}$

As AC is perpendicular to l, $\begin{pmatrix} 1 \\ p + 3 \\ -6 \end{pmatrix} \bullet \begin{pmatrix} -3 \\ 5 \\ -3 \end{pmatrix} = 0$

$$-3 + 5p + 15 + 18 = 0$$

$$5p = -30$$

$$p = -6$$

[4 marks]

(d) $\left|\overrightarrow{AC}\right| = \left|\begin{pmatrix} 1 \\ -6 + 3 \\ -6 \end{pmatrix}\right| = \sqrt{1^2 + (-3)^2 + (-6)^2} = \sqrt{46}$ or 6.78

[2 marks]

Answer 4

(a)

$$\begin{pmatrix} 9 \\ 13 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

Upper row : $9 + \lambda = 2 + 2\mu \Rightarrow \lambda - 2\mu = -7$ (1)

Middle row : $13 + 4\lambda = -1 + \mu \Rightarrow 4\lambda - \mu = -14$ (2)

Solve simultaneously : $\lambda - 2\mu = -7$ (1)

$8\lambda - 2\mu = -28$ (2) $\times$ 2

$-7\lambda = 21$

$\lambda = -3 \quad \therefore \ \mu = 4\lambda + 14$

$\mu = 2$

Crucial !

Check the lower row : LHS $= -3 - 2\lambda$ RHS $= 1 + \mu$

$= -3 + 6$ $\quad = 1 + 2$

$= 3$ $\quad = 3$

As LHS = RHS the lines intersect.

To get the point of intersection,

$$\begin{pmatrix} 9 \\ 13 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} = \begin{pmatrix} 9 \\ 13 \\ -3 \end{pmatrix} - 3 \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} 6 \\ 1 \\ 3 \end{pmatrix} \quad \text{i.e. } (6,\ 1,\ 3)$$

[5 marks]

(b) Using the scalar product $\begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

$$2 + 4 - 2 = \sqrt{1^2 + 4^2 + (-2)^2}\ \sqrt{2^2 + 1^2 + 1^2}\ \cos\theta$$

$$\cos\theta = \frac{4}{\sqrt{21}\ \sqrt{6}}$$

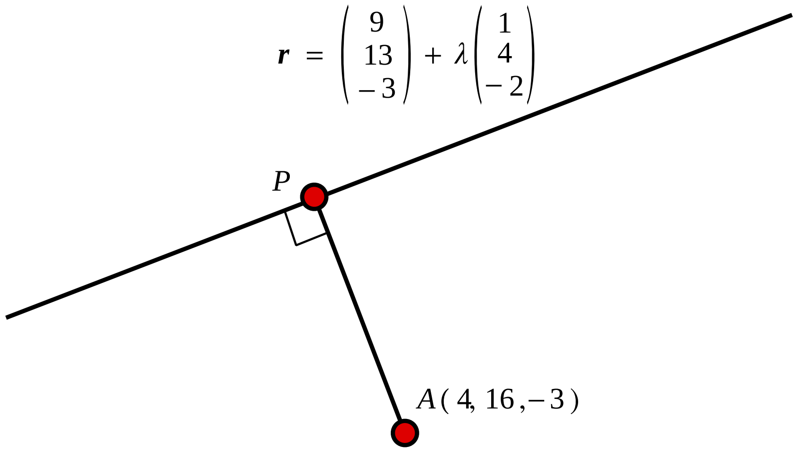
$$= \frac{4}{3\sqrt{14}}$$

$$\theta = 69.1°$$

[3 marks]

(c)

$$\boldsymbol{r} = \begin{pmatrix} 9 \\ 13 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}$$



There must be a special value of λ that moves a point along the line from the start point of (9, 13, − 3) to the point P. Call this special value λ'.

Thus, $P\ (9 + \lambda',\ 13 + 4\lambda',\ -3 - 2\lambda')$

$$\overrightarrow{AP} = \boldsymbol{p} - \boldsymbol{a}$$

$$= \begin{pmatrix} 9 + \lambda' \\ 13 + 4\lambda' \\ -3 - 2\lambda' \end{pmatrix} - \begin{pmatrix} 4 \\ 16 \\ -3 \end{pmatrix}$$

$$= \begin{pmatrix} 5 + \lambda' \\ -3 + 4\lambda' \\ -2\lambda' \end{pmatrix}$$

Making use of the scalar product,

$$\begin{pmatrix} 5 + \lambda' \\ -3 + 4\lambda' \\ -2\lambda' \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}$$

$$5 + \lambda' - 12 + 16\lambda' + 4\lambda' = 0$$

$$-7 + 21\lambda' = 0$$

$$\lambda' = \frac{1}{3}$$

$$P\ (9 + \lambda',\ 13 + 4\lambda',\ -3 - 2\lambda') = P\left(9\frac{1}{3},\ 14\frac{1}{3},\ -3\frac{2}{3}\right)$$

