3.4 Answers

3.4.1 Solution to 3.3 Example (Teaching Video)

Step 1 : Find the determinant of of **A**, *det* **A**

$$\begin{vmatrix} 2 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{vmatrix} = 2\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} - 1\begin{vmatrix} 3 & 1 \\ 2 & 2 \end{vmatrix} + 1\begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix}$$

$$= 2 \times 3 - 1 \times 4 + 1 \times (-1)$$

$$= 1$$

Step 2 : Form, **M**, the matrix of minors of **A** by replacing each of the nine elements of the matrix **A** with that element's minor.

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 3 & 1 \\ 2 & 2 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} 3 & 4 & -1 \\ 1 & 2 & 0 \\ -1 & -1 & 1 \end{pmatrix}$$

Step 3 : Form, **C**, the matrix of cofactors by reversing the sign of some elements of the matrix of minors according to the pattern matrix,

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$+$ indicates no change whereas $-$ indicate change

$$\mathbf{C} = \begin{pmatrix} 3 & -4 & -1 \\ -1 & 2 & 0 \\ -1 & 1 & 1 \end{pmatrix}$$

Step 4 : Write down, $\mathbf{C}^{T}$, the transpose of the matrix of cofactors.

$$\mathbf{C}^{T} = \begin{pmatrix} 3 & -1 & -1 \\ -4 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix}$$

Step 5 : $\mathbf{A}^{-1} = \dfrac{1}{det\ \mathbf{A}} \mathbf{C}^{T}$

$$\mathbf{A}^{-1} = \frac{1}{1}\begin{pmatrix} 3 & -1 & -1 \\ -4 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -1 & -1 \\ -4 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix}$$

[6 marks]

3.4.2 Solutions to 3.3 Exercise

Answer 1

$$\mathbf{A}^{-1}\mathbf{A} = \begin{pmatrix} 3 & -1 & -1 \\ -4 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 6-3-2 & 3-2-1 & 3-1-2 \\ -8+6+2 & -4+4+1 & -4+2+2 \\ -2+0+2 & -1+0+1 & -1+0+2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \mathbf{I}$$

This is not a surprise as for any matrix $\mathbf{X}$, which has an inverse $\mathbf{X}^{-1}$,

$$\mathbf{X}\,\mathbf{X}^{-1} = \mathbf{X}^{-1}\,\mathbf{X} = \mathbf{I}$$

where $\mathbf{I}$ is the identity matrix

[3 marks]

Answer 2

Step 1 : Find the determinant of of **W**, *det* **W**

$$\begin{vmatrix} -4 & 5 & 2 \\ -5 & 6 & 2 \\ 8 & -9 & -3 \end{vmatrix} = -4\begin{vmatrix} 6 & 2 \\ -9 & -3 \end{vmatrix} - 5\begin{vmatrix} -5 & 2 \\ 8 & -3 \end{vmatrix} + 2\begin{vmatrix} -5 & 6 \\ 8 & -9 \end{vmatrix}$$

$$= (-4) \times 0 - 5 \times (-1) + 2 \times (-3)$$

$$= -1$$

Step 2 : Form, **M**, the matrix of minors of **A** by replacing each of the nine elements of the matrix **A** with that element's minor.

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} 6 & 2 \\ -9 & -3 \end{vmatrix} & \begin{vmatrix} -5 & 2 \\ 8 & -3 \end{vmatrix} & \begin{vmatrix} -5 & 6 \\ 8 & -9 \end{vmatrix} \\ \begin{vmatrix} 5 & 2 \\ -9 & -3 \end{vmatrix} & \begin{vmatrix} -4 & 2 \\ 8 & -3 \end{vmatrix} & \begin{vmatrix} -4 & 5 \\ 8 & -9 \end{vmatrix} \\ \begin{vmatrix} 5 & 2 \\ 6 & 2 \end{vmatrix} & \begin{vmatrix} -4 & 2 \\ -5 & 2 \end{vmatrix} & \begin{vmatrix} -4 & 5 \\ -5 & 6 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} 0 & -1 & -3 \\ 3 & -4 & -4 \\ -2 & 2 & 1 \end{pmatrix}$$

Step 3 : Form, **C**, the matrix of cofactors by reversing the sign of some elements of the matrix of minors according to the pattern matrix,

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$+$ indicates no change whereas $-$ indicate change

$$\mathbf{C} = \begin{pmatrix} 0 & 1 & -3 \\ -3 & -4 & 4 \\ -2 & -2 & 1 \end{pmatrix}$$

Step 4 : Write down, $\mathbf{C}^T$, the transpose of the matrix of cofactors.

$$\mathbf{C}^T = \begin{pmatrix} 0 & -3 & -2 \\ 1 & -4 & -2 \\ -3 & 4 & 1 \end{pmatrix}$$

