10.4 Answers

10.4.1 Solution to 10.2 Example

(a) The angle of the line with the positive *x*-axis is given by,

$$\theta = \arctan\left(\frac{1}{\sqrt{3}}\right)$$

$$= 30° \qquad \text{or } \frac{\pi}{6} \text{ radians}$$

The matrix to reflect in the line $y = \frac{1}{\sqrt{3}}x$ is thus,

$$\mathbf{M} = \begin{pmatrix} \cos 60 & \sin 60 \\ \sin 60 & -\cos 60 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

Using this matrix to move the point,

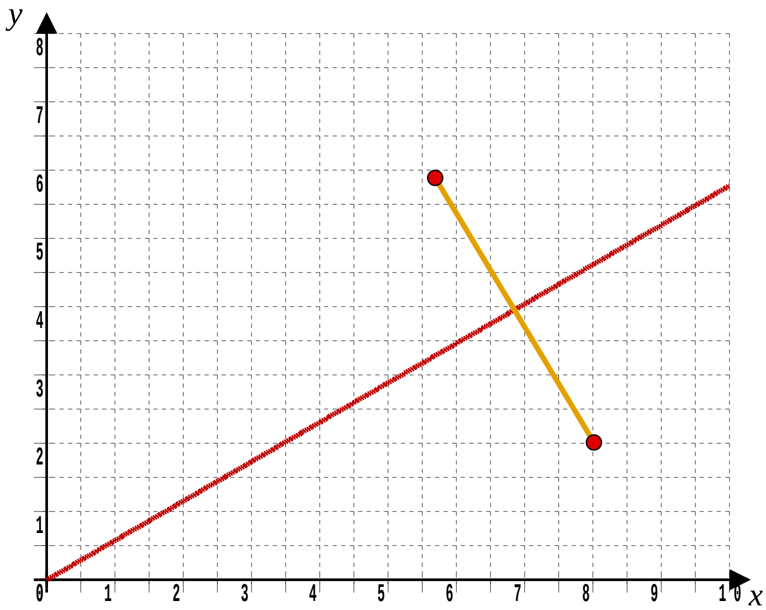
$$\frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}\begin{pmatrix} 8 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}\begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 + \sqrt{3} \\ 4\sqrt{3} - 1 \end{pmatrix}$$

$$(8,\ 2) \rightarrow (4 + \sqrt{3},\ 4\sqrt{3} - 1)$$

[4 marks]

(b) (8, 2) → (5.7, 5.9) for plotting on the graph.



[1 mark]

10.4.2 Solutions to 10.3 Exercise

Answer 1

(i) $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$: Reflection in the line $y = x$

$\mathbf{B} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$: Rotation of 270° about (0, 0)

[4 marks]

(ii)
$$\mathbf{AB}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix}$$
$$= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix}$$
$$= \begin{pmatrix} -p \\ q \end{pmatrix}$$
$$(p,\ q) \rightarrow (-p,\ q)$$

[2 marks]

Answer 2

(i)
$$\begin{pmatrix} cos\ 2\theta & sin\ 2\theta \\ sin\ 2\theta & -cos\ 2\theta \end{pmatrix}\begin{pmatrix} cos\ 2\theta & sin\ 2\theta \\ sin\ 2\theta & -cos\ 2\theta \end{pmatrix}$$
$$= \begin{pmatrix} cos^2\ 2\theta + sin^2\ 2\theta & cos\ 2\theta\ sin\ 2\theta - sin\ 2\theta\ cos\ 2\theta \\ sin\ 2\theta\ cos\ 2\theta - cos\ 2\theta\ sin\ 2\theta & sin^2\ 2\theta + cos^2\ 2\theta \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

[2 marks]

(ii) Reflecting any point in a mirror, and then reflecting it back is the same as doing nothing, as indicated by the identity matrix answer in part (i)

[1 mark]

(iii)
$$\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}^{11} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

[2 marks]

Answer 3

(a) A matrix is singular when its determinant is zero.