[6 marks]

Answer 5

(a) $\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a} = \begin{pmatrix} 7 \\ 14 \\ 5 \end{pmatrix} - \begin{pmatrix} 4 \\ 8 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 6 \end{pmatrix}$ or $\overrightarrow{AB} = 3\boldsymbol{i} + 6\boldsymbol{j} + 6\boldsymbol{k}$

[2 marks]

(b) $\overrightarrow{AO} \bullet \overrightarrow{AB} = \begin{pmatrix} -4 \\ -8 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 6 \\ 6 \end{pmatrix}$

$$(-4) \times 3 + (-8) \times 6 + 1 \times 6 = \sqrt{16 + 64 + 1}\ \sqrt{9 + 36 + 36}\ cos\ \theta$$

$$-12 - 48 + 6 = \sqrt{81}\ \sqrt{81}\ cos\ \theta$$

$$cos\ \theta = -\frac{2}{3}$$ Direction Important, see lesson 5

[3 marks]

(c) The line given is $\boldsymbol{r} = \begin{pmatrix} 0 \\ 0 \\ -9 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

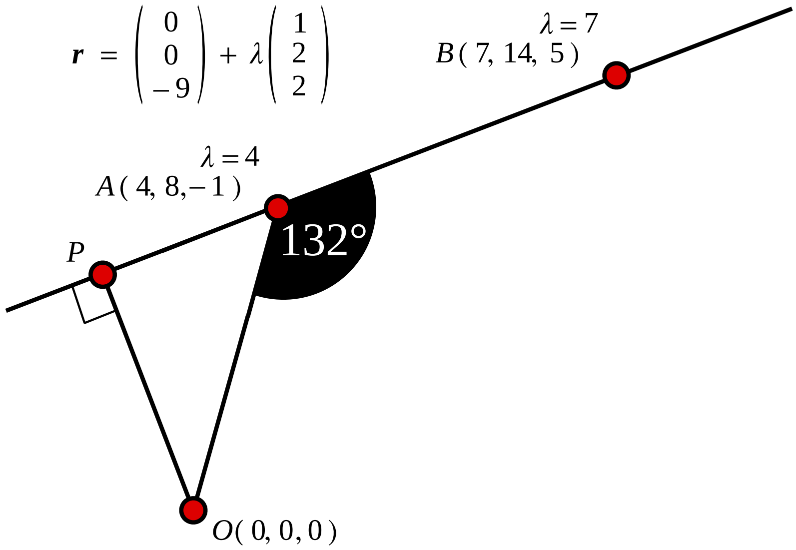
By letting $\lambda = 4$ point A is reached

By letting $\lambda = 7$ point B is reached

$\therefore$ The point P represented by the line $\boldsymbol{r}$ passes through A and B

[3 marks]

(d)



There must be a special value of λ that moves a point along the line from the start point of $(0, \ 0, -9)$ to the point P. Call this special value λ'.

Thus, $P \ (0 + \lambda', \ 0 + 2\lambda', -9 + 2\lambda')$

$$\overrightarrow{OP} = \boldsymbol{p} - \boldsymbol{o}$$

$$= \begin{pmatrix} \lambda' \\ 2\lambda' \\ -9 + 2\lambda' \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda' \\ 2\lambda' \\ -9 + 2\lambda' \end{pmatrix}$$

Making use of the scalar product,

$$\begin{pmatrix} \lambda' \\ 2\lambda' \\ -9 + 2\lambda' \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

$$\lambda' + 4\lambda' - 18 + 4\lambda' = 0$$

$$-18 + 9\lambda' = 0$$

$$\lambda' = 2$$

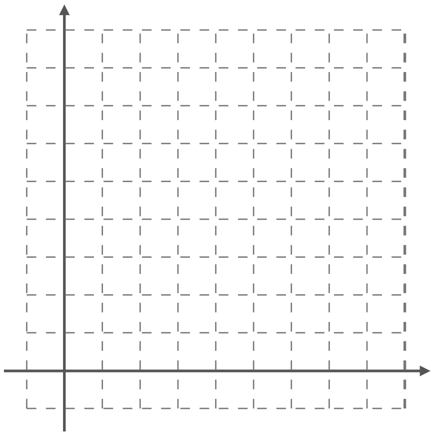
[3 marks]

$$P \ (0 + \lambda', \ 0 + 2\lambda', -9 + 2\lambda') = P(2, \ 4, -5)$$

[2 marks]

5.1 Which Angle From The Scalar Product ?

$\triangle ABC$ is formed from the points A (2, 3), B(5, 1) and C (4, 7)
Determine the size of $\angle$CAB in degrees correct to one decimal place.