Step 5 : $\mathbf{W}^{-1} = \dfrac{1}{det\ \mathbf{A}}\mathbf{C}^T$

$$\mathbf{W}^{-1} = \frac{1}{(-1)}\begin{pmatrix} 0 & -3 & -2 \\ 1 & -4 & -2 \\ -3 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 3 & 2 \\ -1 & 4 & 2 \\ 3 & -4 & -1 \end{pmatrix}$$

[6 marks]

Answer 3

Step 1 : Find the determinant of of **R**, *det* **R**

$$\begin{vmatrix} 3 & 2 & -2 \\ -2 & k & 0 \\ -1 & -3 & 3 \end{vmatrix} = 3\begin{vmatrix} k & 0 \\ -3 & 3 \end{vmatrix} - 2\begin{vmatrix} -2 & 0 \\ -1 & 3 \end{vmatrix} - 2\begin{vmatrix} -2 & k \\ -1 & -3 \end{vmatrix}$$

$$= 3 \times 3k - 2 \times (-6) - 2 \times (6 + k)$$

$$= 9k + 12 - 12 - 2k$$

$$= 7k$$

Step 2 : Form, **M**, the matrix of minors of **A** by replacing each of the nine elements of the matrix **A** with that element's minor.

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} k & 0 \\ -3 & 3 \end{vmatrix} & \begin{vmatrix} -2 & 0 \\ -1 & 3 \end{vmatrix} & \begin{vmatrix} -2 & k \\ -1 & -3 \end{vmatrix} \\ \begin{vmatrix} 2 & -2 \\ -3 & 3 \end{vmatrix} & \begin{vmatrix} 3 & -2 \\ -1 & 3 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ -1 & -3 \end{vmatrix} \\ \begin{vmatrix} 2 & -2 \\ k & 0 \end{vmatrix} & \begin{vmatrix} 3 & -2 \\ -2 & 0 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ -2 & k \end{vmatrix} \end{pmatrix} = \begin{pmatrix} 3k & -6 & 6 + k \\ 0 & 7 & -7 \\ 2k & -4 & 3k + 4 \end{pmatrix}$$

Step 3 : Form, **C**, the matrix of cofactors by reversing the sign of some elements of the matrix of minors according to the pattern matrix,

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$+$ indicates no change whereas $-$ indicate change

$$\mathbf{C} = \begin{pmatrix} 3k & 6 & 6 + k \\ 0 & 7 & 7 \\ 2k & 4 & 3k + 4 \end{pmatrix}$$

Step 4 : Write down, $\mathbf{C}^{\mathrm{T}}$, the transpose of the matrix of cofactors.

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} 3k & 0 & 2k \\ 6 & 7 & 4 \\ 6 + k & 7 & 3k + 4 \end{pmatrix}$$

Step 5 : $\mathbf{W}^{-1} = \dfrac{1}{det\ \mathbf{A}} \mathbf{C}^{\mathrm{T}}$

$$\mathbf{W}^{-1} = \frac{1}{7k}\begin{pmatrix} 3k & 0 & 2k \\ 6 & 7 & 4 \\ 6 + k & 7 & 3k + 4 \end{pmatrix}$$

[6 marks]

Answer 4

$$\mathbf{S} = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 4 & 1 \\ 2 & -1 & 0 \end{pmatrix} \Rightarrow \mathbf{S}^{-1} = \begin{pmatrix} -1 & 1 & 1 \\ -2 & 2 & 1 \\ 8 & -7 & -4 \end{pmatrix}$$

[2 marks]

Answer 5

(i)

$$\mathbf{A} = \mathbf{A}^{-1}$$

$$\mathbf{A}\,\mathbf{A} = \mathbf{A}\,\mathbf{A}^{-1} \qquad \text{Pre-multiply both sides by A}$$

$$\mathrm{A}^{2} = \mathbf{I} \qquad \square$$

[2 marks]

(ii) From part (i) if $\mathbf{A} = \mathbf{A}^{-1}$ then $\mathbf{A}^{2} = \mathrm{I}$

$$\therefore \begin{pmatrix} 5 & a & 4 \\ b & -7 & 8 \\ 2 & -2 & c \end{pmatrix} \begin{pmatrix} 5 & a & 4 \\ b & -7 & 8 \\ 2 & -2 & c \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 25 + ab + 8 & 5a - 7a - 8 & 20 + 8a + 4c \\ 5b - 7b + 16 & ab + 49 - 16 & 4b - 56 + 8c \\ 10 - 2b + 2c & 2a + 14 - 2c & 8 - 16 + c^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$a_{2\,1} = 0$$

$$5a - 7a - 8 = 0$$

$$-2a = 8 \Rightarrow a = -4$$

$$a_{1\,2} = 0$$

$$5b - 7b + 16 = 0$$

$$-2b = -16 \Rightarrow b = 8$$

$$a_{2\,3} = 0$$

$$4b - 56 + 8c = 0$$

$$32 - 56 + 8c = 0$$

$$8c = 24 \Rightarrow c = 3$$

[6 marks]