$$|\mathbf{A}| = 0$$

$$\begin{vmatrix} 1 & 2 \\ 1 & k \end{vmatrix} = 0$$

$$k - 2 = 0$$

$$k = 2$$

[1 mark]

(b) $\mathbf{B} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$: Reflection in the *y*-axis

[1 mark]

(c) **(i)** Verify that $\mathbf{A}^{-1}\mathbf{B}^{-1} = (\mathbf{BA})^{-1}$

$$\text{LHS} = \mathbf{A}^{-1}\mathbf{B}^{-1}$$

$$= \begin{pmatrix} 1 & 2 \\ 1 & k \end{pmatrix}^{-1} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}^{-1}$$

$$= \frac{1}{k-2}\begin{pmatrix} k & -2 \\ -1 & 1 \end{pmatrix} \frac{1}{(-1)}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \frac{1}{k-2}\begin{pmatrix} k & -2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{showing B is self inverse}$$

$$= \frac{1}{k-2}\begin{pmatrix} -k & -2 \\ 1 & 1 \end{pmatrix} \qquad k \neq 2$$

$$\text{RHS} = (\mathbf{BA})^{-1}$$

$$= \left(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & k \end{pmatrix} \right)^{-1}$$

$$= \begin{pmatrix} -1 & -2 \\ 1 & k \end{pmatrix}^{-1}$$

$$= \frac{1}{-k+2}\begin{pmatrix} k & 2 \\ -1 & -1 \end{pmatrix}$$

$$= \frac{1}{k-2}\begin{pmatrix} -k & -2 \\ 1 & 1 \end{pmatrix} \qquad k \neq 2$$

$$= \text{LHS}$$

Thus, for matrices **A** and **B** the claimed result is true.

[4 marks]

(ii) If it is true that $\mathbf{M}^{-1}\mathbf{N}^{-1} = (\mathbf{NM})^{-1}$

then $\mathbf{NM} \times \mathbf{M}^{-1}\mathbf{N}^{-1}$ will yield the identity matrix, $\mathbf{I}$

$$\begin{aligned} \text{LHS} &= \mathbf{NM} \times \mathbf{M}^{-1}\mathbf{N}^{-1} \\ &= \mathbf{N}\left(\mathbf{M}\ \mathbf{M}^{-1}\right)\mathbf{N}^{-1} \\ &= \mathbf{N}\,\mathbf{I}\,\mathbf{N}^{-1} \\ &= \mathbf{N}\,\mathbf{N}^{-1} \\ &= \mathbf{I} \\ &= \text{RHS} \end{aligned}$$

So the claimed result is indeed true.

[4 marks]

Answer 4

(i) Reflection in the y-axis : $\mathbf{M}_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

[1 mark]

(ii) Rotation of $\dfrac{\pi}{2}$ radians about the origin : $\mathbf{M}_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

[1 mark]

(iii) $\mathbf{M}_3 = \mathbf{M}_1\,\mathbf{M}_2$ (Notice the order)

$$= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

[1 mark]

(iv) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$: Reflection in the line $y = x$

[1 mark]

Answer 5

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

[4 marks]

Answer 6

(a) **(i)** $\mathbf{M}^2 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

[1 mark]

(ii) $\mathbf{M} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$: Reflection in the line $y = -x$

[1 mark]

(iii) The opposite of a reflection is the same reflection.
For example, a point is reflected, and then reflected back to its original location.

[1 mark]

(b) $|\mathbf{N}| = \begin{vmatrix} k+1 & 0 \\ k & k+2 \end{vmatrix}$

$= (k+1)(k+2) - k \times 0$

$= (k+1)(k+2)$

If **N** represents a reflection then $|\mathbf{N}| = -1$

$\therefore$ we require $(k+1)(k+2) = -1$

$k^2 + 3k + 2 = -1$

$k^2 + 3k + 3 = 0$

But this has no solutions (it has a negative discriminant)

So **N** can never represent a reflection.

[4 marks]