[3 marks]

5.2 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 50

Question 1

You are given that,

$$\overrightarrow{AB} = \begin{pmatrix} -4 \\ 1 \\ -8 \end{pmatrix} \quad \text{and} \quad \overrightarrow{BC} = \begin{pmatrix} 3 \\ -2 \\ -6 \end{pmatrix}$$

(i) Write down the vector $\overrightarrow{BA}$

[1 mark]

(ii) Determine $\left|\overrightarrow{BA}\right|$

[1 mark]

(iii) Determine $\left|\overrightarrow{BC}\right|$

[1 mark]

(iv) Determine $\angle ABC$ in degrees, accurate to 1 decimal place.

[2 marks]

(v) Find the area of $\triangle ABC$

[1 mark]

Question 2

$\triangle ABC$ is formed from the points $A\,(-4, 7, 3)$, $B(-8, 5, 2)$ and $C\,(9, 1, -6)$

Determine the size of $\angle BAC$ in degrees correct to one decimal place.

[3 marks]

Question 3

C4 Examination Question from January 2007, Q7

The point A has position vector $\boldsymbol{a} = 2\,\boldsymbol{i} + 2\,\boldsymbol{j} + \boldsymbol{k}$ and the point B has position vector $\boldsymbol{b} = \boldsymbol{i} + \boldsymbol{j} - 4\,\boldsymbol{k}$ relative to an origin O.

(a) Find the position vector of the point C, with position vector $\boldsymbol{c}$ given by $\boldsymbol{c} = \boldsymbol{a} + \boldsymbol{b}$

[1 mark]

(b) Show that $OACB$ is a rectangle, and find its exact area

[6 marks]

The diagonals of the rectangle, *AB* and *OC* meet at the point *D*

(c) Write down the position vector of the point *D*

[1 mark]

(d) Find the size of the angle *ADC*

[6 marks]

Question 4

C4 Examination Question from June 2010, Q7 (edited)

The line l_1 has equation;

$$\mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

where λ is a scalar parameter.

The line l_2 has equation;

$$\mathbf{r} = \begin{pmatrix} 0 \\ 9 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix}$$

where μ is a scalar parameter.

Given that l_1 and l_2 meet at the point C, find

(a) the coordinates of C

[3 marks]

The point A is the point on l_1 where $\lambda = 0$ and the point B is the point on l_2 where $\mu = -1$

(b) Find the size of the angle ACB.
Give your answer in degrees to 2 decimal places.

[4 marks]

(c) Hence, or otherwise, find the area of the triangle ABC

[5 marks]

Question 5

C4 Examination Question from January 2012, Q7

Relative to a fixed origin O, the point A has position vector $2\,\boldsymbol{i} - \boldsymbol{j} + 5\,\boldsymbol{k}$,
the point B has position vector $5\,\boldsymbol{i} + 2\,\boldsymbol{j} + 10\,\boldsymbol{k}$
and the point D has position vector $-\,\boldsymbol{i} + \boldsymbol{j} + 4\,\boldsymbol{k}$

The line l passes through the points A and B

(a) Find the vector $\overrightarrow{AB}$

[2 marks]

(b) Find a vector equation for the line l

[2 marks]

(c) Show that the size of the angle BAD is 109°, to the nearest degree.

[4 marks]

The points A, B and D, together with a point C, are the vertices of the parallelogram $ABCD$, where $\overrightarrow{AB} = \overrightarrow{DC}$

(d) Find the position vector of C

[2 marks]

(e) Find the area of the parallelogram $ABCD$, giving your answer to 3 significant figures.

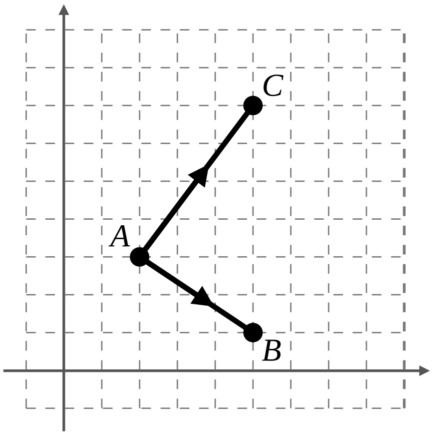
[3 marks]

(f) Find the shortest distance from the point D to the line l, giving your answer to 3 significant figures.

[2 marks]

5.3 Answers

5.3.1 Solution to the introductory example



To get the correct angle, $\angle CAB$, it is necessary[†] to get both direction vectors ***away*** from the point A

Thus the calculation proceeds as follows...

$$\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a} = \begin{pmatrix} 5 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

$$\overrightarrow{AC} = \boldsymbol{c} - \boldsymbol{a} = \begin{pmatrix} 4 \\ 7 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} 3 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

$$\begin{aligned} 3 \times 2 + (-2) \times 4 &= \sqrt{3^2 + (-2)^2}\ \sqrt{2^2 + 4^2}\ \cos\theta \\ -2 &= \sqrt{13}\ \sqrt{20}\ \cos\theta - 0.124 \\ \cos\theta &= -0.124 \qquad \therefore \quad \theta = 97.1° \end{aligned}$$

[3 marks]

† Having both direction vectors towards the point A also works but having one towards A and one away from A will get the angle's compliment.
i.e. $180 - \angle CAB$.
Often examinations questions are about directions between lines in which case the angle wanted is not as tightly specified, and such questions usually ask for the acute angle between the lines. i.e. The angle less than 90°.

5.3.2 Solutions to 5.2 Exercise

Answer 1

(i) $\overrightarrow{BA} = \begin{pmatrix} 4 \\ -1 \\ 8 \end{pmatrix}$

[1 mark]

(ii) $\left|\overrightarrow{BA}\right| = 9$

[1 mark]

(iii) $\left|\overrightarrow{BC}\right| = 7$

[1 mark]

(iv)

$$\overrightarrow{BA} \bullet \overrightarrow{BC} = \begin{pmatrix} 4 \\ -1 \\ 8 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ -2 \\ -6 \end{pmatrix}$$

$$12 + 2 - 48 = 9 \times 7 \times cos\ (\angle ABC)$$

$$cos\ (\angle ABC) = \frac{-34}{63}$$

$$\angle ABC = 122.7°$$

[2 marks]

(v)

$$\text{Area } \Delta = 0.5 \times 7 \times 9 \times sin\ (122.7°)$$

$$= 26.5$$

[2 marks]

Answer 2

$$\overrightarrow{AB} = \begin{pmatrix} -8 \\ 5 \\ 2 \end{pmatrix} - \begin{pmatrix} -4 \\ 7 \\ 3 \end{pmatrix} = \begin{pmatrix} -4 \\ -2 \\ -1 \end{pmatrix}$$

$$\overrightarrow{AC} = \begin{pmatrix} 9 \\ 1 \\ -6 \end{pmatrix} - \begin{pmatrix} -4 \\ 7 \\ 3 \end{pmatrix} = \begin{pmatrix} 13 \\ -6 \\ -9 \end{pmatrix}$$

$$\overrightarrow{AB} \bullet \overrightarrow{AC} = \begin{pmatrix} -4 \\ -2 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} 13 \\ -6 \\ -9 \end{pmatrix}$$

$$-52 + 12 + 9 = \sqrt{21}\ \sqrt{286}\ cos\,(\angle BAC)$$

$$cos\ (\angle BAC) = \frac{-31}{77.498}$$

$$\angle ABC = 113.6°$$

[4 marks]

Answer 3

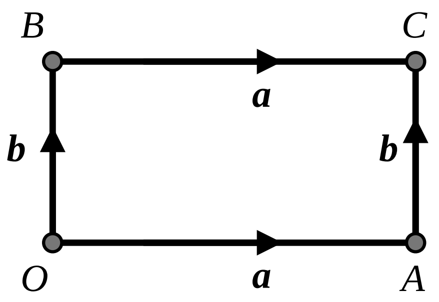
(a)

$$\boldsymbol{c} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ -4 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ -3 \end{pmatrix}$$

$$c = 3\boldsymbol{i} + 3\boldsymbol{j} - 3\boldsymbol{k}$$

[1 mark]

(b) As $\boldsymbol{c} = \boldsymbol{a} + \boldsymbol{b}$ we could simply show that any one of the four vertex angles is 90° and it follows that all four angles are 90° and thus that *OABC* is a rectangle.



Alternatively, show that $\boldsymbol{a} \bullet \boldsymbol{b} = 0$

Thus,

$$a \bullet b = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 1 \\ -4 \end{pmatrix}$$

$$= 2 + 2 - 4$$

$$= 0$$

$\therefore$ As the scalar product is zero each angle of quadrilateral *OACB* is 90° and so *OACB* is a rectangle. ☐

$$Area = \sqrt{2^2 + 2^2 + 1^2} \times \sqrt{1^2 + 1^2 + (-4)^2}$$

$$= 3 \times \sqrt{18}$$

$$= 9\sqrt{2} \qquad * \text{ Exact area asked for } *$$

[6 marks]

(c)

$$\overrightarrow{OD} = \frac{1}{2}\overrightarrow{OC}$$

$$= \frac{1}{2}(3\boldsymbol{i} + 3\boldsymbol{j} - 3\boldsymbol{k})$$

$$= 1.5\boldsymbol{i} + 1.5\boldsymbol{j} - 1.5\boldsymbol{k}$$

[1 mark]

(d)

$$\overrightarrow{DA} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 1.5 \\ 1.5 \\ -1.5 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0.5 \\ 2.5 \end{pmatrix}$$

$$\overrightarrow{DC} = \begin{pmatrix} 3 \\ 3 \\ -3 \end{pmatrix} - \begin{pmatrix} 1.5 \\ 1.5 \\ -1.5 \end{pmatrix} = \begin{pmatrix} 1.5 \\ 1.5 \\ -1.5 \end{pmatrix}$$

$$\overrightarrow{DA} \bullet \overrightarrow{DC} = \begin{pmatrix} 0.5 \\ 0.5 \\ 2.5 \end{pmatrix} \bullet \begin{pmatrix} 1.5 \\ 1.5 \\ -1.5 \end{pmatrix}$$

$$0.75 + 0.75 - 3.75 = \sqrt{6.75}\ \sqrt{6.75}\ cos\,(\angle ADC)$$

$$cos\ (\angle ADC) = \frac{-2.25}{6.75}$$

$$cos\ (\angle ADC) = \frac{-1}{3}$$

$$\angle ADC = 109.47122°$$

[6 marks]

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 28

6.1 Test

Question 1

Two points, V and W have coordinates $V\ (9, 8, -7)$ and $W\ (11, 11, -1)$
In addition, O is the origin with coordinates $O\ (0, 0, 0)$.

(i) Determine the vector $\overrightarrow{VW}$

[2 marks]

(ii) Calculate $\left|\overrightarrow{VW}\right|$

[2 marks]

(iii) Show that $\overrightarrow{VW} \bullet \overrightarrow{VO} = 0$

[2 marks]

(iv) What does part (c) tell you about $\triangle OVW$?

[1 mark]

Question 2

(i) Determine the vector equation of the line, l, that passes through the points A and B with coordinates $A\,(2, 8, 3)$ and $B\,(7, 5, 1)$

[3 marks]

(ii) State, giving a reason if the point $C\,(-8, 2, 7)$ is on the line , l

[2 marks]

Question 3

With respect to an origin O, the position vectors of the points L and M are $\boldsymbol{i} - \boldsymbol{j} + 3\,\boldsymbol{k}$ and $2\,\boldsymbol{i} - 4\,\boldsymbol{j} + 2\,\boldsymbol{k}$ respectively.

(i) Write down the vector $\overrightarrow{LM}$

[1 mark]

(ii) Show that $|\overrightarrow{OL}| = |\overrightarrow{LM}|$

[2 marks]

(iii) Find $\angle OLM$, giving your answer to the nearest tenth of a degree

[3 marks]

Question 4

C4 Examination Question from January 2008, Q6

The points A and B have position vector $2\,\boldsymbol{i} + 6\,\boldsymbol{j} - \boldsymbol{k}$ and $3\,\boldsymbol{i} + 4\,\boldsymbol{j} + \boldsymbol{k}$ respectively.

The line l_1 passes through the points A and B

(a) Find the vector $\overrightarrow{AB}$

[2 marks]

(b) Find a vector equation for the line l_1

[2 marks]

A second line l_2 passes through the origin and is parallel to the vector $\boldsymbol{i} + \boldsymbol{k}$

The line l_1 meets the line l_2 at the point C

(c) Find the acute angle between l_1 and l_2

[3 marks]

(d) Find the position vector of the point C

[4 marks]

6.2 Answers

Answer 1

(i) $\overrightarrow{VW} = \boldsymbol{w} - \boldsymbol{v} = \begin{pmatrix} 11 \\ 11 \\ -1 \end{pmatrix} - \begin{pmatrix} 9 \\ 8 \\ -7 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}$

[2 marks]

(ii) $\left|\overrightarrow{VW}\right| = \sqrt{2^2 + 3^2 + 6^2} = \sqrt{49} = 7$

[2 marks]

(iii) To show, $\overrightarrow{VW} \bullet \overrightarrow{VO}$

$$\begin{aligned} \text{LHS} &= \overrightarrow{VW} \bullet \overrightarrow{VO} \\ &= \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix} \bullet \begin{pmatrix} -9 \\ -8 \\ 7 \end{pmatrix} \\ &= -18 - 24 + 42 \\ &= 0 \\ &= \text{RHS} \quad \square \end{aligned}$$

[2 marks]

(iv) $\triangle OVW$ is right angled at V

[1 mark]

Answer 2

(i) $\boldsymbol{r} = \begin{pmatrix} 2 \\ 8 \\ 3 \end{pmatrix} + \lambda\left(\begin{pmatrix} 7 \\ 5 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 8 \\ 3 \end{pmatrix}\right) \Rightarrow \boldsymbol{r} = \begin{pmatrix} 2 \\ 8 \\ 3 \end{pmatrix} + \lambda\begin{pmatrix} 5 \\ -3 \\ -2 \end{pmatrix}$ for example

[3 marks]

(ii) Try $\lambda = -2$,

$$\begin{aligned} \boldsymbol{r} &= \begin{pmatrix} 2 \\ 8 \\ 3 \end{pmatrix} + \lambda\begin{pmatrix} 5 \\ -3 \\ -2 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ 8 \\ 3 \end{pmatrix} - 2\begin{pmatrix} 5 \\ -3 \\ -2 \end{pmatrix} \\ &= \begin{pmatrix} -8 \\ 14 \\ 7 \end{pmatrix} \end{aligned}$$

$\therefore$ $(-8,\ 2,\ 7)$ is not on the line, l

[2 marks]

Answer 3

(i) $\overrightarrow{LM} = \boldsymbol{m} - \boldsymbol{l} = \begin{pmatrix} 2 \\ -4 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix}$

[2 marks]

(ii) $\left|\overrightarrow{OL}\right| = \sqrt{1^2 + (-1)^2 + 3^2} = \sqrt{11}$

$$\left|\overrightarrow{LM}\right| = \sqrt{1^2 + (-3)^2 + (-1)^2} = \sqrt{11}$$

$$\therefore \left|\overrightarrow{OL}\right| = \left|\overrightarrow{LM}\right| \qquad \square$$

[2 marks]

(iii) Making use of the scalar product,

$$\overrightarrow{LO} \bullet \overrightarrow{LM}$$

$$\begin{pmatrix} -1 \\ 1 \\ -3 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix}$$

$$-1 - 3 + 3 = \sqrt{11}\ \sqrt{11}\ cos\ \theta$$

$$cos\ \theta = -\frac{1}{11}$$

$$\theta = 95.2°$$

[3 marks]

Answer 4

(a) $\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 6 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$

[2 marks]

(b) $l_1 : \boldsymbol{r} = \begin{pmatrix} 2 \\ 6 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ for example

[2 marks]

(c) Making use of the scalar product,

$$\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$1 + 0 + 2 = \sqrt{1^2 + (-2)^2 + 2^2}\ \sqrt{1^2 + 0^2 + 1^2}\ cos\theta$$

$$cos\ \theta = \frac{3}{\sqrt{9}\ \sqrt{2}}$$

$$= \frac{1}{\sqrt{2}}$$

$$\theta = 45°$$

[3 marks]

(d)

$$\begin{pmatrix} 2 \\ 6 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = \mu \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Middle row : $6 - 2\lambda = 0 \Rightarrow \lambda = 3$

To get point C, $\begin{pmatrix} 2 \\ 6 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \\ -1 \end{pmatrix} + 3 \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$

$$= \begin{pmatrix} 5 \\ 0 \\ 5 \end{pmatrix} \text{ i.e. } C\,(5,\ 0,\ 5)$$

[4 marks]